Mathematics Interdisciplinary Research **4** (2019) 193 – 211

A Multiplicative Version of Forgotten Topological Index

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The multiplicative forgotten topological index of G are defined as:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3,$$

where $d_G(v)$ is the degree of the vertex v of G. In this paper, we present upper bounds for the multiplicative forgotten topological index of several graph operations such as sum, Cartesian product, corona product, composition, strong product, disjunction and symmetric difference in terms of the $F-$index and the first Zagreb index of their components. Also, we give explicit formulas for this new graph invariant under two graph operations such as union and Tensor product. Moreover, we obtain the expressions for this new graph invariant of subdivision graphs and vertex – semitotal graphs. Finally, we compare the discriminating ability of indices.

Keywords: Topological index, multiplicative forgotten topological index, graph operations, subdivision graphs, vertex-semitotal graphs.

2010 Mathematics Subject Classification: 05C92.

How to cite this article
A. Yousefi, A. Iranmanesh, A. A. Dobrynin and A. Tehranian, A multiplicative version of forgotten topological index, *Math. Interdisc. Res.* **4** (2019) 193-211.

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Academic Editor: Ali Reza Ashrafi
Received 18 March 2019, Accepted 23 September 2019
DOI: 10.22052/mir.2019.176557.1126

1. Introduction

Throughout this paper, we only consider finite, undirected and simple graphs. In chemical graph theory, the vertices of molecular graph correspond to the atoms in the molecule and the edges correspond to the covalent bonds. From definition of the molecular graph, it is clear that the molecular graph is a nontrivial, connected, finite, undirected and simple graph. A single number, representing a chemical structure, in graph-theoretical terms, is called a *topological descriptor*. It must be a structural invariant, i.e., it does not depend on the labeling or the pictorial representation of a graph. If such a topological descriptor correlates with a molecular property, it is named *molecular index* or *topological index*. In fact, a topological index is a numerical descriptor of the molecular structure derived from the molecular graph. Different topological indices are used for quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) [12, 13, 23, 29]. In [21], Gutman and Trinajstić introduced the most famous vertex-degree based topological indices and named them as the *first Zagreb index* and *second Zagreb index*. These topological indices were elaborated in [20]. For a (molecular) graph G with the vertex set $V(G)$ and the edge set $E(G)$, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of G are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2, \qquad and \qquad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(v)$ denotes the degree of the vertex v of G which is the number of edges incident to v.

The first Zagreb index can also expressed as [14]:

$$M_1(G) = \sum_{uv \in E(G)} \left[d_G(u) + d_G(v) \right] .$$

For more information on the Zagreb indices and their applications see [2, 6, 24, 27, 29, 32].

In [21], beside the first Zagreb index, another topological index defined as:

$$F(G) = \sum_{v \in V(G)} d_G(v)^3.$$

However this index, except (implicity) in a few works about the general first Zagreb index [25, 26] and the zeroth-order general Randić index [22], was not further studied till then, except in a recent article by Furtula and Gutman [15], where they reinvestigated this index and studied some basic properties of this index. They proposed that $F(G)$ be named *forgotten topological index*, or shortly the "$F-$index". In fact, we can rewrite the $F-$index as [14]:

$$F(G) = \sum_{uv \in E(G)} \left[d_G(u)^2 + d_G(v)^2 \right].$$

For more information on the $F-$index see [8, 10, 16–18].

We now define a new graph invariant and name it as the *multiplicative forgotten topological index*. The multiplicative forgotten topological index of a graph G is denoted by $\Pi_F(G)$ and defined as follows:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3.$$

Many interesting graphs are composed of simpler graphs via various graph operations. It is, hence, important to understand how certain invariants of such graph operations are related to the corresponding invariants of their components. In [24], Khalifeh et al. presented some exact formulas for computing the Zagreb indices of some graph operations. In [10], De et al. gave some explicit expressions of the $F-$index of different graph operations and applied their results to compute the $F-$index for some important classes of molecular graphs and nano-structures. In [5], Azari and Iranmanesh obtained some lower bounds for the multiplicative sum Zagreb index of several graph operations in terms of the multiplicative sum Zagreb index and the multiplicative Zagreb indices of their components. For more information on computing topological indices of graph operations see [1, 3, 4, 7, 11].

In papers [9, 19, 30], the authors obtained the expressions for Zagreb indices and coindices of derived graphs. This motivates us to find expressions for the multiplicative forgotten topological index of some derived graphs.

The goal of this paper is to present upper bounds for the multiplicative forgotten topological index of several graph operations such as sum, Cartesian product, corona product, composition, strong product, disjunction and symmetric difference in terms of the $F-$index and the first Zagreb index of their components. Also, another goal of this paper is to give explicit formulas for this new graph invariant under two graph operations such as union and Tensor product. Moreover, another goal of this paper is to obtain the expressions for this new graph invariant of subdivision graphs and vertex-semitotal graphs. Finally, we compare the discriminating ability of indices.

2. The Multiplicative Forgotten Topological Index of some Graph Operations

We begin this section with one standard inequality as follows:

Lemma 2.1. (AM-GM inequality) Let $x_1, \ldots, x_n$ be nonnegative numbers. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n},$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Throughout this section, let G_1 and G_2 be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Also, let P_n, C_n and K_n denote a path graph, cycle graph and complete graph with n number of vertices, respectively.

2.1. Union

The simplest operation we consider here is a union of two graphs. The union $G_1 \cup G_2$ of the graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The degree of a vertex v of $G_1 \cup G_2$ is given by:

$$d_{G_1 \cup G_2}(v) = \begin{cases} d_{G_1}(v) & v \in V(G_1), \\ \\ d_{G_2}(v) & v \in V(G_2). \end{cases}$$

Theorem 2.2. The multiplicative forgotten topological index of $G_1 \cup G_2$ is equal to:

$$\Pi_F(G_1 \cup G_2) = \Pi_F(G_1)\Pi_F(G_2).$$

Proof. Let $G = G_1 \cup G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3 = \prod_{v \in V(G_1)} d_{G_1}(v)^3 \prod_{v \in V(G_2)} d_{G_2}(v)^3 = \Pi_F(G_1)\, \Pi_F(G_2).$$

$\square$

Example 2.3. Consider three graphs P_n, C_n and K_n. We have:

1. $\Pi_F(P_n) = 8^{n-2},\ n \geq 2,$
2. $\Pi_F(C_n) = 8^n,\ n \geq 3,$
3. $\Pi_F(K_n) = (n-1)^{3n},\ n \geq 1,$
4. $\Pi_F(P_n \cup C_m) = 8^{n+m-2}.$

2.2. Tensor Product

The Tensor product $G_1 \otimes G_2$ of the graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1$ is adjacent with v_1 in $G_1]$ and $[u_2$ is adjacent with v_2 in $G_2]$. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \otimes G_2$ is given by:

$$d_{G_1 \otimes G_2}(v) = d_{G_1}(v_1) d_{G_2}(v_2).$$

Theorem 2.4. The multiplicative forgotten topological index of $G_1 \otimes G_2$ is equal to:

$$\Pi_F(G_1 \otimes G_2) = \Pi_F(G_1)^{n_2} \Pi_F(G_2)^{n_1}.$$

Proof. Let $G = G_1 \otimes G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\begin{aligned}
\Pi_F(G) &= \Pi_{(v_1,v_2) \in V(G)} d_G((v_1, v_2))^3 \\
&= \Pi_{v_1 \in V(G_1)} \Pi_{v_2 \in V(G_2)} (d_{G_1}(v_1) d_{G_2}(v_2))^3 \\
&= [\Pi_{v_1 \in V(G_1)} d_{G_1}(v_1)^3]^{n_2} \times [\Pi_{v_2 \in V(G_2)} d_{G_2}(v_2)^3]^{n_1} \\
&= \Pi_F(G_1)^{n_2} \Pi_F(G_2)^{n_1}.
\end{aligned}$$

□

Example 2.5.

1. $\Pi_F(P_n \otimes P_m) = 64^{mn-n-m}$,
2. $\Pi_F(C_n \otimes C_m) = 64^{mn}$,
3. $\Pi_F(K_n \otimes K_m) = [(n-1)(m-1)]^{3mn}$,
4. $\Pi_F(P_n \otimes C_m) = 64^{mn-m}$,
5. $\Pi_F(C_n \otimes K_m) = [2(m-1)]^{3mn}$.

2.3. Sum (Join)

The sum $G_1 + G_2$ of the graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{u_1 u_2 \mid u_1 \in V(G_1), u_2 \in V(G_2)\}$. The degree of a vertex v of $G_1 + G_2$ is given by:

$$d_{G_1+G_2}(v) = \begin{cases} d_{G_1}(v) + n_2 & v \in V(G_1), \\ d_{G_2}(v) + n_1 & v \in V(G_2). \end{cases}$$

Theorem 2.6. The multiplicative forgotten topological index of G_1+G_2 satisfies the following inequality:

$$\begin{aligned}\Pi_F(G_1+G_2) &\leq [\frac{F(G_1)+3n_2M_1(G_1)+6m_1n_2^2+n_1n_2^3}{n_1}]^{n_1}\\ &\times [\frac{F(G_2)+3n_1M_1(G_2)+6m_2n_1^2+n_2n_1^3}{n_2}]^{n_2}.\end{aligned}$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G=G_1+G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G)=\Pi_{v\in V(G)}d_G(v)^3=\Pi_{v\in V(G_1)}(d_{G_1}(v)+n_2)^3\Pi_{v\in V(G_2)}(d_{G_2}(v)+n_1)^3.$$

Now by Lemma 2.1,

$$\begin{aligned}\Pi_F(G) &\leq [\frac{\sum_{v\in V(G_1)}(d_{G_1}(v)+n_2)^3}{n_1}]^{n_1}\times[\frac{\sum_{v\in V(G_2)}(d_{G_2}(v)+n_1)^3}{n_2}]^{n_2}\\ &= [\frac{\sum_{v\in V(G_1)}(d_{G_1}(v)^3+3n_2d_{G_1}(v)^2+3n_2^2d_{G_1}(v)+n_2^3)}{n_1}]^{n_1}\\ &\times [\frac{\sum_{v\in V(G_2)}(d_{G_2}(v)^3+3n_1d_{G_2}(v)^2+3n_1^2d_{G_2}(v)+n_1^3)}{n_2}]^{n_2}\\ &= [\frac{F(G_1)+3n_2M_1(G_1)+3n_2^2(2m_1)+n_2^3n_1}{n_1}]^{n_1}\\ &\times [\frac{F(G_2)+3n_1M_1(G_2)+3n_1^2(2m_2)+n_1^3n_2}{n_2}]^{n_2}.\end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $u_1,v_1\in V(G_1)$ and $u_2,v_2\in V(G_2)$,

$$(d_{G_1}(u_1)+n_2)^3=(d_{G_1}(v_1)+n_2)^3$$

and

$$(d_{G_2}(u_2)+n_1)^3=(d_{G_2}(v_2)+n_1)^3.$$

This implies that for every $u_1,v_1\in V(G_1)$ and $u_2,v_2\in V(G_2)$,

$$d_{G_1}(u_1)=d_{G_1}(v_1),\qquad d_{G_2}(u_2)=d_{G_2}(v_2).$$

Thus, both G_1 and G_2 are regular graphs. □

Example 2.7. Consider two cycle graphs C_n and C_m. We have:

$$\Pi_F(G_1+G_2)=(n+2)^{3m}(m+2)^{3n}.$$

2.4. Cartesian Product

The Cartesian product $G_1 \times G_2$ of the graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2$ and $u_1v_1 \in E(G_1)]$. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \times G_2$ is given by:

$$d_{G_1 \times G_2}(v) = d_{G_1}(v_1) + d_{G_2}(v_2).$$

Theorem 2.8. The multiplicative forgotten topological index of $G_1 \times G_2$ satisfies the following inequality:

$$\Pi_F(G_1 \times G_2) \le [\frac{n_2F(G_1) + n_1F(G_2) + 6(m_2M_1(G_1) + m_1M_1(G_2))}{n_1n_2}]^{n_1n_2}.$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \times G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \prod_{(v_1,v_2)\in V(G)} d_G((v_1, v_2))^3 = \prod_{v_1\in V(G_1)} \prod_{v_2\in V(G_2)} (d_{G_1}(v_1) + d_{G_2}(v_2))^3.$$

Now by Lemma 2.1,

$$\begin{aligned}\Pi_F(G) \le& [\frac{\sum_{v_1\in V(G_1)} \sum_{v_2\in V(G_2)} (d_{G_1}(v_1) + d_{G_2}(v_2))^3}{n_1n_2}]^{n_1n_2}\\ =& \frac{1}{(n_1n_2)^{n_1n_2}}[\sum_{v_1\in V(G_1)} \sum_{v_2\in V(G_2)} (d_{G_1}(v_1)^3\\ &+ 3d_{G_1}(v_1)^2d_{G_2}(v_2) + 3d_{G_2}(v_2)^2d_{G_1}(v_1) + d_{G_2}(v_2)^3)]^{n_1n_2}\\ &[\frac{n_2F(G_1) + 3(2m_2)M_1(G_1) + 3M_1(G_2)(2m_1) + n_1F(G_2)}{n_1n_2}]^{n_1n_2}.\end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1, u_2), (v_1, v_2) \in V(G)$,

$$(d_{G_1}(u_1) + d_{G_2}(u_2))^3 = (d_{G_1}(v_1) + d_{G_2}(v_2))^3.$$

This implies that for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$,

$$d_{G_1}(u_1) = d_{G_1}(v_1), \qquad d_{G_2}(u_2) = d_{G_2}(v_2).$$

Thus, both G_1 and G_2 are regular graphs. □

Example 2.9. Consider a cycle graph C_n and a complete graph K_m. We have:

$$\Pi_F(C_n \times K_m) = (m + 1)^{3mn}.$$

2.5. Corona Product

The corona product $G_1 \circ G_2$ of the graphs G_1 and G_2 is a graph obtained by taking one copy of G_1 and n_1 copies of G_2 and joining the $i-$th vertex of G_1 to every vertex in $i-$th copy of G_2 for $1 \leq i \leq n_1$. The $i-$th copy of G_2 will be denoted by $G_{2,i}$, $1 \leq i \leq n_1$. Different topological indices of the corona product of two graphs have already been studied in [28, 31]. The degree of a vertex $v \in V(G_1 \circ G_2)$ is given by:

$$d_{G_1 \circ G_2}(v) = \begin{cases} d_{G_1}(v) + n_2 & v \in V(G_1), \\ & \\ d_{G_2}(v) + 1 & v \in V(G_{2,i}). \end{cases}$$

Theorem 2.10. The multiplicative forgotten topological index of $G_1 \circ G_2$ satisfies the following inequality:

$$\begin{aligned} \Pi_F(G_1 \circ G_2) &\leq [\frac{F(G_1) + 3n_2 M_1(G_1) + 6m_1 n_2^2 + n_1 n_2^3}{n_1}]^{n_1} \\ &\times [\frac{F(G_2) + 3M_1(G_2) + 6m_2 + n_2}{n_2}]^{n_1 n_2}. \end{aligned}$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \circ G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F\left(G\right) = \prod_{v \in V(G)} d_G(v)^3 = \prod_{v \in V(G_1)} \left(d_{G_1}\left(v\right) + n_2\right)^3 \times \left[\prod_{v \in V(G_2)} \left(d_{G_2}\left(v\right) + 1\right)^3\right]^{n_1}.$$

Now by Lemma 2.1,

$$\begin{aligned} \Pi_F(G) &\leq [\frac{\sum_{v \in V(G_1)} (d_{G_1}(v) + n_2)^3}{n_1}]^{n_1} \times [\frac{\sum_{v \in V(G_1)} (d_{G_2}(v) + 1)^3}{n_2}]^{n_1 n_2} \\ &= [\frac{\sum_{v \in V(G_1)} [d_{G_1}(v)^3 + 3n_2 d_{G_1}(v)^2 + 3n_2^2 d_{G_1}(v) + n_2^3]}{n_1}]^{n_1} \\ &\times [\frac{\sum_{v \in V(G_2)} (d_{G_2}(v)^3 + 3d_{G_2}(v)^2 + 3d_{G_2}(v) + 1)}{n_2}]^{n_1 n_2} \\ &= [\frac{F(G_1) + 3n_2 M_1(G_1) + 3n_2^2(2m_1) + n_2^3 n_1}{n_1}]^{n_1} \\ &\times [\frac{F(G_2) + 3M_1(G_2) + 3(2m_2) + n_2}{n_2}]^{n_1 n_2}. \end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$,

$$(d_{G_1}(u_1)+n_2)^3=(d_{G_1}(v_1)+n_2)^3$$

and

$$(d_{G_2}(u_2)+1)^3=(d_{G_2}(v_2)+1)^3.$$

This implies that for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$,

$$d_{G_1}(u_1)=d_{G_1}(v_1), \qquad d_{G_2}(u_2)=d_{G_2}(v_2).$$

Thus, both G_1 and G_2 are regular graphs.

□

Example 2.11. Consider a cycle graph C_n and a complete graph K_m. We have

$$\Pi_F(C_n \circ K_m)=m^{3mn}(m+2)^{3n}.$$

2.6. Composition

The composition $G_1[G_2]$ of the graphs G_1 and G_2 is a graph with the vertex set $V(G_1)\times V(G_2)$, and vertex (u_1,u_2) is adjacent with vertex (v_1,v_2) whenever [u_1 is adjacent with v_1 in G_1] or [$u_1=v_1$ and u_2 is adjacent with v_2 in G_2]. The degree of a vertex $v=(v_1,v_2)$ of $G_1[G_2]$ is given by:

$$d_{G_1[G_2]}(v)=n_2d_{G_1}(v_1)+d_{G_2}(v_2).$$

Theorem 2.12. The multiplicative forgotten topological index of $G_1[G_2]$ satisfies the following inequality:

$$\Pi_F(G_1[G_2])\le [\frac{n_2^4F(G_1)+n_1F(G_2)+6n_2(n_2m_2M_1(G_1)+m_1M_1(G_2))}{n_1n_2}]^{n_1n_2}.$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G=G_1[G_2]$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G)=\prod_{(v_1,v_2)\in V(G)} d_G((v_1,v_2))^3=\prod_{v_1\in V(G_1)}\prod_{v_2\in V(G_2)}(n_2d_{G_1}(v_1)+d_{G_2}(v_2))^3.$$

Now by Lemma 2.1,

$$\begin{aligned}\Pi_F(G)\le&[\frac{\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}(n_2d_{G_1}(v_1)+d_{G_2}(v_2))^3}{n_1n_2}]^{n_1n_2}\\ =&\frac{1}{(n_1n_2)^{n_1n_2}}[\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}(n_2^3d_{G_1}(v_1)^3\\ &+3n_2^2d_{G_1}(v_1)^2d_{G_2}(v_2)+3n_2d_{G_1}(v_1)d_{G_2}(v_2)^2+d_{G_2}(v_2)^3)]^{n_1n_2}\\ =&[\frac{n_2^4F(G_1)+6n_2^2m_2M_1(G_1)+6n_2m_1M_1(G_2)+n_1F(G_2)}{n_1n_2}]^{n_1n_2}.\end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1,u_2),(v_1,v_2)\in V(G)$,

$$(n_2 d_{G_1}(u_1) + d_{G_2}(u_2))^3 = (n_2 d_{G_1}(v_1) + d_{G_2}(v_2))^3.$$

This implies that both G_1 and G_2 are regular graphs.

□

Example 2.13. Consider two cycle graphs C_n and C_m. We have:

$$\Pi_F(C_n[C_m]) = 2^{3mn}(m+1)^{3mn}.$$

2.7. Strong Product

The strong product $G_1 \boxtimes G_2$ of the graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[\ u_2 = v_2$ and $u_1v_1 \in E(G_1)]$ or $[u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)]$. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \boxtimes G_2$ is given by:

$$d_{G_1 \boxtimes G_2}(v) = d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1) d_{G_2}(v_2).$$

Theorem 2.14. The multiplicative forgotten topological index of $G_1 \boxtimes G_2$ satisfies the following inequality:

$$\begin{aligned} \Pi_F(G_1 \boxtimes G_2) \quad &\leq \quad \frac{1}{(n_1n_2)^{n_1n_2}}[F(G_1)F(G_2) + (n_2 + 6m_2 + 3M_1(G_2))F(G_1) \\ &+ \quad (n_1 + 6m_1 + 3M_1(G_1))F(G_2) + 6(m_2M_1(G_1) + m_1M_1(G_2) \\ &+ \quad M_1(G_1)M_1(G_2))]^{n_1n_2}. \end{aligned}$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \boxtimes G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\begin{aligned} \Pi_F(G) &= \prod_{(v_1,v_2)\in V(G)} d_G((v_1, v_2))^3 \\ &= \prod_{v_1\in V(G_1)} \prod_{v_2\in V(G_2)} [d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)\, d_{G_2}(v_2)]^3. \end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}
\Pi_F(G) &\leq [\frac{\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}[d_{G_1}(v_1)+d_{G_2}(v_2)+d_{G_1}(v_1)d_{G_2}(v_2)]^3}{n_1n_2}]^{n_1n_2} \\
&= \frac{1}{(n_1n_2)^{n_1n_2}}[\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}[d_{G_1}(v_1)^3+d_{G_2}(v_2)^3 \\
&+ 3d_{G_1}(v_1)^2d_{G_2}(v_2)+3d_{G_1}(v_1)d_{G_2}(v_2)^2+3d_{G_1}(v_1)^3d_{G_2}(v_2) \\
&+ 6d_{G_1}(v_1)^2d_{G_2}(v_2)^2+3d_{G_1}(v_1)d_{G_2}(v_2)^3+3d_{G_1}(v_1)^3d_{G_2}(v_2)^2 \\
&+ 3d_{G_1}(v_1)^2d_{G_2}(v_2)^3+d_{G_1}(v_1)^3d_{G_2}(v_2)^3]]^{n_1n_2} \\
&= \frac{1}{(n_1n_2)^{n_1n_2}}[n_2F(G_1)+n_1F(G_2)+6m_2M_1(G_1)+6m_1M_1(G_2) \\
&+ 6m_2F(G_1)+6M_1(G_1)M_1(G_2)+6m_1F(G_2)+3M_1(G_2)F(G_1) \\
&+ 3M_1G_1)F(G_2)+F(G_1)F(G_2)]^{n_1n_2}.
\end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1,u_2),(v_1,v_2)\in V(G)$,

$$(d_{G_1}(u_1)+d_{G_2}(u_2)+d_{G_1}(u_1)d_{G_2}(u_2))^3=(d_{G_1}(v_1)+d_{G_2}(v_2)+d_{G_1}(v_1)d_{G_2}(v_2))^3.$$

This implies that for every $u_1,v_1\in V(G_1)$ and $u_2,v_2\in V(G_2)$,

$$d_{G_1}(u_1)=d_{G_1}(v_1),d_{G_2}(u_2)=d_{G_2}(v_2).$$

Thus, both G_1 and G_2 are regular graphs. □

2.8. Disjunction

The disjunction $G_1\vee G_2$ of two graphs G_1 and G_2 is a graph with the vertex set $V(G_1)\times V(G_2)$ in which (u_1,u_2) is adjacent with (v_1,v_2) whenever [u_1 is adjacent with v_1 in G_1] or [u_2 is adjacent with v_2 in G_2]. The degree of a vertex $v=(v_1,v_2)$ of $G_1\vee G_2$ is given by:

$$d_{G_1\vee G_2}(v)=n_2d_{G_1}(v_1)+n_1d_{G_2}(v_2)-d_{G_1}(v_1)d_{G_2}(v_2).$$

Theorem 2.15. The multiplicative forgotten topological index of $G_1\vee G_2$ satisfies the following inequality:

$$\begin{aligned}
\Pi_F(G_1\vee G_2) &\leq \frac{1}{(n_1n_2)^{n_1n_2}}[(n_2^4-6n_2^2m_2+3n_2M_1(G_2))F(G_1) \\
&+ (n_1^4-6n_1^2m_1+3n_1M_1(G_1))F(G_2)-F(G_1)F(G_2) \\
&+ 6n_1n_2(n_2m_2M_1(G_1)+n_1m_1M_1(G_2)-M_1(G_1)M_1(G_2))]^{n_1n_2}.
\end{aligned}$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \vee G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\begin{aligned}\Pi_F(G) &= \prod_{(v_1,v_2)\in V(G)} d_G((v_1,v_2))^3 \\ &= \prod_{v_1\in V(G_1)} \prod_{v_2\in V(G_2)} [n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - d_{G_1}(v_1)\, d_{G_2}(v_2)]^3.\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}\Pi_F(G) \leq& [\frac{\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}[n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - d_{G_1}(v_1) d_{G_2}(v_2)]^3}{n_1 n_2}]^{n_1 n_2} \\ =& \frac{1}{(n_1n_2)^{n_1n_2}}[\sum_{v_1\in V(G_1)}\sum_{v_2\in V(G_2)}[n_2^3 d_{G_1}(v_1)^3 + 3n_2^2 n_1 d_{G_1}(v_1)^2 d_{G_2}(v_2) \\ &+3n_2 n_1^2 d_{G_1}(v_1) d_{G_2}(v_2)^2 + n_1^3 d_{G_2}(v_2)^3 - 3n_2^2 d_{G_1}(v_1)^3 d_{G_2}(v_2) \\ &-6n_2 n_1 d_{G_1}(v_1)^2 d_{G_2}(v_2)^2 - 3n_1^2 d_{G_1}(v_1) d_{G_2}(v_2)^3 + 3n_2 d_{G_1}(v_1)^3 d_{G_2}(v_2)^2 \\ &+3n_1 d_{G_1}(v_1)^2 d_{G_2}(v_2)^3 - d_{G_1}(v_1)^3 d_{G_2}(v_2)^3]]^{n_1n_2} \\ =& \frac{1}{(n_1n_2)^{n_1n_2}}[n_2^3 n_2 F(G_1) + 3n_2^2 n_1 (2m_2) M_1(G_1) + 3n_2 n_1^2 (2m_1) M_1(G_2) \\ &+n_1^3 n_1 F(G_2) - 3n_2^2(2m_2)F(G_1) - 6n_2n_1 M_1(G_1)M_1(G_2) \\ &-3n_1^2(2m_1)F(G_2) + 3n_2 M_1(G_2)F(G_1) \\ &+3n_1 M_1(G_1)F(G_2) - F(G_1)F(G_2)]^{n_1n_2}.\end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1, u_2), (v_1, v_2) \in V(G)$,

$$\begin{aligned}(n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - d_{G_1}(u_1) d_{G_2}(u_2))^3 = (n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) \\ - d_{G_1}(v_1) d_{G_2}(v_2))^3.\end{aligned}$$

This implies that for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$, $d_{G_1}(u_1) = d_{G_1}(v_1)$ and $d_{G_2}(u_2) = d_{G_2}(v_2)$. Thus, both G_1 and G_2 are regular graphs. □

2.9. Symmetric Difference

The symmetric difference $G_1 \oplus G_2$ of two graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2) whenever [u_1 is adjacent with v_1 in G_1] or [u_2 is adjacent with v_2 in G_2], but not both. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \oplus G_2$ is given by:

$$d_{G_1\oplus G_2}(v) = n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - 2 d_{G_1}(v_1)\, d_{G_2}(v_2).$$

Theorem 2.16. The multiplicative forgotten topological index of $G_1 \oplus G_2$ satisfies the following inequality:

$$\begin{aligned} \Pi_F(G_1 \oplus G_2) &\leq \frac{1}{(n_1 n_2)^{n_1 n_2}} [(n_2^4 - 12 n_2^2 m_2 + 12 n_2 M_1(G_2)) F(G_1) \\ &+ (n_1^4 - 12 n_1^2 m_1 + 12 n_1 M_1(G_1)) F(G_2) - 8 F(G_1) F(G_2) \\ &+ 6 n_1 n_2 (n_2 m_2 M_1(G_1) + n_1 m_1 M_1(G_2) - 2 M_1(G_1) M_1(G_2))]^{n_1 n_2}. \end{aligned}$$

The equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \oplus G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\begin{aligned} \Pi_F(G) &= \prod_{(v_1,v_2)\in V(G)} d_G((v_1,v_2))^3 \\ &= \prod_{v_1\in V(G_1)} \prod_{v_2\in V(G_2)} [n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - 2 d_{G_1}(v_1)\, d_{G_2}(v_2)]^3. \end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned} \Pi_F(G) &\leq \left[\frac{\sum_{v_1\in V(G_1)} \sum_{v_2\in V(G_2)} [n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - 2 d_{G_1}(v_1) d_{G_2}(v_2)]^3}{n_1 n_2}\right]^{n_1 n_2} \\ &= \frac{1}{(n_1 n_2)^{n_1 n_2}} [(n_2^4 - 12 n_2^2 m_2 + 12 n_2 M_1(G_2)) F(G_1) + (n_1^4 - 12 n_1^2 m_1 \\ &+ 12 n_1 M_1(G_1)) F(G_2) - 8 F(G_1) F(G_2) + 6 n_1 n_2 (n_2 m_2 M_1(G_1) \\ &+ n_1 m_1 M_1(G_2) - 2 M_1(G_1) M_1(G_2))]^{n_1 n_2}. \end{aligned}$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1,u_2), (v_1,v_2) \in V(G)$,

$$\begin{aligned} (n_2 d_{G_1}(u_1) &+ n_1 d_{G_2}(u_2) - 2 d_{G_1}(u_1) d_{G_2}(u_2))^3 = (n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) \\ &- 2 d_{G_1}(v_1) d_{G_2}(v_2))^3. \end{aligned}$$

This implies that both G_1 and G_2 are regular graphs. □

3. The Multiplicative Forgotten Topological Index of Subdivision Graphs and Vertex-Semitotal Graphs

In papers [9, 19, 30], the authors obtained the expressions for Zagreb indices and coindices of derived graphs. This motivates us to find expressions for the multiplicative forgotten topological index of some derived graphs.

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *subdivision graph* $S = S(G)$ of G is a graph that is obtained by inserting a new vertex of degree two on each edge of G. The *vertex–semitotal graph* $T_1 = T_1(G)$ of G is a graph with the vertex set $V(T_1) = V(S)$ and the edge set $E(T_1) = E(S) \cup E(G)$.

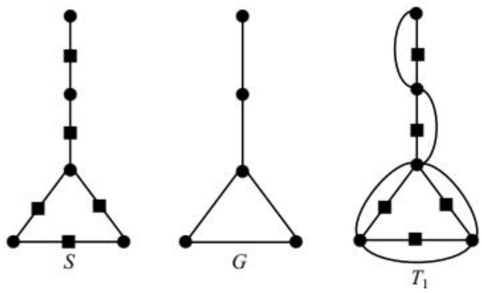

Figure 1: Two graphs derived from the graph G. The vertices of these derived graphs, corresponding to the vertices of the parent graph G, are indicated by circles. The vertices of these graphs, corresponding to the edges of the parent graph G are indicated by squares.

Theorem 3.1. Let S be the subdivision graph of the graph G and $|E(G)| = m$. Then

$$\Pi_F(S) = 8^m \Pi_F(G).$$

Proof. By the definition of the multiplicative forgotten topological index and subdivision graph of G, we have:

$$\Pi_F(S) = \prod_{v \in V(S)} d_S(v)^3 = \prod_{v \in V(G)} d_G(v)^3 \prod_{uv \in E(G)} 2^3 = 8^m \Pi_F(G).$$

□

Theorem 3.2. Let T_1 be the vertex–semitotal graph of the graph G and $|V(G)| = n$ and $|E(G)| = m$. Then

$$\Pi_F(T_1) = 8^{n+m} \Pi_F(G).$$

Proof. By the definition of the multiplicative forgotten topological index and vertex–semitotal graph of G, we have:

$$\Pi_F(T_1) = \prod_{v \in V(T_1)} d_{T_1}(v)^3 = \prod_{v \in V(G)} (2d_G(v))^3 \prod_{uv \in E(G)} 2^3 = 8^{n+m} \Pi_F(G).$$

□

The *k-subdivision graph* $Sk = Sk(G)$ of G is a graph that is obtained by inserting k new vertex of degree two on each edge of G.

Theorem 3.3. Let Sk be the k-subdivision of the graph G and $|E(G)| = m$. Then

$$\Pi_F(Sk) = 8^{mk}\Pi_F(G) = 64^m \Pi_F(P_k)^m \Pi_F(G).$$

Proof. By construction of the k-subdivision of G and Example 2.3, we have:

$$\begin{aligned}\Pi_F(Sk) &= \prod_{v\in V(Sk)} d_{Sk}(v)^3 = \prod_{v\in V(G)} d_G(v)^3 \prod_{uv\in E(G)} 2^{3k} \\ &= 8^{mk}\Pi_F(G) = \left(64 \cdot 8^{k-2}\right)^m \Pi_F(G).\end{aligned}$$

□

In general case, one can use a new graph H for edge subdivisions of G. There are some simple constructions in which vertex degrees of graphs don't change.

Let H be an arbitrary graph and $xy \in E(H)$. The *H'-subdivision graph* $SH = SH(G)$ of G is a graph that is obtained by inserting graph $H' = H - xy$ on each edge uv of G such that $ux,\ vy\ \in E(SH)$. For instance, k-subdivision can be considered as C_k-subdivision of a graph, $k \geq 3$.

Theorem 3.4. Let SH be the H'-subdivision of G and $|E(G)| = m$. Then

$$\Pi_F(SH) = \Pi_F(H)^m \Pi_F(G).$$

Proof. By the construction of edge subdivisions of G, we have:

$$\begin{aligned}\Pi_F(SH) &= \prod_{v\in V(SH)} d_{SH}(v)^3 = \prod_{v\in V(G)} d_G(v)^3 \prod_{uv\in E(G)} \Pi_F(H) \\ &= \Pi_F(H)^m \Pi_F(G).\end{aligned}$$

□

Suppose that the graph H has two vertices of degree one (graph-monomer). To keep vertex degrees, one can subdivide every edge of G by a chain consisting of k copies of H (one edge separates two consecutive non-terminal copies of H). Then one can write for the resulting graph SP:

$$\begin{aligned}\Pi_F(SP) &= \prod_{v\in V(SP)} d_{SP}(v)^3 = \prod_{v\in V(G)} d_G(v)^3 \prod_{uv\in E(G)} \Pi_F(H)^k \\ &= \Pi_F(H)^{mk} \Pi_F(G).\end{aligned}$$

4. Comparison of the Discriminating Ability Indices

Topological indices are often used for quantifying structure of molecular graphs. A good ability of an index to distinguish between nonisomorphic graphs is important for chemical applications. Here we consider index Π_F in compare with the other degree-based indices M_1, M_2 and F. A topological index is called degenerate if it possesses the same value for more than one graph. A set of graphs with the same value of a given index forms a degeneracy class. The discriminating ability of an index can be characterized by the average size of degeneracy classes:

$$\{\text{average size}\} = \{\text{number of graphs}\} \ / \ \{\text{number of the degeneracy classes}\}.$$

Table 1: Size of average degeneracy classes of indices for small $n-$vertex graph.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	trees															
M_1	2	2	4	7	6	26	55	118	263	168	1380	797	7742	3830	8946	20231
F	2	2	3	6	5	20	39	81	166	92	840	363	4424	1622	3517	7884
Π_F	2	2	3	6	5	20	39	81	166	92	805	358	4129	1506	3178	6702
M_2	1	2	2	3	3	9	20	39	85	75	429	338	2382	1656	3691	8718
	unicyclic graphs															
M_1	2	4	7	14	30	65	148	341	818	1971	4792	11590				
F	2	3	6	10	21	42	85	182	397	876	2009	4710				
Π_F	2	3	6	10	19	41	85	182	393	862	1923	4339				
M_2	1	2	3	6	13	27	61	139	327	772	1876	4636				
	bicyclic graphs															
M_1	3	7	17	42	107	285	761	2034	5569	15228	41119					
F	2	5	11	26	60	145	353	866	2228	5779	15290					
Π_F	2	5	10	23	58	140	344	858	2103	5359	13346					
M_2	1	4	6	16	39	105	275	760	2046	5577	15138					

Table 1 contains comparative data for trees, uni- and bicyclic graphs of small order. Since the set of possible values of index Π_F very quickly increases, one can expect that the sizes of its average degeneracy classes will be less than for the other indices. However, we observe this for graphs of sufficiently large order. For instance, this is valid for trees starting from 19 vertices (see data in bold in Table 1 for Π_F and M_2). Topological index M_2 has the better discriminating power for most considered classes of graphs of small order.

Acknowledgment. The Third author was supported by the Russian Foundation for Basic Research (project numbers 19-01-00682, 17-51-560008) and the second author was supported by Iranian National Science Foundation (project number 96004167).

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] A. R. Ashrafi, T. Došlić and A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.* **158** (2010) 1571–1578.

[2] M. Azari and A. Iranmanesh, Chemical graphs constructed from rooted product and their Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 901–919.

[3] M. Azari and A. Iranmanesh, Computation of the edge Wiener indices of the sum of graphs, *Ars Combin.* **100** (2011) 113–128.

[4] M. Azari and A. Iranmanesh, Computing the eccentric– distance sum for graph operations, *Discrete Appl. Math.* **161** (2013) 2827–2840.

[5] M. Azari and A. Iranmanesh, Some inequalities for the multiplicative sum Zagreb index of graph operations, *J. Math. Inequal.* **9** (2015) 727–738.

[6] M. Azari, A. Iranmanesh and I. Gutman, Zagreb indices of bridge and chain graphs, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 921–938.

[7] M. Azari, A. Iranmanesh and A. Tehranian, A method for calculating an edge version of the Wiener number of a graph operation, *Util. Math.* **87** (2012) 151–164.

[8] B. Basavanagoud and V. R. Desai, Forgotten topological index and hyper − Zagreb index of generalized transformation graphs, *Bulletin of Mathematical Sciences and Applications* **14** (2016) 1–6.

[9] B. Basavanagoud, I. Gutman and C. S. Gali, On second Zagreb index and coindex of some derived graphs, *Kragujevac J. Sci.* **37** (2015) 113–121.

[10] N. De, S. M. A. Nayeem and A. Pal, $F-$index of some graph operations, *Discrete Math. Algorithms Appl.* **8** (2016) 1650025, 17 pp.

[11] N. De, S. M. A. Nayeem and A. Pal, Reformulated first Zagreb index of some graph operations, *Mathematics*, **3** (2015) 945-960.

[12] J. Devillers and A. T. Balaban (Eds.), *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, 1999.

[13] M. V. Diudea, *QSPR/ QSAR Studies by Molecular Descriptors*, Nova Sci. Publ., Huntington, NY, 2000.

[14] T. Došlić, B. Furtula, A. Graovac, I. Gutman, S. Moradi and Z. Yarahmadi, On vertex – degree – based molecular structure descriptors, *MATCH Commun. Math. Comput. Chem.* **66** (2011) 613–626.

[15] B. Furtula and I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015) 1184–1190.

[16] B. Furtula, I. Gutman, Z. Kovijanić Vukićević, G. Lekishvili and G. Popivoda, On an old/new degree – based topological index, *Bulletin T.CXLVIII de l′Académie serbe des sciences et des arts* (2015) 19–31.

[17] W. Gao, M. R. Farahani and L. Shi, The forgotten topological index of some drug structures, *Acta Medica Mediterranea* **32** (2016) 579–585.

[18] S. Ghobadi and M. Ghorbaninejad, The forgotten topological index of four operations on some special graphs, *Bulletin of Mathematical Sciences and Applications* **16** (2016) 89–95.

[19] I. Gutman, B. Furtula, Ž. Kovijanić Vukićević and G. Popivoda, On Zagreb indices and coindices, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 5–16.

[20] I. Gutman, B. Ruščić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.* **62** (1975) 33–99.

[21] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total π–electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[22] Y. Hu, X. Li, Y. Shi, T. Xu and I. Gutman, On molecular graphs with smallest and greatest zeroth–order general Randić index, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 425–434.

[23] M. Karelson, *Molecular Descriptors in QSAR/QSPR*, Wiley–Interscience, New York, 2000.

[24] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.* **157** (2009) 804–811.

[25] X. Li and H. Zhao, Trees with the first three smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 57–62.

[26] X. Li and J. Zheng, A unified approach to the extremal trees for different indices, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 195–208.

[27] S. Nikolić, G. Kovačević, A. Miličević, and N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta* **76** (2003) 113–124.

[28] K. Pattabiraman and P. Kandan, Weighted PI index of corona product of graphs, *Discret. Math. Algorithms Appl.* **6** (2014) 1450055, 9 pp.

[29] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley – VCH, Weinheim, 2000.

[30] M. Wang and H. Hua, More on Zagreb coindices of composite graphs, *Int. Math. Forum* **7** (2012) 669–673.

[31] Z. Yarahmadi and A. R. Ashrafi, The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs, *Filomat* **26** (2012) 467–472.

[32] B. Zhou, Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **52** (2004) 113–118.

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