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Distinguishing Number and Distinguishing Index of the Join of Two Graphs

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Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has an vertex labeling (edge labeling) with d labels that is preserved only by a trivial automorphism. In this paper we study the distinguishing number and the distinguishing index of the join of two graphs G and H, i.e., $G+H$. We prove that $0 \leq D(G+H)-max\{D(G), D(H)\} \leq z$, where z depends on the number of some induced subgraphs generated by some suitable partitions of $V(G)$ and $V(H)$. Let G^k be the k-th power of G with respect to the join product. We prove that if G is a connected graph of order $n \geq 2$, then G^k has the distinguishing index 2, except $D'(K_2+K_2) = 3$.

Keywords: Distinguishing index, distinguishing number, join.

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1. Introduction

Let $G = (V, E)$ be a graph with n vertices. We use the standard graph notation in [6]. An automorphism of a graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge–vertex connectivity. The set of all automorphisms of G, with the operation of composition of permutations, is a permutation group on V and is denoted by Aut(G). A labeling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is *r-distinguishing*, if no non-trivial automorphism of G preserves all of the vertex labels. Formally, ϕ is r-distinguishing if for every non-trivial

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$\sigma \in \mathrm{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph G is the minimum number r such that G has a labeling that is r-distinguishing. This number was defined by Albertson and Collins [2]. Similar to this definition, Kalinowski and Pilśniak [8] have defined the *distinguishing index* $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. Observe that $D(G) = 1$ for the asymmetric graphs G and $D(G) = |V(G)|$, if and only if $G = K_n$. It is immediate that $D(P_n) = 2$ $(n \geq 2)$, where P_n is the n-vertex path. A classical result gives that for the cycle with n vertices, C_n, $D(C_n) = 3$ if $n = 3, 4, 5$ and $D(C_n) = 2$ $(n \geq 6)$. Also for complete bipartite graph when $q > p$, $D(K_{p,q}) = q$, $D(K_{n,n}) = n + 1$ for $n \geq 3$, for the n-cube Q_n, $D(Q_n) = 2$, for $n \geq 4$ and $D(Q_n) = 3$ for $n = 2, 3$ ([4]). The distinguishing index of some graphs is exhibited in [8]. The distinguishing number and index of the Cartesian product and the Cartesian powers of graphs has been thoroughly investigated [1, 9, 5]. Pilśniak studied the Nordhaus-Gaddum bounds for the distinguishing index in [10]. Also the distinguishing number of the hypercube has been investigated in [4]. Recently, we studied the distinguishing number and distinguishing index of corona product of two graphs [3].

We say that $G = (V, E)$ is a *join* graph if G is the complete union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In other words, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv | u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2, we write $G = G_1 + G_2$. For simple connected graph G, and $v \in V$, the neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. The nonadjacent vertices to v in G is $V(G) \setminus N(v)$ and denoted by $\overline{N(v)}$. A subgraph H of G is an induced, if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. We denote the induced subgraph by a set $X \subseteq V$, by $G[X]$.

In the next section, we study the distinguishing number of the join of two graphs. In Section 3, we present two upper bounds for the distinguishing index of the join of two graphs and show that they are sharp.

2. The Distinguishing Number of the Join of Two Graphs

In this section, we study the distinguishing number of the join of two graphs. We begin with the following theorem which gives a lower bound for the distinguishing number of this kind of graphs:

Theorem 2.1. Let G_1 and G_2 be two connected graphs. Then

$$\max\{D(G_1), D(G_2)\} \leq D(G_1 + G_2) \leq D(G_1) + D(G_2).$$

Proof. By contradiction, suppose that $D(G_1 + G_2) < \max\{D(G_1), D(G_2)\}$. Without loss of generality, we can assume that $D(G_1 + G_2) < D(G_2)$. In this case, the vertices of the graph G_2 can be labeled with less than $D(G_2)$ labels, in which a nontrivial automorphism f_2 of G_2 preserves the labeling of G_2. On the other hand, there exists the following nontrivial automorphism h of $G_1 + G_2$ preserving the labeling of $G_1 + G_2$, which is contradiction.

$$h(v) = \begin{cases} v & \textit{If } v \in V(G_1), \\ f_2(v) & \textit{If } v \in V(G_2). \end{cases}$$

To prove $D(G_1 + G_2) \leq D(G_1) + D(G_2)$, we first label G_1 in a distinguishing way with $D(G_1)$ labels, next we label the vertices of G_2 with the labels $\{D(G_1) + 1, \ldots, D(G_1) + D(G_2)\}$ in a distinguishing way. This labeling is distinguishing because if f is an arbitrary automorphism of $G_1 + G_2$ preserving the labeling, then with respect to the label of vertices of G_1 and G_2 we get that the restriction of automorphism f to G_i is G_i, where $i = 1, 2$, i.e., $f|_{G_1} = G_1$ and $f|_{G_2} = G_2$, and so $f|_{G_i}$ is an automorphism of G_i for $i = 1, 2$. Since both G_1 and G_2 have been labeled in a distinguishing way, so we have $f|_{G_1} = id_{G_1}$ and $f|_{G_2} = id_{G_2}$. Therefore, f is the identity automorphism of $G_1 + G_2$. □

To obtain a better upper bound for the distinguishing number of the join of two arbitrary graphs G_1 and G_2, we partition the vertices of $G_1 + G_2$ such that every automorphism of $G_1 + G_2$ maps the classes to each other. This partition is as follows:

Let G_1 and G_2 be two graphs and $G = G_1 + G_2$. Let v_1 be an arbitrary vertex of G_1. First put $A_1 = \overline{N_G(v_1)}$ (note that $\overline{N_G(v_1)} \subseteq V(G_1)$). We add all nonadjacent sets of the vertices of G (say v) such that their nonadjacent sets satisfy $\overline{N_G(v)} \cap A_1 \neq \emptyset$, to A_1 and denote again the new set by A_1 (if $v \in G$ and $\overline{N_G(v)} \cap A_1 \neq \emptyset$ then $v \in G_1$). We continue this process until there is no vertex in G with this property.

Let v_2 be a vertex of G_1 such that $v_2 \notin A_1$. Put $A_2 = \overline{N_G(v_2)}$ and similar to construction of A_1, add suitable nonadjacent sets of a vertex to A_2 and repeat this action. It is clear that after a finite number of steps, the vertices of G_1 partition to A_i's. With a similar argument we suppose that the vertices of G_2 partition to some sets, say, B_j's. Without loss of generality, we assume that the vertices of G are partitioned into $k + k'$ equivalence classes as follows (the notation v is used for the vertices of G_1 and the notation w is used for the vertices of G_2):

$$\begin{aligned} A_1 &= \overline{N_G(v_1)} \cup \cdots \cup \overline{N_G(v_{t_1})}, \\ A_2 &= \overline{N_G(v_{t_1+1})} \cup \cdots \cup \overline{N_G(v_{t_1+t_2})}, \\ &\vdots \\ A_k &= \overline{N_G(v_{t_1+\ldots+t_{k-1}+1})} \cup \cdots \cup \overline{N_G(v_{t_1+\ldots+t_k})}, \end{aligned}$$

$$\begin{aligned}
B_1 &= \overline{N_G(w_1)} \cup \cdots \cup \overline{N_G(w_{t'_1})},\\
B_2 &= \overline{N_G(w_{t'_1+1})} \cup \cdots \cup \overline{N_G(w_{t'_1+t'_2})},\\
&\vdots\\
B_{k'} &= \overline{N_G(w_{t'_1+\ldots+t'_{k'-1}+1})} \cup \ldots \cup \overline{N_G(w_{t'_1+\ldots+t'_{k'}})}.
\end{aligned} \tag{1}$$

Lemma 2.2. Let G_1 and G_2 be two graphs and $G = G_1 + G_2$. Suppose that $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_{k'}\}$ are two partitions of the vertices G_1 and G_2 as stated in (1), respectively. If f is an automorphism of G, then f is a permutation on the set $\mathcal{A} \cup \mathcal{B}$.

Proof. Let $u_1, u'_1 \in V(G)$ and $f(u_1) = u'_1$. Since an automorphism preserves adjacency relation, $f(\overline{N_G(u_1)}) = \overline{N_G(u'_1)}$.

Now let $u_1, u_2, u'_1, u'_2 \in V(G)$, $f(\overline{N_G(u_1)}) = \overline{N_G(u'_1)}$ and $f(\overline{N_G(u_2)}) = \overline{N_G(u'_2)}$. Then, we have

$$\overline{N_G(u_1)} \cap \overline{N_G(u_2)} \neq \emptyset \Leftrightarrow \overline{N_G(u'_1)} \cap \overline{N_G(u'_2)} \neq \emptyset.$$

By induction, if $u_1, \ldots, u_s, u'_1, \ldots, u'_s \in V(G)$ and $f(\overline{N_G(u_i)}) = \overline{N_G(u'_i)}$, where $1 \leq i \leq s$, then we have

$$\Big(\overline{N_G(u_1)} \cup \cdots \cup \overline{N_G(u_s)}\Big) \cap \overline{N_G(u_s)} \neq \emptyset \Leftrightarrow \Big(\overline{N_G(u'_1)} \cup \cdots \cup \overline{N_G(u'_s)}\Big) \cap \overline{N_G(u'_s)} \neq \emptyset.$$

By the above illustrations and definitions of A_i and B_j with $1 \leq i \leq k$ and $1 \leq j \leq k'$, we can conclude that f is a permutation on $\mathcal{A} \cup \mathcal{B}$. □

Corollary 2.3. Let G_1 and G_2 be two graphs and $G = G_1 + G_2$. Suppose that $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_{k'}\}$ are two partitions of the vertices G_1 and G_2 as stated in (1), respectively and put $\mathcal{A} \cup \mathcal{B} = \mathcal{C} = \{C_1, \ldots, C_{k+k'}\}$. If f is an automorphism of G and $f(C_i) = C_j$, for some $i, j \in \{1, \ldots, k + k'\}$, then the induced subgraphs $G[C_i]$ and $G[C_j]$ are isomorphic.

Before stating and proving the main theorems, we need some additional information about G_1 and G_2. Let G_1 and G_2 be two graphs and $G = G_1 + G_2$ such that $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_{k'}\}$ are two partitions of the vertices G_1 and G_2 as stated in (1), respectively. Now we put $H = \big\{G[A_1], \ldots, G[A_k]\big\}$ and $H' = \big\{G[B_1], \ldots, G[B_{k'}]\big\}$. Some of the induced subgraphs in each H and H' are isomorphic. We put all isomorphic induced subgraphs in H and also H', in a set and denote them by $\mathcal{A}_i$ and $\mathcal{B}_j$, respectively. In fact, we partitioned the two sets H, H' into t, t' disjoint sets $\mathcal{A}_1, \ldots, \mathcal{A}_t$ and $\mathcal{B}_1, \ldots, \mathcal{B}_{t'}$ such that $|\mathcal{A}_i| = n_i$ and $|\mathcal{B}_j| = m_j$ with $n_i, m_j \geq 1$, $1 \leq i \leq t$ and $1 \leq j \leq t'$ as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \{G[A_1], \ldots, G[A_{n_1}]\}, \\
\mathcal{A}_2 &= \{G[A_{n_1+1}], \ldots, G[A_{n_1+n_2}]\}, \\
&\vdots \\
\mathcal{A}_t &= \{G[A_{n_1+\cdots+n_{t-1}+1}], \ldots, G[A_{n_1+\cdots+n_t}]\}, \\
\mathcal{B}_1 &= \{G[B_1], \ldots, G[B_{m_1}]\}, \\
\mathcal{B}_2 &= \{G[B_{m_1+1}], \ldots, G[B_{m_1+m_2}]\}, \\
&\vdots \\
\mathcal{B}_{t'} &= \{G[B_{m_1+\cdots+m_{t'-1}+1}], \ldots, G[B_{m_1+\cdots+m_{t'}}]\}.
\end{aligned} \tag{2}$$

It is possible that some of the elements in $\mathcal{A}_i$ are isomorphic to some elements in a $\mathcal{B}_j$, where $1 \leq i \leq t$ and $1 \leq j \leq t'$ (note that if an element of $\mathcal{A}_i$ is isomorphic to an element of $\mathcal{B}_j$ then all elements of $\mathcal{A}_i$ have this property). Let q be the number of $\mathcal{A}_i$ for which there exist some $\mathcal{B}_j$ that the elements of $\mathcal{A}_i$ are isomorphic to elements of $\mathcal{B}_j$. Then we can partition the set $H \cup H'$ into disjoint sets $\Gamma_1, \ldots, \Gamma_{t+t'-q}$ as follows (we use new notation for vertices of G, if necessary):

$$\begin{cases} \Gamma_i = \mathcal{A}_i \cup \mathcal{B}_i & 1 \leq i \leq q, \\ \Gamma_{q+i} = \mathcal{A}_{q+i} & 1 \leq i \leq t-q, \\ \Gamma_{t+i} = \mathcal{B}_{t+i} & 1 \leq i \leq t'-q, \end{cases} \tag{3}$$

where $0 \leq q \leq \min\{t, t'\}$ (see Figure 1).

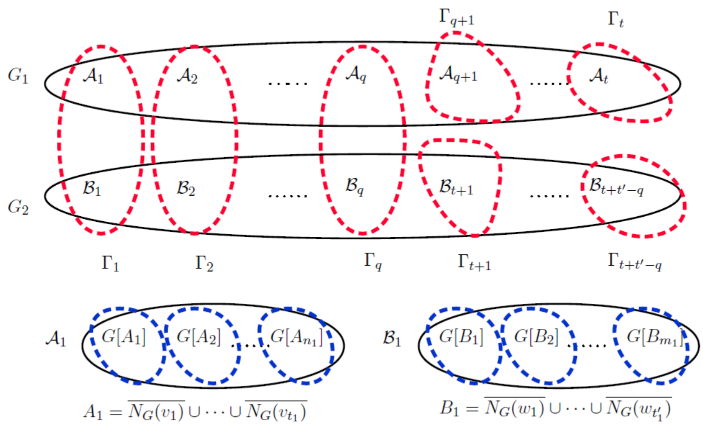

Figure 1: The partition of $G_1 + G_2$.

Remark 1. Using the partition of $H \cup H'$ in (3), Lemma 2.2 and Corollary 2.3, we can conclude that if $f \in \mathrm{Aut}(G_1 + G_2)$ then $f|_{\Gamma_i} \in \mathrm{Aut}(\Gamma_i)$ for $1 \leq i \leq q$, where $0 \leq q \leq \min\{t, t'\}$.

Now we are ready to state and prove the main result on the distinguishing number of the join of two graphs.

Theorem 2.4. Let G_1 and G_2 be two non-isomorphic graphs and $G = G_1 + G_2$.

(i) If $q = 0$, then $D(G_1 + G_2) = \max\{D(G_1), D(G_2)\}$.

(ii) If $q \neq 0$ and $z = \min\big\{\max\{n_1, \ldots, n_q\}, \max\{m_1, \ldots, m_q\}\big\}$, then

$$D(G_1 + G_2) \leq \max\{D(G_1), D(G_2)\} + z.$$

Proof. (i) If $q = 0$, then there is no element of H isomorphic to an element of H'. By Corollary 2.3, if $f \in \mathrm{Aut}(G)$, then $f|_{G_1} \in \mathrm{Aut}(G_1)$ and $f|_{G_2} \in \mathrm{Aut}(G_2)$, and so $D(G_1 + G_2) \leq \max\{D(G_1), D(G_2)\}$. Therefore, by Theorem 2.1 we have the result.

(ii) Let $d = \max\{D(G_1), D(G_2)\}$. We shall present a distinguishing labeling with $d + z$ labels. Without loss of generality, we can assume that $z = m_1$, and so $\mathcal{B}_1 = \{G[B_1], \ldots, G[B_{m_1}]\}$.

First, we label both G_1 and G_2 with $D(G_1)$ and $D(G_2)$ labels in a distinguishing way, respectively. Now to obtain a distinguishing labeling of $G_1 + G_2$, we change the labels of the vertices G_2 as follows:

- We change the label of an arbitrary vertex of $G[B_i]$ to $d + i$, for every $1 \leq i \leq m_1$.

We do similar above process on $\mathcal{B}_2, \ldots, \mathcal{B}_{t'}$ (note that if $z = n_{i_k}$ for some $k \in \{1, \ldots, q\}$ then we should do the similar work on G_1). By Lemma 2.2, Corollary 2.3 and the distinguishing labeling in both G_1 and G_2, we can conclude that presented labeling is distinguishing. Since we used $\max\{D(G_1), D(G_2)\} + z$ labels, the inequality follows.

□

Remark 2. The value of z in Theorem 2.4 (ii) can be zero or sufficiently large, depending on the structure of graphs G_1 and G_2. As an example, consider the complete k-partite graph $K_{d,\ldots,d}$ as G_1 and G_2 and $G = K_{d,\ldots,d} + K_{d,\ldots,d}$, then using notations in (2), $\mathcal{A}_i = \mathcal{B}_i = \emptyset$ for $2 \leq i \leq t$, and $\mathcal{A}_1 = \mathcal{B}_1 = \{G[A_1], \ldots, G[A_k]\}$, where A_i is the i-th part of $K_{d,\ldots,d}$. Therefore $z = k$ and so, z can be sufficiently large.

Now we shall show that the inequality in Theorem 2.4 (ii) is sharp.

Corollary 2.5. Let $n > m, n > m'$ and $m \neq m'$. The distinguishing number of $K_{n,m} + K_{n,m'}$ is $n + 1$.

Proof. Let $X = \{v_1, \ldots, v_n\}$ and $Y = \{w_1, \ldots, w_m\}$ be two parts of $K_{n,m}$, and $X' = \{v'_1, \ldots, v'_n\}$ and $Y' = \{w'_1, \ldots, w'_{m'}\}$ be two parts of $K_{n,m'}$. Suppose that $G = K_{n,m} + K_{n,m'}$. Using the partition in (1) we can write:

$$A_1 = \overline{N_G(v_1)} = \{v_1, \ldots, v_n\}, \quad A_2 = \overline{N_G(w_1)} = \{w_1, \ldots, w_m\}.$$

Since the number of elements in A_1 and A_2 are distinct, $G[A_1] \ncong G[A_2]$. Then by the partition in (2) we have $\mathcal{A}_1 = \{G[A_1]\}$ and $\mathcal{A}_2 = \{G[A_2]\}$, and so $n_1 = n_2 = 1$. Now by similar argument we can write:

$$B_1 = \overline{N_G(v'_1)} = \{v'_1, \ldots, v'_n\}, \quad B_2 = \overline{N_G(w'_1)} = \{w'_1, \ldots, w'_{m'}\}.$$

Then $\mathcal{B}_1 = \{G[B_1]\}$ and $\mathcal{B}_2 = \{G[B_2]\}$, and so $m_1 = m_2 = 1$. Since the induced subgraphs have no edges, $G[A_1] \cong G[B_1]$. With respect to the partition in (3) we have

$$\Gamma_1 = \mathcal{A}_1 \cup \mathcal{B}_1 = \{G[A_1], G[B_1]\}, \ \Gamma_2 = \mathcal{A}_2 = \{G[A_2]\}, \ \Gamma_3 = \mathcal{B}_2 = \{G[B_2]\}.$$

It is clear that for every labeling by n labels we can find a labeling preserving automorphism of Γ_1. So we can find an automorphism of G with this property. Consider the following labeling by $n + 1$ labels:

We assign to the vertices in A_1 the labels $1, \ldots, n$ and to the vertices in B_1 the labels $1, \ldots, n-1, n+1$. We label the vertices in A_2 with the labels $1, \ldots, m$ and the vertices in B_2 with the labels $1, \ldots, m'$. By Remark 1, this labeling is distinguishing, and so $D(K_{n,m} + K_{n,m'}) = n + 1$. □

Theorem 2.6. Let $n_1, ..., n_t$ be the number of elements of classes stated in (2). We have

$$D(G) \leq D(G + G) \leq D(G) + \max\{n_1, \ldots, n_t\}.$$

Proof. Let G_1 and G_2 be two isomorphic graphs and denote both of them by G, then the left side inequality is identified by Theorem 2.1. To prove the right side of inequality, we present a distinguishing labeling as follows:

Without loss of generality we can assume that $n_1 = \max\{n_1, \ldots, n_t\}$. First, we label G and its copy with $D(G)$ labels in a distinguishing way. To obtain a distinguishing labeling for $G + G$ we change the labels of the vertices of G as follows:

- We change the label of an arbitrary vertex of $(G + G)[A_i]$ to $D(G) + i$, for every $1 \leq i \leq n_1$.

So the labels of vertices of $\mathcal{A}_1$ were changed. We do similar process on $\mathcal{A}_2, \ldots, \mathcal{A}_t$. By Lemma 2.2, Corollary 2.3 and the distinguishing labeling in both G and its copy, we can conclude that presented labeling is distinguishing. Since we used $D(G) + \max\{n_1, \ldots, n_t\}$ labels, the right side inequality follows. □

Remark 3. With similar argument as in the proof of Corollary 2.5, we can show that the inequality in Theorem 2.6 is sharp for the star graphs $K_{1,n}$. In fact $D(K_{1,n} + K_{1,n}) = n + 1$ where $D(K_{1,n}) = n$ and $\max\{n_1, \ldots, n_t\} = 1$.

3. The Distinguishing Index of the Join of Two Graphs

In this section we study the distinguishing index for the join of two graphs. We say that a graph G is almost spanned by a subgraph H if $G - v$ is spanned by H for some $v \in V(G)$. We need the following lemmas in this section.

Lemma 3.1. [10] If a graph G is spanned or almost spanned by a subgraph H, then $D'(G) \leq D'(H) + 1$.

Lemma 3.2. [10] Let G be a graph of order $n \geq 7$ with a Hamiltonian path, then $D'(G) \leq 2$.

By these two lemmas, we can obtain the following upper bounds for the distinguishing index of the join of two graphs.

Theorem 3.3. Let G and H be two graphs of orders n and m, respectively. Then $D'(G + H) \leq D'(K_{n,m}) + 1$.

Proof. Since the complete bipartite graph $K_{n,m}$, is a spanning subgraph $G + H$, we can conclude the result by Lemma 3.1. □

Theorem 3.4. If G has n vertices and H has m vertices, such that $4 \leq n \leq m \leq 2n$, then $D'(G + H) \leq 2$.

Proof. We use the complete bipartite $K_{n,m}$ subgraph to find an asymmetric spanning subgraph of $G + H$. Now we have the result by Lemma 3.1. □

Theorem 3.5. Let G and H be two graphs of orders n and m, respectively, such that $\delta(G) \leq \delta(H)$. If $\min\{\delta(G) + m, \delta(H) + n\} \geq \frac{n+m-1}{2}$ and $m + n \geq 7$, then $D'(G + H) \leq 2$.

Proof. It is known that if the minimum degree of a graph of order n is at least $\frac{n-1}{2}$, then graph has a Hamiltonian path. Since the minimum degree of $G + H$ is $\min\{\delta(G) + m, \delta(H) + n\}$, so the result follows by Theorem 3.2. □

Corollary 3.6. If G is a graph of order $n \geq 2$, then $D'(G^k) = 2$ for any $k \geq 2$, except $D'(K_2 + K_2) = 3$.

Proof. For $k = 2$, we have $\delta(G + G) = \delta(G) + n \geq \frac{2n-1}{2} = \frac{|G+G|-1}{2}$, and hence $G + G$ has a Hamiltonian path. If $n \geq 4$, then $2n = |G + G| \geq 7$, and so we have $D'(G+G) \leq 2$, by Lemma 3.2. On the other hand, since the automorphism group of graph $G + G$ is non-trivial, so $D'(G + G) \geq 2$. Therefore $D'(G + G) = 2$. If $n = 3$, then it is easy to see that $D'(G+G) = 2$. Now a simple induction argument together with Theorem 3.5 yield that $D'(G^k) = 2$, for any $k \geq 2$. □

To obtain an upper bound for $D'(G_1+G_2)$ we consider (3) which is a partition of $H \cup H'$, i.e., $\Gamma_1, \ldots, \Gamma_{t+t'-q}$. Note that the elements of $\mathcal{A}_i$ are isomorphic to elements of $\mathcal{B}_i$ for $1 \leq i \leq q$, where $0 \leq q \leq \min\{t, t'\}$. If $G_1 \cong G_2$ then $t = t'$ and the elements of $\mathcal{A}_i$ are isomorphic to elements of $\mathcal{B}_i$ for $1 \leq i \leq t$.

Let E_i be the set of edges of $G_1 + G_2$ such that the end points of its edges are in Γ_i for $1 \leq i \leq t + t' - q$. We add the set E_i to the set of edges Γ_i and denote the obtained new graph by Γ'_i. The following result gives an upper bound for $D'(G_1 + G_2)$ based on the distinguishing index of Γ'_i.

Theorem 3.7. Let G_1 and G_2 be two graphs such that $G_1 + G_2$ has been partitioned to the set of induced subgraphs $\Gamma_1, \ldots, \Gamma_{t+t'-q}$ as (3). Then

$$D'(G_1 + G_2) \leq \max\{D'(\Gamma'_1), \ldots, D'(\Gamma'_{t+t'-q})\}.$$

Proof. We label the edges of the graph Γ'_i $(1 \leq i \leq t + t' - q)$ by $D'(\Gamma'_i)$ labels in a distinguishing way. We assign the remaining edges the label 1. By Remark 1, this labeling is distinguishing. The number of labels that have been used here is

$$\max\{D'(\Gamma'_1), \ldots, D'(\Gamma'_{t+t'-q})\}.$$

So, we have the result. □

Now, we like to present another upper bound for $D'(G_1+G_2)$. For this purpose, we state some preliminaries.

Let X_i, $i \in I$ (I is the index set) be the set of complete bipartite graphs $K_{|V(\Gamma_s)|,|V(\Gamma_{s'})|}$ satisfying the following two conditions:

- The two parts of each element of X_i should be distinct.
- The set of all parts that have been used as parts of elements of X_i should be $\{V(\Gamma_1), \ldots, V(\Gamma_{t+t'-q})\}$.

Let $\varepsilon_i = \max\{D'(K_{|V(\Gamma_s)|,|V(\Gamma_{s'})|}) : K_{|V(\Gamma_s)|,|V(\Gamma_{s'})|} \in X_i\}$. Then we have the following theorem:

Theorem 3.8. Let G_1 and G_2 be two graphs such that $G_1 + G_2$ has been partitioned to induced subgraphs $\Gamma_1, \ldots, \Gamma_{t+t'-q}$ as (3). Then $D'(G_1 + G_2) \leq \min\{\varepsilon_i\}_{i\in I}$.

Proof. We label the edges of each complete bipartite graph in X_i in distinguishing way (by $D'(K_{|V(\Gamma_s)|,|V(\Gamma_{s'})|})$ labels) and assign to the remaining edges the label 1. Since all parts $\Gamma_1, \ldots, \Gamma_{t+t'-q}$ have been used in building of the complete bipartite graphs in X_i and by Remark 1, this labeling is distinguishing. Therefore $D'(G_1 + G_2) \leq \min\{\varepsilon_i\}_{i\in I}$. □

Remark 4. By setting $\lambda_1 = \max\{D'(\Gamma'_1), \ldots, D'(\Gamma'_{t+t'-q})\}$ and $\lambda_2 = \min\{\varepsilon_i\}_{i\in I}$ and by Theorem 3.7 and 3.8 we have $D'(G_1 + G_2) \leq \min\{\lambda_1, \lambda_2\}$. This raises the question "which upper bound is better, λ_1 or λ_2?". We show that for some graphs, the upper bound λ_1 is better than λ_2 and for some graphs, the situation is different. We present two examples and these examples show also that the upper bounds of the Theorem 3.7 and 3.8 are sharp.

Since the line graph of $K_{k,n}$ is isomorphic to Cartesian product $K_k \Box K_n$, so $\mathrm{Aut}(K_{k,n})$ coincides with $\mathrm{Aut}(K_k \Box K_n)$. Therefore the distinguishing index of the complete bipartite graphs which is needed in the solution of Example 3.10 can be translated to distinguishing number of Cartesian product of complete graphs.

Theorem 3.9. [7] Let k, n, d be integers so that $d \geq 2$ and $(d-1)^k < n \leq d^k$. Then

$$D(K_k \Box K_n) = \begin{cases} d & \text{If } n \leq d^k - \lceil \log_d k \rceil - 1, \\ d+1 & \text{If } n \geq d^k - \lceil \log_d k \rceil + 1. \end{cases}$$

If $n = d^k - \lceil \log_d k \rceil$ then $D(K_k \Box K_n)$ is either d or $d+1$ and can be computed recursively in $O(\log^*(n))$ time.

Example 3.10. The upper bound in Theorem 3.7 is better than the upper bound in Theorem 3.8 for the $D'(P_n + P_m)$ with $n, m \geq 2$ and $n \neq m$.

Solution. Set $G = P_n + P_m$. Suppose that $V(P_n) = \{v_1, \ldots, v_n\}$ and $V(P_m) = \{w_1, \ldots, w_m\}$. With these notations we have $A_1 = \overline{N_G(v_1)} = \{v_1, \ldots, v_n\}$ and $B_1 = \overline{N_G(w_1)} = \{w_1, \ldots, w_m\}$. Thus $\mathcal{A}_1 = \{G[A_1]\}$ and $\mathcal{B}_1 = \{G[B_1]\}$. Since $n \neq m$, so $\Gamma_1 = \mathcal{A}_1 = \{G[A_1]\}$, $\Gamma_2 = \mathcal{B}_1 = \{G[B_1]\}$ and $q = 0$. Also, $\Gamma'_1 = P_n$ and $\Gamma'_2 = P_m$. If we label both Γ'_1 and Γ'_2 by two labels in a distinguishing way (note that $D'(P_n) = D'(P_m) = 2$) then we have a distinguishing labeling with two labels by Remark 1.

It is easy to see that $D'(P_n + P_m) = \lambda_1 = 2$, and so the inequality of Theorem 3.7 is sharp. On the other hand, using the notation of Theorem 3.8 we have $I = \{1\}$, and so $X_1 = \{K_{|V(\Gamma_1)|,|V(\Gamma_2)|}\}$. By Theorem 3.9 it is clear that $\varepsilon_1 = D'(K_{V(\Gamma_1),V(\Gamma_2)})$ is not equal with 2 for all $m, n \geq 2$. Therefore the upper bound λ_1 is better than λ_2 for $D'(P_n + P_m)$.

Here we shall present two graphs for which the upper bound in Theorem 3.8 is better than the upper bound in Theorem 3.7. We recall that the friendship graph F_n is the join of K_1 with nK_2. In other words, F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex (see Figure 2). The following theorem gives the distinguishing index of the friendship graph F_n.

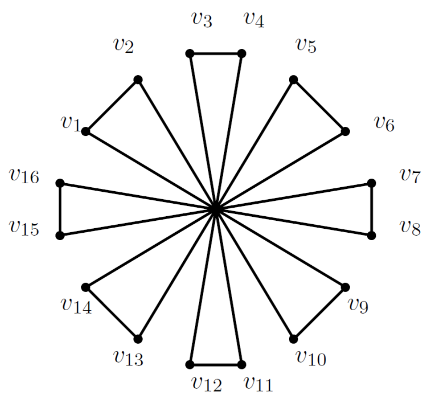

Figure 2: The graph F_8.

Theorem 3.11. [3] Let $a_n = 1 + 27n + 3\sqrt{81n^2 + 6n}$. The distinguishing index of the friendship graph F_n $(n \geq 2)$ is

$$D'(F_n) = \lceil \frac{1}{3}(a_n)^{\frac{1}{3}} + \frac{1}{3(a_n)^{\frac{1}{3}}} + \frac{1}{3} \rceil.$$

Example 3.12. The upper bound in Theorem 3.8 is better than the upper bound in Theorem 3.7 for $D'(F_n + F_m)$, where $2 \leq n < m$.

Solution. Suppose that $G = F_n + F_m$. The central vertices of F_n and F_m are denoted by x_0 and y_0, respectively. Any two adjacent vertices of F_n (except central vertex x_0) are denoted by x_{2i-1} and x_{2i} where $i = 1, \ldots, n$. The corresponding vertices of F_m are denoted by y_{2j-1} and y_{2j} where $j = 1, \ldots, m$.

By the partition in (1) we can write $A_1 = \overline{N_G(x_0)} = \{x_0\}$ and $A_2 = \overline{N_G(x_1)} \cup \overline{N_G(x_2)} = \{x_1, \ldots, x_{2n}\}$. Also $B_1 = \overline{N_G(y_0)} = \{y_0\}$ and $B_2 = \overline{N_G(y_1)} \cup \overline{N_G(y_2)} = \{y_1, \ldots, y_{2m}\}$. By the partition in (2), $\mathcal{A}_1 = \{G[A_1]\}$ and $\mathcal{A}_2 = \{G[A_2]\}$, also $\mathcal{B}_1 = \{G[B_1]\}$ and $\mathcal{B}_2 = \{G[B_2]\}$. Let $m \neq n$. By (3) and the hypothesis $m \neq n$, we have $\Gamma_1 = \mathcal{A}_1 \cup \mathcal{B}_1 = \{G[A_1], G[B_1]\}$, $\Gamma_2 = \mathcal{A}_2 = \{G[A_2]\}$ and $\Gamma_3 = \mathcal{B}_2 = \{G[B_2]\}$, and so $q = 1$. By notation of Theorem 3.8, one of the sets X_i

is $X_1 = \{K_{|V(\Gamma_1)|,|V(\Gamma_2)|}, K_{|V(\Gamma_2)|,|V(\Gamma_3)|}\}$. By Theorem 3.9, we have

$$\begin{aligned}\varepsilon_1 &= \max\{D'(K_{|V(\Gamma_1)|,|V(\Gamma_2)|}), D'(K_{|V(\Gamma_2)|,|V(\Gamma_3)|})\}\\ &= \max\{D'(K_{2,2n}), D'(K_{2n,2m})\}.\end{aligned}$$

Thus $\lambda_2 \leq \varepsilon_1$. On the other hand, Γ'_i, $i \in \{2,3\}$ is the union of graphs P_2, and so the distinguishing index of graphs Γ'_2 and Γ'_3 has not been defined. Therefore the upper bound λ_2 is better than λ_1.

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