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On n-A-Con-Cos Groups and Determination of some n-A-Con-Cos Groups

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Abstract

In this paper, we introduce the concept of n-A-con-cos groups, $n \geq 2$, mention some properties of them and determine all finite abelian groups with at most two direct factors. As a consequence, it is proved that dihedral groups D_{2m} in which m has at most two prime factors are n-A-con-cos.

Keywords: $n^{th}-$autocommutator subgroup, finite abelian groups, dihedral groups, n-A-con-cos groups.

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1. Introduction

Let $Aut(G)$ denote the automorphism group of a given group G. For any element $g \in G$ and $\alpha \in Aut(G)$, the autocommutator of g and α is defined to be $[g, \alpha] = g^{-1}\alpha(g)$. The absolute centre and autocommutator subgroup of a group G are defined as follows:

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \ \forall \alpha \in Aut(G)\},$$
$$K(G) = [G, Aut(G)] = \langle [g, \alpha] \mid g \in G, \ \alpha \in Aut(G)\rangle.$$

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We define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \ldots, \alpha_{n-1}], \alpha_n],$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G)$. The autocommutator subgroup of weight $n+1$ is defined in the following way:

$$\begin{aligned} K_n(G) &= [K_{n-1}(G), Aut(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_n] \mid g \in G,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G) \rangle. \end{aligned}$$

Clearly, $K_n(G)$ is a characteristic subgroup of G for all $n \geq 1$. The following series of subgroups

$$G \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_n(G) \supseteq \cdots .$$

is called the lower autocentral series of G (See also [3, 5, 7] and [8]).

A group G is called A-nilpotent, if the lower autocentral series of G ends in the identity subgroup after a finite number of steps. (See also [6]).

Let G be a group and $a, b \in G$. Then a and b are said to be fused, if there exists $\alpha \in Aut(G)$ such that $\alpha(a) = b$. (See [4]). Arora and Karan [1], defined a fusion relation in G as follows: Two elements a and b are related if they are fused. One can easily check that fusion relation is an equivalence relation. $\overline{cl(a)} = \{\alpha(a) \mid \alpha \in Aut(G)\}$ denotes the fusion class of a in G. They also defined Auto con-cos groups. We mention the definition of it:

Let G be a group and K be a proper characteristic subgroup of G, then we have two partitions of G, one is coset partition and another one is fusion class partition. If these two partitions coincide in $G - K$, that is $\overline{cl(g)} = gK$, for all $g \in G - K$, then we call the group G as Auto con-cos group.

In this paper, we introduce the new notion of n-A-con-cos groups for natural number n, where $n \geq 2$ and classify all finite abelian groups with at most two direct factors. It is also proved that dihedral groups D_{2m}, where m has at most two prime factors, are n-A-con-cos groups.

2. Main Results

We start this section by definition of n-A-con-cos groups.

Definition 2.1. A group G would be known as n-A-con-cos, if $K_n(G) < G$ and for all $g \in G - K_n(G)$ and $\alpha_1, \ldots, \alpha_{n-1} \in Aut(G)$, where $[g, \alpha_1, \ldots, \alpha_{n-1}] \neq 1$ we have

$$\overline{cl([g, \alpha_1, \ldots, \alpha_{n-1}])} = [g, \alpha_1, \ldots, \alpha_{n-1}]K_n(G) - 1.$$

The following theorem is useful in our investigation on con-cos-groups.

Theorem 2.2. Let G be a group and $K_n(G) = K_{n-1}(G) < G$ and $K_{n-1}(G)$ be the union of two fusion classes. Then the group G is n-A-con-cos.

Proof. Let $K_n(G) = K_{n-1}(G) = 1 \cup \overline{cl(x)}$, where $1 \neq x \in G$. Hence for every $g \in G - K_n(G)$ and $\alpha_1, \ldots, \alpha_{n-1} \in Aut(G)$, where $[g, \alpha_1, \ldots, \alpha_{n-1}] \neq 1$ we have $[g, \alpha_1, \ldots, \alpha_{n-1}] \in K_{n-1}(G) - 1 = \overline{cl(x)}$. Therefore,

$$\overline{cl([g, \alpha_1, \ldots, \alpha_{n-1}])} = \overline{cl(x)} = K_n(G) - 1,$$

and $[g, \alpha_1, \ldots, \alpha_{n-1}]K_n(G) - 1 = K_n(G) - 1$, which implies that

$$\overline{cl([g, \alpha_1, \ldots, \alpha_{n-1}])} = [g, \alpha_1, \ldots, \alpha_{n-1}]K_n(G) - 1.$$

Hence the group G is n-A-con-cos. □

For instance, the group $C_3 \rtimes C_4 = \langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^2 \rangle$ is 3-A-con-cos, since

$$K_3(C_3 \rtimes C_4) = K_2(C_3 \rtimes C_4) = < y >= 1 \cup \overline{cl(y)}.$$

Theorem 2.3. Let G be an A-nilpotent group, where $|G| > 2$ and $K_n(G) = 1$. Then the group G is n-A-con-cos.

Proof. For every $x \in G$ and $\alpha_1, \ldots, \alpha_n \in Aut(G)$, we have $[x, \alpha_1, \ldots, \alpha_n] = 1$. Hence $[x, \alpha_1, \ldots, \alpha_{n-1}]^{-1}\alpha_n([x, \alpha_1, \ldots, \alpha_{n-1}]) = 1$, and so

$$\overline{cl([x, \alpha_1, \ldots, \alpha_{n-1}])} = \{[x, \alpha_1, \ldots, \alpha_{n-1}]\}.$$

Also, for every $g \in G - K_n(G)$ and $\alpha_1, \ldots, \alpha_{n-1} \in Aut(G)$, $[g, \alpha_1, \ldots, \alpha_{n-1}] \neq 1$, we have

$$[g, \alpha_1, \ldots, \alpha_{n-1}]K_n(G) - 1 = \{[g, \alpha_1, \ldots, \alpha_{n-1}]\}.$$

This proves the result. □

For instance, the cyclic group C_4 is 2-A-con-cos, since C_4 is A-nilpotent and $K_2(C_4) = 1$. Furthermore, the dihedral group D_8 is 3-A-con-cos, since by Corollary 2.4 of [6], D_8 is A-nilpotent and $K_3(D_8) = 1$.

Remark 1. Let G be a finite abelian group of odd order. Then by Corollary 2.4 of [6], $K_n(G) = G$, for any natural number n. Hence G is not n-A-con-cos.

The following theorem is one of the main results of this paper.

Theorem 2.4. Let $n \geq 2$ be a natural number. Then the finite n-A-con-cos abelian groups with at most two direct factors are:

i) C_{2^t} for $1 \leq t \leq n+1$,

ii) $C_{2^t} \times C_p$ for $1 \leq t \leq n-1$,

iii) $C_{2^t} \times C_2$ for $2 \leq t \leq n+1$,

vi) $C_{2^t} \times C_{2^s}$ for $t \leq n+1$ and $2 \leq s \leq t-2$,

v) $C_{2^t} \times C_{2^s}$ for even number n, where $t \leq n-1$, $s \leq \frac{n}{2}$ and $t = s+1 \geq 3$,

iv) $C_{2^t} \times C_{2^s}$ for odd number n, where $t \leq n-1$, $t \leq \frac{n+1}{2}$ and $t = s+1 \geq 3$,

where p is an odd prime number and t, s are natural numbers.

Proof. By Remark 1, we should investigate finite abelian groups with at most two direct factors of even order.

Clearly the group C_{2^t} is n-A-con-cos, for $1 \leq t \leq n-1$. The group C_{2^n} is n-A-con-cos, since it is A-nilpotent and $K_n(C_{2^n}) = 1$.

Let $C_{2^{n+1}} = \langle x \mid x^{2^{n+1}} = 1\rangle$. By Lemma 2.2 of [5],

$$K_n(C_{2^{n+1}}) = C_{2^{n+1}}^{2^n} =< x^{2^n} >, \quad K_{n-1}(C_{2^{n+1}}) = C_{2^{n+1}}^{2^{n-1}} =< x^{2^{n-1}} > .$$

So for every $g \in C_{2^{n+1}} - K_n(C_{2^{n+1}})$ and $\alpha_1, \ldots, \alpha_{n-1} \in Aut(C_{2^{n+1}})$, $[g, \alpha_1, \ldots, \alpha_{n-1}] \neq 1$, we have

$$[g, \alpha_1, \ldots, \alpha_{n-1}] \in K_{n-1}(C_{2^{n+1}}) - 1 = \{x^{2^{n-1}}, x^{2^n}, x^{3.2^{n-1}}\}.$$

Clearly

$$\{x^{2^{n-1}}, x^{3.2^{n-1}}\} = \overline{cl(x^{3.2^{n-1}})} = \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}} K_n(C_{2^{n+1}}) - 1,$$

and

$$\{x^{2^n}\} = \overline{cl(x^{2^n})} = x^{2^n} K_n(C_{2^{n+1}}) - 1.$$

Hence the group $C_{2^{n+1}}$ is n-A-con-cos.

Suppose that $t \geq n+2$ and $C_{2^t} = \langle x \mid x^{2^t} = 1\rangle$. Then $K_n(C_{2^t}) =< x^{2^n} >$, and hence $x \in C_{2^t} - K_n(C_{2^t})$. Consider $\alpha, \beta \in Aut(C_{2^t})$ with $\alpha(x) = x^{2^{t-n+1}+1}$ and $\beta(x) = x^3$. It is easy to check that $[x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{t-1}}$. By Theorem 2.2 of [7], $x^{2^{t-1}} \in L(C_{2^t})$. Hence, $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$ but $x^{2^{t-1}} K_n(C_{2^t}) - 1 = K_n(C_{2^t}) - 1$ has $2^{t-n} - 1$ elements, and so

$$\overline{cl([x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] K_n(C_{2^t}) - 1.$$

Thus the group C_{2^t} is not n-A-con-cos, for $t \geq n+2$.

In what follows, we investigate the group $C_{2^t} \times C_{p^s}$ for natural numbers t, s with the presentation

$$\langle x, y \mid x^{2^t} = y^{p^s} = [x, y] = 1\rangle.$$

There are five cases:

Case 1: $1 \leq t \leq n-1$ and $s = 1$. By Lemma 2.1 and Lemma 2.2 of [5],

$$K_n(C_{2^t} \times C_p) = K_n(C_{2^t}) \times K_n(C_p) = 1 \times C_p = C_p$$

and $K_{n-1}(C_{2^t} \times C_p) = C_p$. Clearly $C_p =< y >= 1 \cup \overline{cl(y)}$. So by Theorem 2.2, the group $C_{2^t} \times C_p$ is n-A-con-cos, for $1 \leq t \leq n-1$.

Case 2: $t = n$ and $s = 1$. We know that $K_n(C_{2^n} \times C_p) =< y >$. Thus $x \in (C_{2^n} \times C_p) - K_n(C_{2^n} \times C_p)$. Consider the automorphism α of $C_{2^n} \times C_p$ with $\alpha(x) = x^3$ and $\alpha(y) = y$. Then

$$[x, \underbrace{\alpha, \ldots, \alpha}_{n-1-times}] = x^{2^{n-1}}$$

and $\overline{cl(x^{2^{n-1}})} = \{x^{2^{n-1}}\}$, but $x^{2^{n-1}} K_n(C_{2^n} \times C_p) - 1$ has p elements. Hence the group $C_{2^n} \times C_p$ is not n-A-con-cos.

Case 3: $t \geq n+1$ and $s = 1$. In this case $K_n(C_{2^t} \times C_p) =< x^{2^n} > \times < y >$. So, $x \in (C_{2^t} \times C_p) - K_n(C_{2^t} \times C_p)$. Consider $\alpha, \beta \in Aut(C_{2^t} \times C_p)$ with $\alpha(x) = x^{2^{t-n+1}+1}$, $\alpha(y) = y$, $\beta(x) = x^3$ and $\beta(y) = y$. Then

$$[x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{t-1}}$$

and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$ but $x^{2^{t-1}} K_n(C_{2^t} \times C_p) - 1$ has $2^{t-n}p - 1$ elements. Hence the group $C_{2^t} \times C_p$ is not n-A-con-cos, for $t \geq n+1$.

Case 4: $t = 1$ and $s \geq 2$. Note that $K_n(C_2 \times C_{p^s}) =< y >$. So, $xy \in (C_2 \times C_{p^s}) - K_n(C_2 \times C_{p^s})$. Consider $\alpha \in Aut(C_2 \times C_{p^s})$ with $\alpha(x) = x, \alpha(y) = y^2$. Therefore,

$$[xy, \underbrace{\alpha, \ldots, \alpha}_{n-1-times}] = y.$$

Clearly $\overline{cl(y)}$ has $\phi(p^s)$ elements, where ϕ is the Euler's phi function, but $yK_n(C_2 \times C_{p^s}) - 1$ has $p^s - 1$ elements. Since $s \geq 2$ we conclude that the group $C_2 \times C_{p^s}$ is not n-A-con-cos, for $s \geq 2$.

Case 5: $t \geq 2$ and $s \geq 2$. In this case we have $K_n(C_{2^t} \times C_{p^s}) =< y >$ for $t \leq n$ and $K_n(C_{2^t} \times C_{p^s}) =< x^{2^n} > \times < y >$ for $t \geq n+1$. Hence $xy \in (C_{2^t} \times C_{p^s}) - K_n(C_{2^t} \times C_{p^s})$. Consider $\alpha \in Aut(C_{2^t} \times C_{p^s})$ with $\alpha(x) = x$ and $\alpha(y) = y^2$. This shows that

$$[xy, \underbrace{\alpha, \ldots, \alpha}_{n-1-times}] = y.$$

Clearly $\overline{cl(y)}$ has $\phi(p^s)$ elements but $yK_n(C_{2^t} \times C_{p^s}) - 1$ has $p^s - 1$ elements for $t \leq n$ and $2^{t-n}p^s - 1$ elements for $t \geq n+1$, which implies that the group $C_{2^t} \times C_{p^s}$ is not n-A-con-cos, for $t, s \geq 2$.

Next we investigate finite abelian 2-groups with two direct factors. Let

$$C_{2^t} \times C_{2^s} = \langle x, y \mid x^{2^t} = y^{2^s} = [x, y] = 1 \rangle,$$

for natural numbers t, s. There are six cases:

Case 1: $t \geq 2$ and $s = 1$. It is easy to check that the group $C_4 \times C_2$ is 2-A-con-cos. Also $K_n(C_4 \times C_2) = 1$, for $n \geq 3$. So the group $C_4 \times C_2$ is n-A-con-cos. The group $C_{2^t} \times C_2$ is n-A-con-cos, for $3 \leq t \leq n$, since $K_n(C_{2^t} \times C_2) = 1$. If $t = n + 1$, then $K_n(C_{2^{n+1}} \times C_2) =< x^{2^n} >$. Clearly $C_8 \times C_2$ is 2-A-con-cos. For $n \geq 3$, $K_{n-1}(C_{2^{n+1}} \times C_2) =< x^{2^{n-1}} >= \{1, x^{2^{n-1}}, x^{2^n}, x^{3.2^{n-1}}\}$. It is easy to check that $\{x^{2^{n-1}}, x^{3.2^{n-1}}\} = \overline{cl(x^{3.2^{n-1}})} = \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}}K_n(C_{2^{n+1}} \times C_2) - 1$, and $\{x^{2^n}\} = \overline{cl(x^{2^n})} = x^{2^n}K_n(C_{2^{n+1}} \times C_2) - 1$. Thus the group $C_{2^{n+1}} \times C_2$ is n-A-con-cos.

If $t \geq n+2$, then $K_n(C_{2^t} \times C_2) =< x^{2^n} >$. So $x \in (C_{2^t} \times C_2) - K_n(C_{2^t} \times C_2)$. Consider $\alpha, \beta \in Aut(C_{2^t} \times C_2)$ with $\alpha(x) = x^{2^{t-n+1}+1}$, $\alpha(y) = y$, $\beta(x) = x^3$ and $\beta(y) = y$. Thus

$$[x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{t-1}}$$

and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$, but $x^{2^{t-1}}K_n(C_{2^t} \times C_2) - 1$ has $2^{t-n} - 1$ elements. Hence the group $C_{2^t} \times C_2$ is not n-A-con-cos, for $t \geq n + 2$.

Case 2: $t = s$. By Theorem 3.1 (ii) of [2], $K_n(C_{2^t} \times C_{2^t}) = C_{2^t} \times C_{2^t}$. Hence the group $C_{2^t} \times C_{2^t}$ is not n-A-con-cos.

Case 3: $t > s \geq 2$ and $t \leq n-1$. If $t \geq s+2$, then the group $C_{2^t} \times C_{2^s}$ is n-A-con-cos, since $K_n(C_{2^t} \times C_{2^s}) = 1$. If $t = s+1$ and n is even, then by Corollary 3.2 of [2], $K_n(C_{2^t} \times C_{2^{t-1}}) =< x^{2^{\frac{n}{2}}} > \times < y^{2^{\frac{n}{2}}} >$. If $\frac{n}{2} \geq t$, then the group $C_{2^t} \times C_{2^{t-1}}$ is n-A-con-cos, since $K_n(C_{2^t} \times C_{2^{t-1}}) = 1$. For $\frac{n}{2} = t-1$, $K_n(C_{2^t} \times C_{2^{t-1}}) =< x^{2^{t-1}} >$ and $K_{n-1}(C_{2^t} \times C_{2^{t-1}}) =< x^{2^{t-1}} > \times < y^{2^{t-2}} >= \{1, x^{2^{t-1}}, y^{2^{t-2}}, x^{2^{t-1}}y^{2^{t-2}}\}$. Clearly $\{x^{2^{t-1}}\} = \overline{cl(x^{2^{t-1}})} = x^{2^{t-1}}K_n(C_{2^t} \times C_{2^{t-1}}) - 1$, and $\{x^{2^{t-1}}y^{2^{t-2}}, y^{2^{t-2}}\}$ $= \overline{cl(x^{2^{t-1}}y^{2^{t-2}})} = \overline{cl(y^{2^{t-2}})} = y^{2^{t-2}}K_n(C_{2^t} \times C_{2^{t-1}}) - 1$, which implies that if n is even and $\frac{n}{2} = t - 1 = s \geq 2$, then the group $C_{2^t} \times C_{2^s}$ is n-A-con-cos.

Next we investigate the group $C_{2^t} \times C_{2^s}$ for $\frac{n}{2} < t - 1$ and $t = n - 1$. Note that $x \in (C_{2^{n-1}} \times C_{2^{n-2}}) - K_n(C_{2^{n-1}} \times C_{2^{n-2}})$. Consider the automorphisms α, β, γ of $C_{2^{n-1}} \times C_{2^{n-2}}$ with $\alpha(x) = x^3$, $\alpha(y) = y$, $\beta(x) = xy$, $\beta(y) = y$, $\gamma(x) = x$ and $\gamma(y) = x^2y$. Thus

$$[x, \underbrace{\alpha, \ldots, \alpha}_{n-3-times}, \beta, \gamma] = x^{2^{n-2}}.$$

It is obvious that $\overline{cl(x^{2^{n-2}})} = \{x^{2^{n-2}}\}$ but $x^{2^{n-2}}K_n(C_{2^{n-1}}\times C_{2^{n-2}})-1 = K_n(C_{2^{n-1}}\times C_{2^{n-2}})-1$ has $2^{n-3}-1$ elements. Hence this group is not n-A-con-cos. Similarity, we can show that the group $C_{2^t}\times C_{2^s}$ is not n-A-con-cos, for $\frac{n}{2} < t-1$ and $t < n-1$. If $t = s+1$ and n is odd, then $K_n(C_{2^t}\times C_{2^{t-1}}) =< x^{2^{\frac{n+1}{2}}} > \times < y^{2^{\frac{n-1}{2}}} >$. For $\frac{n+1}{2} \geq t$, we have $\frac{n-1}{2} \geq t-1$ and so the group $C_{2^t}\times C_{2^{t-1}}$ is n-A-con-cos, since $K_n(C_{2^t}\times C_{2^{t-1}}) = 1$. If $\frac{n+1}{2} < t$ and $t = n-1$, then we have $x \in (C_{2^{n-1}}\times C_{2^{n-2}}) - K_n(C_{2^{n-1}}\times C_{2^{n-2}})$. Consider the automorphisms α, β, γ of $C_{2^{n-1}}\times C_{2^{n-2}}$ with

$$\alpha(x) = x^3,\ \ \alpha(y) = y,\ \ \beta(x) = xy,\ \ \beta(y) = y,\ \ \gamma(x) = x,\ \ \gamma(y) = x^2y.$$

It is easy to check that

$$\overline{cl([x, \underbrace{\alpha,\ldots,\alpha}_{n-3-times}, \beta, \gamma])} \neq [x, \underbrace{\alpha,\ldots,\alpha}_{n-3-times}, \beta, \gamma]K_n(C_{2^{n-1}}\times C_{2^{n-2}}) - 1.$$

Therefore this group is not n-A-con-cos. By a similar argument it can be shown that the group $C_{2^t}\times C_{2^{t-1}}$ is not n-A-con-cos, for $\frac{n+1}{2} < t$ and $t < n-1$.

Case 4: $t > s \geq 2$ and $t = n$. If $t \geq s+2$, then the group $C_{2^n}\times C_{2^s}$ is n-A-con-cos, since $K_n(C_{2^n}\times C_{2^s}) = 1$. If $t = s+1$ and n is even, then $C_{2^n}\times C_{2^{n-1}}$ is not n-A-con-cos, since $K_n(C_{2^n}\times C_{2^{n-1}}) =< x^{2^{\frac{n}{2}}} > \times < y^{2^{\frac{n}{2}}} >$. Thus $x \in (C_{2^n}\times C_{2^{n-1}}) - K_n(C_{2^n}\times C_{2^{n-1}})$ and for the automorphism α of $C_{2^n}\times C_{2^{n-1}}$ with $\alpha(x) = x^3$ and $\alpha(y) = y$,

$$[x, \underbrace{\alpha,\ldots,\alpha}_{n-1-times}] = x^{2^{n-1}}$$

and $\overline{cl(x^{2^{n-1}})} = \{x^{2^{n-1}}\}$ but $x^{2^{n-1}}K_n(C_{2^n}\times C_{2^{n-1}}) - 1$ has $2^{n-1}-1$ elements. If $t = s+1$ and n is odd, then $C_{2^n}\times C_{2^{n-1}}$ is not n-A-con-cos, since $K_n(C_{2^n}\times C_{2^{n-1}}) =< x^{2^{\frac{n+1}{2}}} > \times < y^{2^{\frac{n-1}{2}}} >$ and $x \in (C_{2^n}\times C_{2^{n-1}}) - K_n(C_{2^n}\times C_{2^{n-1}})$. Consider $\alpha \in Aut(C_{2^n}\times C_{2^{n-1}})$ with $\alpha(x) = x^3$, $\alpha(y) = y$. Clearly

$$\overline{cl([x, \underbrace{\alpha,\ldots,\alpha}_{n-1-times}])} \neq [x, \underbrace{\alpha,\ldots,\alpha}_{n-1-times}]K_n(C_{2^n}\times C_{2^{n-1}}) - 1.$$

Case 5: $t > s \geq 2$ and $t = n+1$. If $t \geq s+2$, then $K_n(C_{2^{n+1}}\times C_{2^s}) =< x^{2^n} >$. By assumption $n-2 \geq s-1$. If $n-2 = s-1$, then

$$K_{n-1}(C_{2^{n+1}}\times C_{2^s}) =< x^{2^{n-1}} > \times < y^{2^{n-2}} >,$$

and if $n-2 \geq s$, then $K_{n-1}(C_{2^{n+1}}\times C_{2^s}) =< x^{2^{n-1}} >$. In two cases for every $a \in K_{n-1}(C_{2^{n+1}}\times C_{2^s}) - 1$ we have $\overline{cl(a)} = aK_n(C_{2^{n+1}}\times C_{2^s}) - 1$, and it shows that the group $C_{2^t}\times C_{2^s}$ is n-A-con-cos, for $t = n+1$ and $t \geq s+2 \geq 4$.

Next we investigate the group $C_{2^{n+1}} \times C_{2^n}$, for $t = s + 1$. We have

$$K_n(C_{2^{n+1}} \times C_{2^n}) =< x^{2^{[\frac{n+1}{2}]}} > \times < y^{2^{[\frac{n}{2}]}} > .$$

Thus $x \in (C_{2^{n+1}} \times C_{2^n}) - K_n(C_{2^{n+1}} \times C_{2^n})$. Consider $\alpha, \beta \in Aut(C_{2^{n+1}} \times C_{2^n})$ with $\alpha(x) = x^5$, $\alpha(y) = y$, $\beta(x) = x^3$ and $\beta(y) = y$. We have

$$\overline{cl([x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}]K_n(C_{2^{n+1}} \times C_{2^n}) - 1,$$

which implies that this group is not n-A-con-cos.

Case 6: $t > s \geq 2$ and $t \geq n + 2$. If $t \geq s + 2$, then $K_n(C_{2^t} \times C_{2^s}) =< x^{2^n} > \times < y^{2^{n-1}} >$. So $x \in (C_{2^t} \times C_{2^s}) - K_n(C_{2^t} \times C_{2^s})$. Consider $\alpha, \beta \in Aut(C_{2^t} \times C_{2^s})$ with $\alpha(x) = x^{2^{t-n+1}+1}$, $\alpha(y) = y$, $\beta(x) = x^3$ and $\beta(y) = y$. Then

$$[x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{t-1}}$$

and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$. On the other hand, $x^{2^{t-1}}K_n(C_{2^t} \times C_{2^s}) - 1$ bas $2^{t+s-2n+1} - 1$ elements if $s > n - 1$ and has $2^{t-n} - 1$ elements if $s \leq n - 1$, which implies that

$$\overline{cl([x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}]K_n(C_{2^t} \times C_{2^s}) - 1.$$

Hence the group $C_{2^t} \times C_{2^s}$ is not n-A-con-cos, for $t \geq s + 2 \geq 4$ and $t \geq n + 2$. If $t = s + 1$, then a similar argument as above shows that the group $C_{2^t} \times C_{2^s}$ is not n-A-con-cos. This completes the proof. □

In following theorem, we investigate some dihedral groups.

Theorem 2.5. Let m, n be natural numbers, where m has at most two prime factors and $n \geq 2$. Then the dihedral group D_{2m} is n-A-con-cos if $m = 2^{t-1}$, for natural number t, $3 \leq t \leq n + 1$, or $m = p$ or $m = 2^t p$, for odd prime number p and natural number t, where $1 \leq t \leq n - 2$.

Proof. Let $D_{2m} = \langle x, y \mid x^m = y^2 = (xy)^2 = 1\rangle$ be the dihedral group of order $2m$. At first we assume that $m = 2^{t-1}$, where t is a natural number and $t \geq 2$. Clearly the group $D_4 = C_2 \times C_2$ is not n-A-con-cos. If $3 \leq t \leq n - 1$, then the group D_{2^t} is n-A-con-cos, since by Theorem 1.1 of [2], $K_t(D_{2^t}) =< x^{2^{t-1}} >= 1$ and $K_n(D_{2^t}) \subseteq K_t(D_{2^t})$. If $t = n$ and $n \geq 3$, then D_{2^n} is A-nilpotent and $K_n(D_{2^n}) = 1$. Hence by Theorem 2.3, the group D_{2^n} is n-A-con-cos, for $n \geq 3$. If $t = n + 1$, then $K_n(D_{2^{n+1}}) =< x^{2^{n-1}} >= \{1, x^{2^{n-1}}\}$ and $K_{n-1}(D_{2^{n+1}}) = < x^{2^{n-2}} > = \{1, x^{2^{n-2}}, x^{2^{n-1}}, x^{3.2^{n-2}}\}$. This implies that for every $g \in D_{2^{n+1}} -$

$K_n(D_{2^{n+1}})$ and $\alpha_1, \ldots, \alpha_{n-1} \in Aut(D_{2^{n+1}})$, where $[g, \alpha_1, \ldots, \alpha_{n-1}] \neq 1$ we have $[g, \alpha_1, \ldots, \alpha_{n-1}] \in \{x^{2^{n-2}}, x^{2^{n-1}}, x^{3.2^{n-2}}\}$. Clearly

$$\begin{aligned}\{x^{2^{n-2}}, x^{3.2^{n-2}}\} &= \overline{cl(x^{2^{n-2}})} = \overline{cl(x^{3.2^{n-2}})} = x^{2^{n-2}} K_n(D_{2^{n+1}}) - 1, \\ \{x^{2^{n-1}}\} &= \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}} K_n(D_{2^{n+1}}) - 1,\end{aligned}$$

which implies that the group $D_{2^{n+1}}$ is n-A-con-cos.

If $t \geq n+2$, then $K_n(D_{2^t}) =< x^{2^{n-1}} >$, and so $x \in D_{2^t} - K_n(D_{2^t})$. Consider $\alpha, \beta \in Aut(D_{2^t})$ with $\alpha(x) = x^{2^{t-n}+1}$, $\alpha(y) = y$, $\beta(x) = x^3$ and $\beta(y) = y$. Then

$$[x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{t-2}}.$$

Clearly $\overline{cl(x^{2^{t-2}})} = \{x^{2^{t-2}}\}$. On the other hand, $x^{2^{t-2}} K_n(D_{2^t}) - 1 =< x^{2^{n-1}} > -1$ has $2^{t-n} - 1$ elements. Therefore,

$$\overline{cl([x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [x, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] K_n(D_{2^t}) - 1,$$

which implies that the group D_{2^t} is not n-A-con-cos, for $t \geq n+2$.

Next we investigate the case that $m = p^t$, for odd prime number p and natural number t. The group D_{2p} is n-A-con-cos, since by Theorem 1.1 of [2], $K_{n-1}(D_{2p}) = K_n(D_{2p})$. Also $< x >= 1 \cup \overline{cl(x)}$, and hence the claim follows from Theorem 2.2. If $t \geq 2$, then $K_n(D_{2p^t}) =< x >$, and therefore $y \in D_{2p^t} - K_n(D_{2p^t})$. Consider $\alpha, \beta \in Aut(D_{2p^t})$ with $\alpha(x) = x$, $\alpha(y) = x^{p^t-1}y$, $\beta(x) = x^2$ and $\beta(y) = y$. Then

$$[y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x$$

and $\overline{cl(x)}$ has $\phi(p^t)$ elements. On the other hand, $xK_n(D_{2p^t}) - 1 =< x > -1$ has $p^t - 1$ elements, since $t \geq 2$ and we have

$$\overline{cl([y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] K_n(D_{2p^t}) - 1.$$

Thus the group D_{2p^t} is not n-A-con-cos, for $t \geq 2$.

We now assume that m has two distinct prime factors. Let $m = p^t q^s$, where p, q are distinct odd prime numbers and t, s are natural numbers. Since $K_n(D_{2p^tq^s}) =< x >$, $x^{p^tq^s-1}y \in D_{2p^tq^s} - K_n(D_{2p^tq^s})$. Consider the automorphisms α and β of $D_{2p^tq^s}$ with $\alpha(x) = x$, $\alpha(y) = x^{p^tq^s-1}y$, $\beta(x) = x^2$ and $\beta(y) = y$. It is easy to check that

$$\overline{cl([x^{p^tq^s-1}y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [x^{p^tq^s-1}y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] K_n(D_{2p^tq^s}) - 1.$$

Hence the group $D_{2p^tq^s}$ is not n-A-con-cos.

Finally we investigate the group $D_{2^{t+1}p^s}$, where p is an odd prime number and t, s are natural numbers. We have the following three cases:

Case 1: $2 \leq t+1 \leq n-1$ and $s=1$. In this case $K_{n-1}(D_{2^{t+1}p}) = K_n(D_{2^{t+1}p})$ $=< x^{2^{n-1}} >= 1 \cup \overline{cl(x^{2^{n-1}})}$. So, by Theorem 2.2, the group $D_{2^{t+1}p}$ is n-A-con-cos, for $2 \leq t+1 \leq n-1$.

Case 2: $2 \leq t+1 \leq n-1$ and $s \geq 2$. Since $K_n(D_{2^{t+1}p^s}) =< x^{2^{n-1}} >$, $y \in D_{2^{t+1}p^s} - K_n(D_{2^{t+1}p^s})$. Consider $\alpha, \beta \in Aut(D_{2^{t+1}p^s})$ with $\alpha(x) = x$, $\alpha(y) = xy$, $\beta(x) = x^{2^t p^s - 1}$ and $\beta(y) = y$. Thus

$$[y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{(-1)^{n-1}2^{n-2}}.$$

If n is odd, then

$$[y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{2^{n-2}}$$

and $\overline{cl(x^{2^{n-2}})}$ has $p^s - p^{s-1}$ elements but $x^{2^{n-2}}K_n(D_{2^{t+1}p^s}) - 1 =< x^{2^{n-1}} > -1$ has $p^s - 1$ elements. If n is even, then

$$[y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}] = x^{-2^{n-2}}$$

and $\overline{cl(x^{-2^{n-2}})} = \overline{cl(x^{2^{n-2}})}$ has $p^s - p^{s-1}$ elements but $x^{-2^{n-2}}K_n(D_{2^{t+1}p^s}) - 1$ has $p^s - 1$ elements. Thus the group $D_{2^{t+1}p^s}$ is not n-A-con-cos, for $2 \leq t+1 \leq n-1$ and $s \geq 2$.

Case 3: $t+1 \geq n$ and $s \geq 1$. Since $K_n(D_{2^{t+1}p^s}) =< x^{2^{n-1}} >$, $y \in D_{2^{t+1}p^s} - K_n(D_{2^{t+1}p^s})$. Consider the automorphisms α and β of $D_{2^{t+1}p^s}$ with $\alpha(x) = x$, $\alpha(y) = xy$, $\beta(x) = x^{2^t p^s - 1}$ and $\beta(y) = y$. It is easy to check that

$$\overline{cl([y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])}$$

has $2^{t-n+1}p^{s-1}(p-1)$ elements but

$$[y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}]K_n(D_{2^{t+1}p^s}) - 1$$

has $2^{t-n+1}p^s$ elements. Therefore,

$$\overline{cl([y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}])} \neq [y, \alpha, \underbrace{\beta, \ldots, \beta}_{n-2-times}]K_n(D_{2^{t+1}p^s}) - 1.$$

Hence the group $D_{2^{t+1}p^s}$ is not n-A-con-cos, for $t+1 \geq n$ and $s \geq 1$. This completes the proof. $\square$

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