A Multiplicative Version of Forgotten Topological Index

Asghar Yousefi, Ali Iranmanesh*, Andrey A. Dobrynin and Abolfazl Tehranian

Abstract

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The multiplicative forgotten topological index of $G$ are defined as:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3,$$

where $d_G(v)$ is the degree of the vertex $v$ of $G$. In this paper, we present upper bounds for the multiplicative forgotten topological index of several graph operations such as sum, Cartesian product, corona product, composition, strong product, disjunction and symmetric difference in terms of the $F-$index and the first Zagreb index of their components. Also, we give explicit formulas for this new graph invariant under two graph operations such as union and Tensor product. Moreover, we obtain the expressions for this new graph invariant of subdivision graphs and vertex – semitotal graphs. Finally, we compare the discriminating ability of indices.

Keywords: Topological index, multiplicative forgotten topological index, graph operations, subdivision graphs, vertex-semitotal graphs.

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1. Introduction

Throughout this paper, we only consider finite, undirected and simple graphs. In chemical graph theory, the vertices of molecular graph correspond to the atoms in the molecule and the edges correspond to the covalent bonds. From definition of the molecular graph, it is clear that the molecular graph is a nontrivial, connected, finite, undirected and simple graph. A single number, representing a chemical structure, in graph-theoretical terms, is called a topological descriptor. It must be a structural invariant, i.e., it does not depend on the labeling or the pictorial representation of a graph. If such a topological descriptor correlates with a molecular property, it is named molecular index or topological index. In fact, a topological index is a numerical descriptor of the molecular structure derived from the molecular graph. Different topological indices are used for quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) [12,13,23,29]. In [21], Gutman and Trinajstić introduced the most famous vertex-degree based topological indices and named them as the first Zagreb index and second Zagreb index. These topological indices were elaborated in [20]. For a (molecular) graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of $G$ are defined as follows:

\[ M_1(G) = \sum_{v \in V(G)} d_G(v)^2, \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v), \]

where $d_G(v)$ denotes the degree of the vertex $v$ of $G$ which is the number of edges incident to $v$.

The first Zagreb index can also expressed as [14]:

\[ M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]. \]

For more information on the Zagreb indices and their applications see [2,6,24,27,29,32].

In [21], beside the first Zagreb index, another topological index defined as:

\[ F(G) = \sum_{v \in V(G)} d_G(v)^3. \]

However this index, except (implicitly) in a few works about the general first Zagreb index [25,26] and the zeroth-order general Randić index [22], was not further studied till then, except in a recent article by Furtula and Gutman [15], where they reinvestigated this index and studied some basic properties of this index. They proposed that $F(G)$ be named forgotten topological index, or shortly the “$F$–index”. In fact, we can rewrite the $F$–index as [14]:

\[ F(G) = \sum_{v \in V(G)} d_G(v)^2. \]
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For more information on the $F$–index see [8, 10, 16–18].

We now define a new graph invariant and name it as the multiplicative forgotten topological index. The multiplicative forgotten topological index of a graph $G$ is denoted by $\Pi_F(G)$ and defined as follows:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3.$$ 

Many interesting graphs are composed of simpler graphs via various graph operations. It is, hence, important to understand how certain invariants of such graph operations are related to the corresponding invariants of their components. In [24], Khalifeh et al. presented some exact formulas for computing the Zagreb indices of some graph operations. In [10], De et al. gave some explicit expressions of the $F$–index of different graph operations and applied their results to compute the $F$–index for some important classes of molecular graphs and nano-structures. In [5], Azari and Iranmanesh obtained some lower bounds for the multiplicative sum Zagreb index of several graph operations in terms of the multiplicative sum Zagreb index and the multiplicative Zagreb indices of their components. For more information on computing topological indices of graph operations see [1, 3, 4, 7, 11].

In papers [9, 19, 30], the authors obtained the expressions for Zagreb indices and coindices of derived graphs. This motivates us to find expressions for the multiplicative forgotten topological index of some derived graphs.

The goal of this paper is to present upper bounds for the multiplicative forgotten topological index of several graph operations such as sum, Cartesian product, corona product, composition, strong product, disjunction and symmetric difference in terms of the $F$–index and the first Zagreb index of their components. Also, another goal of this paper is to give explicit formulas for this new graph invariant under two graph operations such as union and Tensor product. Moreover, another goal of this paper is to obtain the expressions for this new graph invariant of subdivision graphs and vertex-semitotal graphs. Finally, we compare the discriminating ability of indices.

2. The Multiplicative Forgotten Topological Index of some Graph Operations

We begin this section with one standard inequality as follows:

$$F(G) = \sum_{uv \in E(G)} \left[ d_G(u)^2 + d_G(v)^2 \right].$$
Lemma 2.1. (AM-GM inequality) Let $x_1, \ldots, x_n$ be nonnegative numbers. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Throughout this section, let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices and $m_1$ and $m_2$ edges, respectively. Also, let $P_n$, $C_n$ and $K_n$ denote a path graph, cycle graph and complete graph with $n$ number of vertices, respectively.

2.1. Union

The simplest operation we consider here is a union of two graphs. The union $G_1 \cup G_2$ of the graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The degree of a vertex $v$ of $G_1 \cup G_2$ is given by:

$$d_{G_1 \cup G_2}(v) = \begin{cases} 
  d_{G_1}(v) & v \in V(G_1), \\
  d_{G_2}(v) & v \in V(G_2). 
\end{cases}$$

Theorem 2.2. The multiplicative forgotten topological index of $G_1 \cup G_2$ is equal to:

$$\Pi_F(G_1 \cup G_2) = \Pi_F(G_1) \Pi_F(G_2).$$

Proof. Let $G = G_1 \cup G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3 = \prod_{v \in V(G_1)} d_{G_1}(v)^3 \prod_{v \in V(G_2)} d_{G_2}(v)^3 = \Pi_F(G_1) \Pi_F(G_2).$$

Example 2.3. Consider three graphs $P_n$, $C_n$ and $K_n$. We have:

1. $\Pi_F(P_n) = 8^{n-2}$, $n \geq 2$,
2. $\Pi_F(C_n) = 8^n$, $n \geq 3$,
3. $\Pi_F(K_n) = (n-1)^{3n}$, $n \geq 1$,
4. $\Pi_F(P_n \cup C_m) = 8^{n+m-2}$. 
2.2. Tensor Product

The Tensor product $G_1 \otimes G_2$ of the graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if $u_1$ is adjacent with $v_1$ in $G_1$ and $u_2$ is adjacent with $v_2$ in $G_2$. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \otimes G_2$ is given by:

$$d_{G_1 \otimes G_2}(v) = d_{G_1}(v_1)d_{G_2}(v_2).$$

**Theorem 2.4.** The multiplicative forgotten topological index of $G_1 \otimes G_2$ is equal to:

$$\Pi_F(G_1 \otimes G_2) = \Pi_F(G_1)^{n_2}\Pi_F(G_2)^{n_1}.$$

**Proof.** Let $G = G_1 \otimes G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \Pi_{(v_1, v_2) \in V(G)}d_G((v_1, v_2))^3 = \Pi_{v_1 \in V(G_1)}\Pi_{v_2 \in V(G_2)}(d_{G_1}(v_1)d_{G_2}(v_2))^3 = [\Pi_{v_1 \in V(G_1)}d_{G_1}(v_1)^3]^{n_2} \times [\Pi_{v_2 \in V(G_2)}d_{G_2}(v_2)^3]^{n_1} = \Pi_F(G_1)^{n_2}\Pi_F(G_2)^{n_1}.\qed$$

**Example 2.5.**

1. $\Pi_F(P_n \otimes P_m) = 64^{mn-m-n}$,
2. $\Pi_F(C_n \otimes C_m) = 64^{mn}$,
3. $\Pi_F(K_n \otimes K_m) = [(n-1)(m-1)]^{3mn}$,
4. $\Pi_F(P_n \otimes C_m) = 64^{mn-m}$,
5. $\Pi_F(C_n \otimes K_m) = [2(m-1)]^{3mn}$.

2.3. Sum (Join)

The sum $G_1 + G_2$ of the graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1), u_2 \in V(G_2)\}$. The degree of a vertex $v$ of $G_1 + G_2$ is given by:

$$d_{G_1+G_2}(v) = \begin{cases} 
    d_{G_1}(v) + n_2 & v \in V(G_1), \\
    d_{G_2}(v) + n_1 & v \in V(G_2). 
\end{cases}$$
Theorem 2.6. The multiplicative forgotten topological index of $G_1 + G_2$ satisfies the following inequality:

$$\Pi_F(G_1 + G_2) \leq \left[ \frac{F(G_1) + 3n_2M_1(G_1) + 6n_1n_2^2 + n_1n_2^3}{n_1} \right]^{n_1} \times \left[ \frac{F(G_2) + 3n_1M_1(G_2) + 6n_2n_1^2 + n_2n_1^3}{n_2} \right]^{n_2}.$$ 

The equality holds if and only if both $G_1$ and $G_2$ are regular graphs.

Proof. Let $G = G_1 + G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \Pi_{v\in V(G)}d_G(v)^3 = \Pi_{v\in V(G_1)}(d_{G_1}(v) + n_2)^3\Pi_{v\in V(G_2)}(d_{G_2}(v) + n_1)^3.$$ 

Now by Lemma 2.1,

$$\Pi_F(G) \leq \left[ \sum_{v\in V(G_1)}(d_{G_1}(v) + n_2)^3 \right]^{n_1} \times \left[ \sum_{v\in V(G_2)}(d_{G_2}(v) + n_1)^3 \right]^{n_2}$$

$$= \left[ \sum_{v\in V(G_1)}(d_{G_1}(v)^3 + 3n_2d_{G_1}(v)^2 + 3n_2^2d_{G_1}(v) + n_2^3) \right]^{n_1}$$

$$\times \left[ \sum_{v\in V(G_2)}(d_{G_2}(v)^3 + 3n_1d_{G_2}(v)^2 + 3n_1^2d_{G_2}(v) + n_1^3) \right]^{n_2}$$

$$= \left[ \frac{F(G_1) + 3n_2M_1(G_1) + 3n_2^2(2m_1) + n_2^3n_1}{n_1} \right]^{n_1}$$

$$\times \left[ \frac{F(G_2) + 3n_1M_1(G_2) + 3n_1^2(2m_2) + n_1^3n_2}{n_2} \right]^{n_2}.$$ 

By Lemma 2.1, the above equality holds if and only if for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$,

$$(d_{G_1}(u_1) + n_2)^3 = (d_{G_1}(v_1) + n_2)^3$$

and

$$(d_{G_2}(u_2) + n_1)^3 = (d_{G_2}(v_2) + n_1)^3.$$ 

This implies that for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$,

$$d_{G_1}(u_1) = d_{G_1}(v_1), \quad d_{G_2}(u_2) = d_{G_2}(v_2).$$ 

Thus, both $G_1$ and $G_2$ are regular graphs. \qed

Example 2.7. Consider two cycle graphs $C_n$ and $C_m$. We have:

$$\Pi_F(G_1 + G_2) = (n + 2)^3m(m + 2)^3n.$$
2.4. Cartesian Product

The Cartesian product \( G_1 \times G_2 \) of the graphs \( G_1 \) and \( G_2 \) is a graph with the vertex set \( V(G_1) \times V(G_2) \), and two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are adjacent if and only if \([u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)]\) or \([u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)]\). The degree of a vertex \( v = (v_1, v_2) \) of \( G_1 \times G_2 \) is given by:

\[
d_{G_1 \times G_2}(v) = d_{G_1}(v_1) + d_{G_2}(v_2).
\]

**Theorem 2.8.** The multiplicative forgotten topological index of \( G_1 \times G_2 \) satisfies the following inequality:

\[
\Pi_F(G_1 \times G_2) \leq \left[ \frac{n_2 F(G_1) + n_1 F(G_2) + 6(m_2 M_1(G_1) + m_1 M_1(G_2))}{n_1 n_2} \right]^{n_1 n_2}.
\]

The equality holds if and only if both \( G_1 \) and \( G_2 \) are regular graphs.

**Proof.** Let \( G = G_1 \times G_2 \). By definition of the multiplicative forgotten topological index, we have:

\[
\Pi_F(G) = \prod_{(v_1, v_2) \in V(G)} d_G(v_1, v_2) = \prod_{v_1 \in V(G_1)} \prod_{v_2 \in V(G_2)} (d_{G_1}(v_1) + d_{G_2}(v_2))^3.
\]

Now by Lemma 2.1,

\[
\Pi_F(G) \leq \left[ \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} (d_{G_1}(v_1) + d_{G_2}(v_2))^3 \right]^{n_1 n_2}
\]

\[
= \frac{1}{(n_1 n_2)^{n_1 n_2}} \left[ \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} (d_{G_1}(v_1))^3 + 3d_{G_1}(v_1)^2 d_{G_2}(v_2) + 3d_{G_2}(v_2)^2 d_{G_1}(v_1) + d_{G_2}(v_2))^3 \right]^{n_1 n_2}
\]

\[
= \frac{n_2 F(G_1) + 3(2m_2 M_1(G_1) + 3M_1(G_2)(2m_1) + n_1 F(G_2))}{n_1 n_2}.
\]

By Lemma 2.1, the above equality holds if and only if for every \((u_1, u_2), (v_1, v_2) \in V(G),\)

\[
(d_{G_1}(u_1) + d_{G_2}(u_2))^3 = (d_{G_1}(v_1) + d_{G_2}(v_2))^3.
\]

This implies that for every \(u_1, v_1 \in V(G_1)\) and \(u_2, v_2 \in V(G_2),\)

\[
d_{G_1}(u_1) = d_{G_1}(v_1), \quad d_{G_2}(u_2) = d_{G_2}(v_2).
\]

Thus, both \( G_1 \) and \( G_2 \) are regular graphs. \(\square\)

**Example 2.9.** Consider a cycle graph \( C_n \) and a complete graph \( K_m \). We have:

\[
\Pi_F(C_n \times K_m) = (m + 1)^{3mn}.
\]
2.5. Corona Product

The corona product \( G_1 \circ G_2 \) of the graphs \( G_1 \) and \( G_2 \) is a graph obtained by taking one copy of \( G_1 \) and \( n_1 \) copies of \( G_2 \) and joining the \( i \)-th vertex of \( G_1 \) to every vertex in \( i \)-th copy of \( G_2 \) for \( 1 \leq i \leq n_1 \). The \( i \)-th copy of \( G_2 \) will be denoted by \( G_{2,i} \), \( 1 \leq i \leq n_1 \). Different topological indices of the corona product of two graphs have already been studied in [28,31]. The degree of a vertex \( v \in V(G_1 \circ G_2) \) is given by:

\[
d_{G_1 \circ G_2}(v) = \begin{cases} 
  d_{G_1}(v) + n_2 & v \in V(G_1), \\
  d_{G_2}(v) + 1 & v \in V(G_{2,i}).
\end{cases}
\]

**Theorem 2.10.** The multiplicative forgotten topological index of \( G_1 \circ G_2 \) satisfies the following inequality:

\[
\Pi_F(G_1 \circ G_2) \leq \left( \frac{F(G_1) + 3n_2M_1(G_1) + 6n_1n_2^2 + n_1n_2^3}{n_1} \right)^{n_1} \times \left( \frac{F(G_2) + 3M_1(G_2) + 6n_2 + n_2^2}{n_2} \right)^{n_1n_2}.
\]

The equality holds if and only if both \( G_1 \) and \( G_2 \) are regular graphs.

**Proof.** Let \( G = G_1 \circ G_2 \). By definition of the multiplicative forgotten topological index, we have:

\[
\Pi_F(G) = \prod_{v \in V(G)} d_G(v)^3 = \prod_{v \in V(G_{2,i})} (d_{G_1}(v) + n_2)^3 \times \left[ \prod_{v \in V(G_{2,i})} (d_{G_2}(v) + 1)^3 \right]^{n_1}.
\]

Now by Lemma 2.1,

\[
\Pi_F(G) \leq \left[ \sum_{v \in V(G_{2,i})} (d_{G_1}(v) + n_2)^3 \right]^{n_1} \times \left[ \sum_{v \in V(G_{2,i})} (d_{G_2}(v) + 1)^3 \right]^{n_1n_2}
\]

\[
= \left[ \sum_{v \in V(G_{2,i})} (d_{G_1}(v)^3 + 3n_2d_{G_1}(v) + 3n_2^2d_{G_1}(v) + n_2^3) \right]^{n_1}
\]

\[
\times \left[ \sum_{v \in V(G_{2,i})} (d_{G_2}(v)^3 + 3d_{G_2}(v)^2d_{G_2}(v) + 3d_{G_2}(v) + 1) \right]^{n_1n_2}
\]

\[
= \left[ \frac{F(G_1) + 3n_2M_1(G_1) + 3n_2^2(2m_1) + n_2^3}{n_1} \right]^{n_1}
\]

\[
\times \left[ \frac{F(G_2) + 3M_1(G_2) + 3(2m_2) + n_2^2}{n_2} \right]^{n_1n_2}.
\]

By Lemma 2.1, the above equality holds if and only if for every \( u_1, v_1 \in V(G_1) \) and \( u_2, v_2 \in V(G_2) \),
\[(d_{G_1}(u_1) + n_2)^3 = (d_{G_1}(v_1) + n_2)^3\]

and
\[(d_{G_2}(u_2) + 1)^3 = (d_{G_2}(v_2) + 1)^3.\]

This implies that for every \(u_1, v_1 \in V(G_1)\) and \(u_2, v_2 \in V(G_2)\),
\[d_{G_1}(u_1) = d_{G_1}(v_1), \quad d_{G_2}(u_2) = d_{G_2}(v_2).\]

Thus, both \(G_1\) and \(G_2\) are regular graphs.

\[\square\]

**Example 2.11.** Consider a cycle graph \(C_n\) and a complete graph \(K_m\). We have
\[\Pi_F(C_n \circ K_m) = m^{3n}n^2(m + 2)^{3n}.\]

### 2.6. Composition

The composition \(G_1[G_2]\) of the graphs \(G_1\) and \(G_2\) is a graph with the vertex set \(V(G_1) \times V(G_2)\), and vertex \((u_1, u_2)\) is adjacent with vertex \((v_1, v_2)\) whenever \(u_1\) is adjacent with \(v_1\) in \(G_1\) or \(u_1 = v_1\) and \(u_2\) is adjacent with \(v_2\) in \(G_2\). The degree of a vertex \(v = (v_1, v_2)\) of \(G_1[G_2]\) is given by:
\[d_{G_1[G_2]}(v) = n_2d_{G_1}(v_1) + d_{G_2}(v_2).\]

**Theorem 2.12.** The multiplicative forgotten topological index of \(G_1[G_2]\) satisfies the following inequality:
\[\Pi_F(G_1[G_2]) \leq \frac{n_2^2F(G_1) + n_1F(G_2) + 6n_2(n_2m_2M_1(G_1) + m_1M_1(G_2))}{n_1n_2}.\]

The equality holds if and only if both \(G_1\) and \(G_2\) are regular graphs.

**Proof.** Let \(G = G_1[G_2]\). By definition of the multiplicative forgotten topological index, we have:
\[\Pi_F(G) = \prod_{(v_1, v_2) \in V(G)} d_G((v_1, v_2))^3 = \prod_{v_1 \in V(G_1)} \prod_{v_2 \in V(G_2)} (n_2d_{G_1}(v_1) + d_{G_2}(v_2))^3.\]

Now by Lemma 2.1,
\[\Pi_F(G) \leq \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} \left(\sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} (n_2^3d_{G_1}(v_1))^3 + 3n_2^2d_{G_1}(v_1)^2d_{G_2}(v_2) + 3n_2d_{G_1}(v_1)d_{G_2}(v_2)^2 + d_{G_2}(v_2)^3\right) n_1n_2\]
\[= \frac{1}{(n_1n_2)^{n_1n_2}} \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} n_2^3d_{G_1}(v_1)^3 + 3n_2^2d_{G_1}(v_1)^2d_{G_2}(v_2) + 3n_2d_{G_1}(v_1)d_{G_2}(v_2)^2 + d_{G_2}(v_2)^3 n_1n_2.\]

By Lemma 2.1, the above equality holds if and only if for every \((u_1, u_2), (v_1, v_2) \in V(G)\),
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\[(n_2d_{G_1}(u_1) + d_{G_2}(u_2))^3 = (n_2d_{G_1}(v_1) + d_{G_2}(v_2))^3.\]

This implies that both \(G_1\) and \(G_2\) are regular graphs.

\[\square\]

**Example 2.13.** Consider two cycle graphs \(C_n\) and \(C_m\). We have:

\[\Pi_F(C_n[C_m]) = 2^{3mn}(m + 1)^{3mn}.\]

### 2.7. Strong Product

The strong product \(G_1 \boxtimes G_2\) of the graphs \(G_1\) and \(G_2\) is a graph with the vertex set \(V(G_1) \times V(G_2)\), and two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are adjacent if and only if \([u_1 = v_1\) and \(u_2v_2 \in E(G_2)]\) or \([u_2 = v_2\) and \(u_1v_1 \in E(G_1)]\) or \([u_1v_1 \in E(G_1)\) and \(u_2v_2 \in E(G_2)]\). The degree of a vertex \(v = (v_1, v_2)\) of \(G_1 \boxtimes G_2\) is given by:

\[d_{G_1 \boxtimes G_2}(v) = d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(v_2).\]

**Theorem 2.14.** The multiplicative forgotten topological index of \(G_1 \boxtimes G_2\) satisfies the following inequality:

\[
\Pi_F(G_1 \boxtimes G_2) \leq \frac{1}{n_1n_2} \left[ F(G_1)F(G_2) + (n_2 + 6m_2 + 3M_1(G_2))F(G_1) \right.
+ \left. (n_1 + 6m_1 + 3M_1(G_1))F(G_2) + 6(n_2m_1(G_1) + m_1M_1(G_2)) \right. \\
+ \left. M_1(G_1)M_1(G_2) \right]^{n_1n_2}.
\]

The equality holds if and only if both \(G_1\) and \(G_2\) are regular graphs.

**Proof.** Let \(G = G_1 \boxtimes G_2\). By definition of the multiplicative forgotten topological index, we have:

\[
\Pi_F(G) = \prod_{(v_1, v_2) \in V(G)} d_G((v_1, v_2))^3 \\
= \prod_{v_1 \in V(G_1)} \prod_{v_2 \in V(G_2)} [d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(v_2)]^3.
\]
Now by Lemma 2.1,

\[
\Pi_F(G) \leq \frac{1}{(n_1n_2)^{n_1n_2}} \left[ \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} [d_{G_1}(v_1) + d_{G_2}(v_2)]^3 \right]_{n_1n_2} \\
+ 3d_{G_1}(v_1)^2d_{G_2}(v_2) + 3d_{G_1}(v_1)d_{G_2}(v_2)^2 + 3d_{G_1}(v_1)^3d_{G_2}(v_2) \\
+ 6d_{G_1}(v_1)^3d_{G_2}(v_2)^2 + 3d_{G_1}(v_1)d_{G_2}(v_2)^3 + 3d_{G_1}(v_1)^3d_{G_2}(v_2)^2 \\
+ \frac{1}{(n_1n_2)^{n_1n_2}} [n_2F(G_1) + n_1F(G_2) + 6n_2M_1(G_1) + 6n_1M_1(G_2) \\
+ 6M_2F(G_1) + 6M_1(G_1)M_1(G_2) + 6M_1F(G_2) + 3M_2F(G_1)]_{n_1n_2} \\
+ 3M_1F(G_1)F(G_2) + F(G_1)F(G_2)]_{n_1n_2}.
\]

By Lemma 2.1, the above equality holds if and only if for every \((u_1, u_2), (v_1, v_2) \in V(G)\),

\[(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2))^3 = (d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)d_{G_2}(v_2))^3.\]

This implies that for every \(u_1, v_1 \in V(G_1)\) and \(u_2, v_2 \in V(G_2)\),

\[d_{G_1}(u_1) = d_{G_1}(v_1), d_{G_2}(u_2) = d_{G_2}(v_2).\]

Thus, both \(G_1\) and \(G_2\) are regular graphs.

\[\square\]

### 2.8. Disjunction

The disjunction \(G_1 \lor G_2\) of two graphs \(G_1\) and \(G_2\) is a graph with the vertex set \(V(G_1) \times V(G_2)\) in which \((u_1, u_2)\) is adjacent with \((v_1, v_2)\) whenever \([u_1\) is adjacent with \(v_1 \in G_1]\) or \([u_2\) is adjacent with \(v_2 \in G_2]\). The degree of a vertex \(v = (v_1, v_2)\) of \(G_1 \lor G_2\) is given by:

\[d_{G_1 \lor G_2}(v) = n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - d_{G_1}(v_1) d_{G_2}(v_2).\]

**Theorem 2.15.** The multiplicative forgotten topological index of \(G_1 \lor G_2\) satisfies the following inequality:

\[
\Pi_F(G_1 \lor G_2) \leq \frac{1}{(n_1n_2)^{n_1n_2}} [(n_2^3 - 6n_2^2m_1 + 3n_2M_1(G_2))F(G_1) \\
+ (n_1^3 - 6n_1^2m_1 + 3n_1M_1(G_1))F(G_2) - F(G_1)F(G_2) \\
+ 6n_1n_2(n_2m_2M_1(G_1) + n_1m_1M_1(G_2) - M_1(G_1)M_1(G_2))]_{n_1n_2}.
\]

The equality holds if and only if both \(G_1\) and \(G_2\) are regular graphs.
Proof. Let $G = G_1 \lor G_2$. By definition of the multiplicative forgotten topological index, we have:

$$\Pi_F(G) = \prod_{(v_1, v_2) \in V(G)} d_G((v_1, v_2))^3$$

$$= \prod_{v_1 \in V(G)} \prod_{v_2 \in V(G)} [n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - d_{G_1}(v_1)d_{G_2}(v_2)]^3.$$

Now by Lemma 2.1,

$$\Pi_F(G) \leq \frac{1}{(n_1n_2)^{n_1n_2}} \left( \sum_{v_1 \in V(G)} \sum_{v_2 \in V(G)} [n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - d_{G_1}(v_1)d_{G_2}(v_2)] \right)^{n_1n_2}$$

$$= \frac{1}{(n_1n_2)^{n_1n_2}} \left[ n_2n_2F(G_1) + 3n_2^2n_1(2m_2)M_1(G_1) + 3n_2n_1^2(2m_1)M_1(G_2) 

+ n_1^3F(G_2) - 3n_2^2(2m_2)F(G_1) - 6n_2n_1M_1(G_1)M_1(G_2) 

- 3n_2^2(2m_1)F(G_2) + 3n_2M_1(G_2)F(G_1) 

+ 3n_1M_1(G_1)F(G_2) - F(G_1)F(G_2) \right]^{n_1n_2}.$$ 

By Lemma 2.1, the above equality holds if and only if for every $(u_1, u_2), (v_1, v_2) \in V(G)$,

$$(n_2d_{G_1}(u_1) + n_1d_{G_2}(u_2) - d_{G_1}(u_1)d_{G_2}(u_2))^3 = (n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - d_{G_1}(v_1)d_{G_2}(v_2))^3.$$ 

This implies that for every $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$, $d_{G_1}(u_1) = d_{G_1}(v_1)$ and $d_{G_2}(u_2) = d_{G_2}(v_2)$. Thus, both $G_1$ and $G_2$ are regular graphs. □

2.9. Symmetric Difference

The symmetric difference $G_1 \oplus G_2$ of two graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ in which $(u_1, u_2)$ is adjacent with $(v_1, v_2)$ whenever $[u_1$ is adjacent with $v_1$ in $G_1$] or $[u_2$ is adjacent with $v_2$ in $G_2$], but not both. The degree of a vertex $v = (v_1, v_2)$ of $G_1 \oplus G_2$ is given by:

$$d_{G_1 \oplus G_2}(v) = n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - 2d_{G_1}(v_1)d_{G_2}(v_2).$$
Theorem 2.16. The multiplicative forgotten topological index of $G_1 \oplus G_2$ satisfies the following inequality:

$$
\Pi_F(G_1 \oplus G_2) \leq \frac{1}{(n_1n_2)^2 \sum_{v_1,v_2} \left[ (n_1^4 - 12n_1^3m_1 + 12n_1M_1(G_1))F(G_1) + (n_2^4 - 12n_2^3m_2 + 12n_2M_1(G_2))F(G_2) - 8F(G_1)F(G_2) - 6n_1n_2(n_2m_2M_1(G_1) + n_1m_1M_1(G_2) - 2M_1(G_1)M_1(G_2)) \right]^{n_1n_2}}.
$$

The equality holds if and only if both $G_1$ and $G_2$ are regular graphs.

Proof. Let $G = G_1 \oplus G_2$. By definition of the multiplicative forgotten topological index, we have:

$$
\Pi_F(G) = \prod_{(u_1,u_2) \in V(G)} d_G((u_1,u_2))^3
= \prod_{v_1 \in V(G_1)} \prod_{v_2 \in V(G_2)} [n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - 2d_{G_1}(v_1)d_{G_2}(v_2)]^3.
$$

Now by Lemma 2.1,

$$
\Pi_F(G) \leq \left[ \sum_{v_1 \in V(G_1)} \sum_{v_2 \in V(G_2)} [n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - 2d_{G_1}(v_1)d_{G_2}(v_2)]^3 \right]^{n_1n_2}
= \frac{1}{(n_1n_2)^2 \sum_{v_1,v_2} \left[ (n_1^4 - 12n_1^3m_1 + 12n_1M_1(G_1))F(G_1) + (n_2^4 - 12n_2^3m_2 + 12n_2M_1(G_2))F(G_2) - 8F(G_1)F(G_2) - 6n_1n_2(n_2m_2M_1(G_1) + n_1m_1M_1(G_2) - 2M_1(G_1)M_1(G_2)) \right]^{n_1n_2}}.
$$

By Lemma 2.1, the above equality holds if and only if for every $(u_1,u_2), (v_1,v_2) \in V(G),

$$
(n_2d_{G_1}(u_1) + n_1d_{G_2}(u_2) - 2d_{G_1}(u_1)d_{G_2}(u_2))^3 = (n_2d_{G_1}(v_1) + n_1d_{G_2}(v_2) - 2d_{G_1}(v_1)d_{G_2}(v_2))^3.
$$

This implies that both $G_1$ and $G_2$ are regular graphs. \hfill \Box

3. The Multiplicative Forgotten Topological Index of Subdivision Graphs and Vertex-Semisemilotal Graphs

In papers [9, 19, 30], the authors obtained the expressions for Zagreb indices and coindices of derived graphs. This motivates us to find expressions for the multiplicative forgotten topological index of some derived graphs.
Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The \textit{subdivision graph} $S = S(G)$ of $G$ is a graph that is obtained by inserting a new vertex of degree two on each edge of $G$. The \textit{vertex–semitotal graph} $T_1 = T_1(G)$ of $G$ is a graph with the vertex set $V(T_1) = V(S)$ and the edge set $E(T_1) = E(S) \cup E(G)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Two graphs derived from the graph $G$. The vertices of these derived graphs, corresponding to the vertices of the parent graph $G$, are indicated by circles. The vertices of these graphs, corresponding to the edges of the parent graph $G$ are indicated by squares.}
\end{figure}

**Theorem 3.1.** Let $S$ be the subdivision graph of the graph $G$ and $|E(G)| = m$. Then

\[ \Pi_F(S) = 8^m \Pi_F(G). \]

**Proof.** By the definition of the multiplicative forgotten topological index and subdivision graph of $G$, we have:

\[ \Pi_F(S) = \prod_{v \in V(S)} d_S(v)^3 = \prod_{v \in V(G)} d_G(v)^3 \prod_{uv \in E(G)} 2^3 = 8^m \Pi_F(G). \]

\[ \square \]

**Theorem 3.2.** Let $T_1$ be the vertex–semitotal graph of the graph $G$ and $|V(G)| = n$ and $|E(G)| = m$. Then

\[ \Pi_F(T_1) = 8^{n+m} \Pi_F(G). \]

**Proof.** By the definition of the multiplicative forgotten topological index and vertex–semitotal graph of $G$, we have:

\[ \Pi_F(T_1) = \prod_{v \in V(T_1)} d_{T_1}(v)^3 = \prod_{v \in V(G)} (2d_G(v))^3 \prod_{uv \in E(G)} 2^3 = 8^{n+m} \Pi_F(G). \]

\[ \square \]
The $k$-subdivision graph $Sk = Sk(G)$ of $G$ is a graph that is obtained by inserting $k$ new vertex of degree two on each edge of $G$.

**Theorem 3.3.** Let $Sk$ be the $k$-subdivision of the graph $G$ and $|E(G)| = m$. Then

$$\Pi_F(Sk) = 8^{mk}\Pi_F(G) = 64^m\Pi_F(P_k)^m\Pi_F(G).$$

**Proof.** By construction of the $k$-subdivision of $G$ and Example 2.3, we have:

$$\Pi_F(Sk) = \prod_{v \in V(Sk)} d_{Sk}(v)^3 = \prod_{v \in V(G)} d_G(v)^3 \prod_{uv \in E(G)} 2^{3k} = 8^{mk}\Pi_F(G) = (64 \cdot 8^{k-2})^m\Pi_F(G).$$

\[\square\]

In general case, one can use a new graph $H$ for edge subdivisions of $G$. There are some simple constructions in which vertex degrees of graphs don’t change.

Let $H$ be an arbitrary graph and $xy \in E(H)$. The $H'$-subdivision graph $SH = SH(G)$ of $G$ is a graph that is obtained by inserting graph $H' = H - xy$ on each edge $uv$ of $G$ such that $ux, vy \in E(SH)$. For instance, $k$-subdivision can be considered as $C_k$-subdivision of a graph, $k \geq 3$.

**Theorem 3.4.** Let $SH$ be the $H'$-subdivision of $G$ and $|E(G)| = m$. Then

$$\Pi_F(SH) = \Pi_F(H)^m\Pi_F(G).$$

**Proof.** By the construction of edge subdivisions of $G$, we have:

$$\Pi_F(SH) = \prod_{v \in V(SH)} d_{SH}(v)^3 = \prod_{v \in V(G)} d_G(v)^3 \prod_{uv \in E(G)} \Pi_F(H) = \Pi_F(H)^m\Pi_F(G).$$

\[\square\]

Suppose that the graph $H$ has two vertices of degree one (graph-monomer). To keep vertex degrees, one can subdivide every edge of $G$ by a chain consisting of $k$ copies of $H$ (one edge separates two consecutive non-terminal copies of $H$). Then one can write for the resulting graph $SP$:

$$\Pi_F(SP) = \prod_{v \in V(SP)} d_{SP}(v)^3 = \prod_{v \in V(G)} d_G(v)^3 \prod_{uv \in E(G)} \Pi_F(H)^k = \Pi_F(H)^{mk}\Pi_F(G).$$
4. Comparison of the Discriminating Ability Indices

Topological indices are often used for quantifying structure of molecular graphs. A good ability of an index to distinguish between nonisomorphic graphs is important for chemical applications. Here we consider index $\Pi_F$ in compare with the other degree-based indices $M_1$, $M_2$ and $F$. A topological index is called degenerate if it possesses the same value for more than one graph. A set of graphs with the same value of a given index forms a degeneracy class. The discriminating ability of an index can be characterized by the average size of degeneracy classes:

$$\text{(average size)} = \frac{\text{number of graphs}}{\text{number of the degeneracy classes}}.$$ 

Table 1: Size of average degeneracy classes of indices for small $n$—vertex graph.

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>20</td>
<td>29</td>
<td>40</td>
<td>52</td>
<td>65</td>
<td>80</td>
<td>97</td>
<td>116</td>
<td>138</td>
<td>162</td>
<td>188</td>
</tr>
<tr>
<td>$M_2$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>20</td>
<td>29</td>
<td>40</td>
<td>52</td>
<td>65</td>
<td>80</td>
<td>97</td>
<td>116</td>
<td>138</td>
<td>162</td>
<td>188</td>
</tr>
<tr>
<td>$F$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>20</td>
<td>29</td>
<td>40</td>
<td>52</td>
<td>65</td>
<td>80</td>
<td>97</td>
<td>116</td>
<td>138</td>
<td>162</td>
<td>188</td>
</tr>
<tr>
<td>$\Pi_F$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>20</td>
<td>29</td>
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<td>65</td>
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<td>97</td>
<td>116</td>
<td>138</td>
<td>162</td>
<td>188</td>
</tr>
</tbody>
</table>

Table 1 contains comparative data for trees, uni- and bicyclic graphs of small order. Since the set of possible values of index $\Pi_F$ very quickly increases, one can expect that the sizes of its average degeneracy classes will be less than for the other indices. However, we observe this for graphs of sufficiently large order. For instance, this is valid for trees starting from 19 vertices (see data in bold in Table 1 for $\Pi_F$ and $M_2$). Topological index $M_2$ has the better discriminating power for most considered classes of graphs of small order.

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A Multiplicative Version of Forgotten Topological Index


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