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Graph Invariants of Deleted Lexicographic Product of Graphs

Bahare Akhavan Mahdavi, *Mostafa Tavakoli* * *and Freydoon Rahbarnia*

Abstract

The deleted lexicographic product $G[H] - nG$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever ($u_1 = u_2$ and v_1 is adjacent with v_2) or ($v_1 \neq v_2$ and u_1 is adjacent with u_2). In this paper, we compute the exact values of the Wiener, vertex PI and Zagreb indices of deleted lexicographic product of graphs. Applications of our results under some examples are presented.

Keywords: Deleted lexicographic product, Wiener index, vertex PI index, Zagreb indices.

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1. Introduction

Throughout this paper all graphs considered are finite, simple and connected. The distance $d_G(u, v)$ between the vertices u and v of a graph G is equal to the length of a shortest path that connects u and v and the diameter of G is the greatest distance between two vertices in G.

The **lexicographic product** was studied first by Felix Hausdorff in 1914 [9] and then studied by Harary and Sabidussi. Feigenbaum and Schäffer [5] proved that the complexity of testing whether an arbitrary graph can be written nontrivially as the composition of two smaller graphs is the same, to within polynomial

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factors, as the complexity of testing whether two graphs are isomorphic. Frelih and Miklavič [6] proposed another lexicographic-like product that is called the **deleted lexicographic product** as follow:

For two graphs G and H with $|V(H)| = n$, the deleted lexicographic product $G[H] - nG$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever ($u_1 = u_2$ and v_1 is adjacent with v_2) or ($v_1 \neq v_2$ and u_1 is adjacent with u_2), Figure 1.

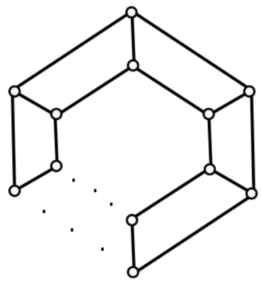

Figure 1: The deleted lexicographic product of C_{2n} and P_2.

The **Graph invariants** are parameters that are preserved under graph isomorphisms. However, they are not usually preserved under graph homomorphisms. A **topological index** is a graph invariant applicable in chemistry.

The **Wiener index**, W, is the first topological index to be used in chemistry [14]. In a graph theoretical language,

$$\mathrm{W}(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v).$$

We encourage the readers to consult [1–4] for more information on the Wiener index.

Suppose G is a graph with vertex and edge sets $V = V(G)$ and $E = E(G)$, respectively, and $e = uv \in E(G)$. The set of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $N_u^G(e)$. The **vertex Padmakar-Ivan index** of the graph G is defined as [10, 11, 13]

$$\mathrm{PI}_v(G) = \sum_{e=uv \in E(G)} \big(|N_u^G(e)| + |N_v^G(e)|\big).$$

The **Zagreb indices** have been introduced by Gutman and Trinajstić as $\mathrm{M}_1(G) = \sum_{u \in V(G)} (deg_G(u))^2$ and $\mathrm{M}_2(G) = \sum_{uv \in E(G)} deg_G(u) deg_G(v)$, where $deg_G(u)$ denotes the degree of vertex u [7, 8, 12].

P_n, K_n and L_n denote the path with n vertices, the complete graph with n vertices and the ladder graph with $2n$ vertices, respectively. Our other notations are standard and taken mainly from the standard books of graph theory.

2. Results

In this section, our main results are presented. We start by a simple lemma that will be used later.

Lemma 2.1. *Let G and H be two graphs with at least two vertices and $W = G[H] - nG$. Then,*

1. *$deg_W((g,h)) = deg_H(h) + (|V(H)|-1)deg_G(g)$,*
2. *If H has at least three vertices, then*

$$d_W((g,h),(g',h')) = \begin{cases} 1 & \text{if } (g=g', hh' \in E(H)) \text{ or } (gg' \in E(G), h \neq h'), \\ 2 & \text{if } (g=g', hh' \notin E(H)) \text{ or } (gg' \in E(G), h = h'), \\ d_G(g,g') & \text{if } (gg' \notin E(G) \text{ and } g \neq g'). \end{cases}$$

3. *If H has exactly two vertices, then*

$$d_W((g,h),(g',h')) = \begin{cases} 1 & \text{if } (g=g') \text{ or } (gg' \in E(G), h \neq h'), \\ d_G(g,g') & \text{if } (g \neq g', h = h', 2 \mid d_G(g,g')) \text{ or } (g \neq g', h \neq h', 2 \nmid d_G(g,g')), \\ d_G(g,g')+1 & \text{if } (g \neq g', h \neq h', 2 \mid d_G(g,g')) \text{ or } (g \neq g', h = h', 2 \nmid d_G(g,g')). \end{cases}$$

Proof. The first statement is easily obtained by the definition of deleted lexicographic product. We prove the statements 2 and 3.
Let H be a graph with more than 2 vertices, and $(g,h),(g',h') \in V(W)$. The first case of relation 2 is clear. Then we suppose $g = g'$ and $hh' \notin E(H)$. In this case, $(g,h)(g'',h'')(g',h')$ is a $((g,h),(g',h'))$-path of length 2 in W where $gg'' \in E(G)$ and $h'' \notin \{h,h'\}$. Thus, assume that $gg' \in E(G)$ and $h = h'$. Therefore, $(g,h)(g',h'')(g',h')$ is a $((g,h),(g',h'))$-path of length 2 in W where $hh'' \in E(H)$.

Now we investigate the third case of relation 2. Suppose $gg' \notin E(G)$ and $g \neq g'$. Since G is a connected graph, then there is a (g,g')-path $gg_1 \ldots g_k g'$ in G. If $h = h'$ and k is an even number, then $(g,h)a_1a_2\ldots a_k(g',h')$ is a $((g,h),(g',h'))$-path of length $d_G(g,g')$ in W where $h'',h''' \in V(H) \setminus \{h\}$, $h'' \neq h'''$, and

$$a_i = \begin{cases} (g_i,h'') & 2 \nmid i \\ (g_i,h''') & 2 \mid i \end{cases}$$

for $1 \leq i \leq k$. Similarly, if $h = h'$ and k is an odd number, then $(g,h)a_1a_2\ldots a_k$ (g',h') is a $((g,h),(g',h'))$-path of length $d_G(g,g')$ in W where $h'' \neq h$ and

$$a_i = \begin{cases} (g_i,h'') & 2 \nmid i \\ (g_i,h) & 2 \mid i \end{cases}$$

for $1 \leq i \leq k$. By a similar argument, in the case that $h \neq h'$ and k is an even number, $(g,h)a_1a_2 \ldots a_k(g',h')$ is a $((g,h),(g',h'))$-path of length $d_G(g,g')$ in W where $h'' \neq h$ and $a_i = \begin{cases} (g_i,h') & 2 \nmid i \\ (g_i,h) & 2 \mid i \end{cases}$ for $1 \leq i \leq k$. Similarly, if $h \neq h'$ and k is an odd number, then $(g,h)a_1a_2 \ldots a_k(g',h')$ is a $((g,h),(g',h'))$-path of length $d_G(g,g')$ in W where $h'' \notin \{h,h'\}$ and $a_i = \begin{cases} (g_i,h'') & 2 \nmid i \\ (g_i,h) & 2 \mid i \end{cases}$ for $1 \leq i \leq k$. This completes the proof of the statement 2.

Now suppose that $V(H) = \{h,h'\}$. Consider two vertices (g,h) and (g',h) of W. Let $gg_1 \ldots g_kg'$ be the shortest (g,g')-path in G. If $d_G(g,g')$ is an even number (in other words, k is an odd number), then we have $((g,h),(g',h))$-path $(g,h)a_1a_2 \ldots a_k(g',h)$ of length $d_G(g,g')$ in W where $a_i = \begin{cases} (g_i,h') & 2 \nmid i \\ (g_i,h) & 2 \mid i \end{cases}$ for $1 \leq i \leq k$. Note that if $d_G(g,g')$ is an odd number, then $(g,h)a_1a_2 \ldots a_k(g',h')(g',h)$ is a $((g,h),(g',h))$-path of length $d_G(g,g')+1$ in W where $a_i = \begin{cases} (g_i,h') & 2 \nmid i \\ (g_i,h) & 2 \mid i \end{cases}$ for $1 \leq i \leq k$. By a similar technique, we can prove the cases in which $h \neq h'$, which completes the proof of the statement 3. $\square$

The next theorem gives a formula for the first Zagreb index of the deleted lexicographic product of G and H in terms of their parameters.

Theorem 2.2. Let G and H be two graphs, then

$$\mathrm{M}_1(G[H]-nG) = \mathrm{M}_1(H)|V(G)|+\mathrm{M}_1(G)|V(H)|(|V(H)|-1)^2+8|E(H)||E(G)|(|V(H)|-1).$$

Proof. By the definition of Zagreb index, and part (1) of Lemma 2.1,

$$\begin{aligned} \mathrm{M}_1(G[H]-nG) &= \sum_{(g,h)\in V(G[H]-nG)} \left(deg_H(h) + (|V(H)|-1)deg_G(g)\right)^2 \\ &= \sum_{(g,h)\in V(G[H]-nG)} \left(deg_H(h)\right)^2 + (|V(H)|-1)^2 \sum_{(g,h)\in V(G[H]-nG)} \left(deg_G(g)\right)^2 \\ &+ 2(|V(H)|-1) \sum_{(g,h)\in V(G[H]-nG)} deg_G(g)deg_H(h) \\ &= \mathrm{M}_1(H)|V(G)| + \mathrm{M}_1(G)|V(H)|(|V(H)|-1)^2 + 8|E(H)||E(G)|(|V(H)|-1). \end{aligned}$$

This completes the proof. $\square$

The next theorem presents a formula for the second Zagreb index of the deleted lexicographic product of $G[H]-nG$ based on the parameters of G and H.

Theorem 2.3. Let G and H be two graphs. Then

$$\begin{aligned}\mathrm{M}_2(G[H]-nG) &= 2|E(G)|(|V(H)|-1)\mathrm{M}_1(H)+|V(G)|\mathrm{M}_2(H)+|E(G)|(4|E(H)|^2-\mathrm{M}_1(H))\\ &\quad+3|E(H)|(|V(H)|-1)^2\mathrm{M}_1(G)+|V(H)|(|V(H)|-1)^3\mathrm{M}_2(G).\end{aligned}$$

Proof. Let G and H be two graphs. For our convenience, we partition the edge set of $G[H]-nG$ into two subsets as follows:

$$\begin{aligned}E_1 &= \{(g,h)(g',h') \mid g=g' \ \text{ and } \ hh'\in E(H)\},\\ E_2 &= \{(g,h)(g',h') \mid h\neq h' \ \text{ and } \ gg'\in E(G)\}.\end{aligned}$$

By the definition of M_2,

$$\begin{aligned}\mathrm{M}_2(G[H]-nG) &= \sum_{E_1} deg_{G[H]-nG}((g,h))deg_{G[H]-nG}((g',h'))\\ &\quad+\sum_{E_2} deg_{G[H]-nG}((g,h))deg_{G[H]-nG}((g',h')).\end{aligned}$$

On the other hand, by this fact that $\sum_{hh'\in E(H)}\big(deg_H(h)+deg_H(h')\big)=\mathrm{M}_1(H)$ and Lemma 2.1,

$$\begin{aligned}&\sum_{(g,h)(g,h')\in E_1} deg_{G[H]-nG}((g,h))deg_{G[H]-nG}((g,h'))\\ &= \sum_{(g,h)(g,h')\in E_1}\big(deg_H(h)+(|V(H)|-1)deg_G(g)\big)\big(deg_H(h')+(|V(H)|-1)deg_G(g)\big)\\ &= \sum_{(g,h)(g,h')\in E_1} deg_H(h)deg_H(h')+(|V(H)|-1)\sum_{(g,h)(g,h')\in E_1} deg_H(h)deg_G(g)\\ &+(|V(H)|-1)\sum_{(g,h)(g,h')\in E_1} deg_H(h')deg_G(g)+(|V(H)|-1)^2\sum_{(g,h)(g,h')\in E_1}\big(deg_G(g)\big)^2\\ &= \sum_{g\in V(G)}\sum_{hh'\in E(H)} deg_H(h)deg_H(h')+(|V(H)|-1)\sum_{g\in V(G)} deg_G(g)\sum_{hh'\in E(H)} deg_H(h)\\ &+(|V(H)|-1)\sum_{g\in V(G)} deg_G(g)\sum_{hh'\in E(H)} deg_H(h')\\ &+(|V(H)|-1)^2\sum_{g\in V(G)}\sum_{hh'\in E(H)}\big(deg_G(g)\big)^2\\ &= |V(G)|\mathrm{M}_2(H)+2|E(G)|(|V(H)|-1)\mathrm{M}_1(H)+|E(H)|(|V(H)|-1)^2\mathrm{M}_1(G).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{E_2} deg_{G[H]-nG}((g,h))deg_{G[H]-nG}((g',h')) &= |E(G)|(4|E(H)|^2-\mathrm{M}_1(H))+2|E(H)|(|V(H)|-1)^2\\ &\quad\times\mathrm{M}_1(G)+|V(H)|(|V(H)|-1)^3\mathrm{M}_2(G).\end{aligned}$$

This completes the proof. $\square$

Theorem 2.4. Let G and H be two graphs and $|V(H)| \geq 3$. Then

$$\begin{aligned}\mathrm{W}(G[H]-nG) = {} & |V(H)|^2(\mathrm{W}(G)-|E(G)|)+|V(H)|(|V(H)|-1)\\ & \times(|V(G)|+|E(G)|)+2|V(H)||E(G)|-|V(G)||E(H)|.\end{aligned}$$

Proof. Let G and H be two graphs and $|V(H)| \geq 3$. We define V_1, V_2 and V_3 as follows:

$$\begin{aligned}V_1 &= \{\{(g,h),(g',h')\} \subseteq V(G[H]-nG) \mid g = g'\},\\ V_2 &= \{\{(g,h),(g',h')\} \subseteq V(G[H]-nG) \mid gg' \in E(G)\},\\ V_3 &= \{\{(g,h),(g',h')\} \subseteq V(G[H]-nG) \mid gg' \notin E(G),\ g \neq g'\}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}W_1 &= \sum_{V_1} d_{G[H]-nG}((g,h),(g',h')) = |V(G)||V(H)|(|V(H)|-1)-|V(G)||E(H)|,\\ W_2 &= \sum_{V_2} d_{G[H]-nG}((g,h),(g',h')) = |E(G)||V(H)|(|V(H)|+1),\\ W_3 &= \sum_{V_3} d_{G[H]-nG}((g,h),(g',h')) = |V(H)|^2\mathrm{W}(G)-|V(H)|^2|E(G)|.\end{aligned}$$

By summation of W_1, W_2 and W_3, the result can be proved. □

By part 3 of Lemma 2.1, it is far from easy to obtain the exact value of $\mathrm{W}(G[H]-nG)$ where $|V(H)| = 2$. However, in the next proposition we compute this invariant only for the case $G \cong C_k$ which is an immediate corollary of Lemma 2.1.

Proposition 2.5. For a cycle C_k, we have

$$\mathrm{W}(C_k[P_2]-2C_k) = \begin{cases} 4n^2(n+1) & \text{if } k = 2n,\\ 4n^3+10n^2+2n-1 & \text{if } k = 2n+1.\end{cases}$$

It is not difficult to check that, if $|V(H)| = 2$, then $\mathrm{W}(G[H]-nG) = 4\mathrm{W}(G)+|V(G)|^2$.

Theorem 2.6. Let G and H be two graphs and $|V(H)| \geq 3$. Then

$$\begin{aligned}\mathrm{PI}_v(G[H]-nG) = {} & |V(G)|(M_1(H)-6t_H)+8|E(G)||E(H)|\\ & +|V(H)|(|V(H)|-1)(2|E(G)|-\mathrm{M}_1(G)+12t_G)\\ & -4|E(G)||E(H)|(|V(H)|-1)+|V(H)|^2(|V(H)|-1)\mathrm{PI}_v(G),\end{aligned}$$

where t_G and t_H denote the number of triangles of G and H, respectively.

Proof. For a graph G, let $t_G(gg')$ denote the number of triangles containing edge gg' of G. So, by definition of deleted lexicographic product,

$$|N_{(g_i,h_l)}((g_i,h_l)(g_j,h_k))| = \begin{cases} deg_H(h_l) + deg_G(g_i) - t_H(h_lh_k) & \text{if } i = j, h_lh_k \in E(H), \\ |V(H)||N_{g_i}(g_ig_j)| - deg_G(g_i) & \\ -deg_H(h_k) + 2t_G(g_ig_j) + 2 & \text{if } i \neq j, h_lh_k \in E(H), \\ |V(H)||N_{g_i}(g_ig_j)| - deg_G(g_i) & \\ -deg_H(h_k) + 2t_G(g_ig_j) & \text{if } i \neq j, h_lh_k \notin E(H). \end{cases}$$

Therefore, $\mathrm{PI}_v(G[H] - nG) = \mathrm{PI}_1 + \mathrm{PI}_2$, where

$$\mathrm{PI}_1 = \sum_{i=1}^{|V(G)|} \sum_{hh' \in E(H)} (|N_{(g_i,h_l)}((g_i,h_l)(g_i,h_k))| + |N_{(g_i,h_k)}((g_i,h_l)(g_i,h_k))|),$$

$$\mathrm{PI}_2 = \sum_{g_ig_j \in E(G)} \sum_{h_l,h_k \in V(H), l \neq k} (|N_{(g_i,h_l)}((g_i,h_l)(g_j,h_k))| + |N_{(g_j,h_k)}((g_i,h_l)(g_j,h_k))|).$$

We know that $\sum h_lh_k \in E(H)t_H(h_lh_k) = 3t_H$ because each triangle has three edges, and so it is counted three times in computing t_H. Also, $\sum_{h_lh_k} \big(deg_H(h_l) + deg_H(h_k)\big) = \mathrm{M}_1(H)$. Thus

$$\begin{aligned} \mathrm{PI}_1 = & \sum_{i=1}^{|V(G)|} \sum_{hh' \in E(H)} \Big(\big(deg_H(h_l) + deg_G(g_i) - t_H(h_lh_k)\big) \\ & + \big(deg_H(h_k) + deg_G(g_i) - t_H(h_lh_k)\big)\Big) \\ = & \sum_{i=1}^{|V(G)|} \sum_{hh' \in E(H)} \big(deg_H(h_l) + deg_H(h_k)\big) + 2 \sum_{i=1}^{|V(G)|} \sum_{hh' \in E(H)} deg_G(g_i) \\ & - 2 \sum_{i=1}^{|V(G)|} \sum_{hh' \in E(H)} t_H(h_lh_k) = |(V(G)|(\mathrm{M}_1(H) - 6t_H) + 4|E(H)||E(G)|. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{PI}_2 = & (2|E(G)| - \mathrm{M}_1(G) + 12t_G)|V(H)|(|V(H)| - 1) \\ & - 2|E(G)|(2|V(H)| - 2)|E(H)| \\ & + \mathrm{PI}_v(G)|V(H)|^2(|V(H)| - 1) + 4|E(H)||E(G)|. \end{aligned}$$

This completes the proof. □

3. Applications

In this section, we apply our results presented in Section 2 for computing the Wiener index, vertex Padmakar-Ivan index, and Zagreb indices of some well-known graphs.

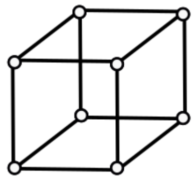

Figure 2: The deleted lexicographic product of C_4 and P_2.

Example 3.1. By using $\mathrm{M}_1(C_n) = \mathrm{M}_2(C_n) = 4n$, $\mathrm{M}_1(P_n) = 4n - 6$, $\mathrm{M}_2(P_2) = 1$ and $\mathrm{M}_2(P_n) = 4(n-2)$ for $n > 2$ [12] and applying Theorems 2.2 and 2.3, we obtain the following formulas:

$$\begin{aligned}\mathrm{M}_1(C_m[P_n] - nC_m) &= m\mathrm{M}_1(P_n) + n(n-1)^2\mathrm{M}_1(C_m) + 8m(n-1)^2\\ &= 4mn^3 - 8mn + 2m,\\ \mathrm{M}_2(C_m[P_n] - nC_m) &= 2m(n-1)\mathrm{M}_1(P_n) + m\mathrm{M}_2(P_n) + m(4(n-1)^2 - \mathrm{M}_1(P_n))\\ &\quad + 3(n-1)^3\mathrm{M}_1(C_m) + n(n-1)^3\mathrm{M}_2(C_m)\\ &= \begin{cases} 4m(n^4 - 3n^2 + n + 1/2) & \text{if } n > 2,\\ 27m & \text{if } n = 2.\end{cases}\end{aligned}$$

On the other hand, by [15], we know $\mathrm{W}(C_m) = \begin{cases} \frac{m^3}{8} & \text{if } 2 \mid m\\ \frac{m^3-m}{8} & \text{if } 2 \nmid m\end{cases}$. Then, by Theorem 2.4, for $n > 2$ we have

$$\mathrm{W}(C_m[P_n] - nC_m)) = \begin{cases} \frac{n^2m^3}{8} + mn^2 - mn + m & \text{if } 2 \mid m,\\ \frac{n^2m^3}{8} + \frac{7mn^2}{8} - mn + m & \text{if } 2 \nmid m.\end{cases}$$

Moreover, by [11], we have $\mathrm{PI}_v(P_n) = n(n-1)$, $\mathrm{PI}_v(C_m) = \begin{cases} m^2 & \text{if } 2 \mid m\\ m(m-1) & \text{if } 2 \nmid m\end{cases}$. Then, by Theorem 2.6, for $n \geq 3$ we have

$$\mathrm{PI}_v(C_m[P_n] - nC_m) = \begin{cases} n^3m^2 - n^2m^2 - 6n^2m + 22nm - 18m & \text{if } 2 \mid m,\\ n^3m^2 - n^2m^2 - mn^3 - 5mn^2 + 22mn - 18m & \text{if } 2 \nmid m.\end{cases}$$

Example 3.2. The n-cube Q_n, $n \geq 1$, is the graph whose vertex set is the set of all n-tuples of 0s and 1s, where two n-tuples are adjacent if they differ in precisely one coordinate. Consider Q_3 shown in Figure 2. This graph is isomorphic to $C_4[P_2] - 2C_4$. By the previous results, we have

$$\mathrm{M}_1(Q_3) = 72,\ \mathrm{M}_2(Q_3) = 108, \mathrm{W}(Q_3) = 48.$$

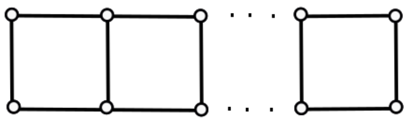

Figure 3: $L_n = P_n[P_2] - 2P_n$.

Example 3.3. Consider the ladder graph L_n shown in Figure 3. It is not difficult to check that L_n is isomorphic to $P_n[P_2] - 2P_n$. So, by Theorem 2.2,

$$\mathrm{M}_1(P_n[P_2] - 2P_n) = 2n + 2(4n - 6) + 8(n - 1) = 18n - 20.$$

Morover, by the previous results and this fact that $\mathrm{M}_2(P_2) = 1$ and $\mathrm{M}_2(P_n) = 4n - 8$, we have $\mathrm{M}_2(P_n[P_2] - 2P_n) = 4(n-1) + n + 4n - 6 + 2(n-1) + 2(4n - 6) + 2(4n - 8) = 27n - 40$.

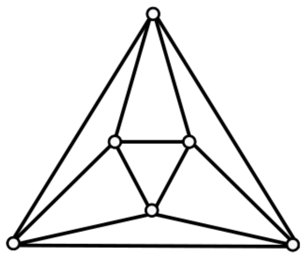

Figure 4: Octahedron graph Γ.

Example 3.4. Consider the octahedron graph Γ shown in Figure 4. This graph is isomorphic to $P_2[C_3] - 3P_2$. So, by Theorem 2.4,

$$\mathrm{W}(P_2[C_3] - 3P_2) = 18.$$

Also, by Theorem 2.6 we have

$$\begin{aligned} \mathrm{PI}_v(\Gamma) = \mathrm{PI}_v(P_2[C_3] - 3P_2) &= |V(P_2)|(M_1(C_3) - 6t_{C_3}) + 8|E(P_2)||E(C_3)| \\ &+ |V(C_3)|(|V(C_3)| - 1)(2|E(P_2)| - \mathrm{M}_1(P_2) + 12t_{P_2}) \quad (1) \\ &- 4|E(P_2)||E(C_3)|(|V(C_3)| - 1) \\ &+ |V(C_3)|^2(|V(C_3)| - 1)\mathrm{PI}_v(P_2). \end{aligned}$$

Then, by replacing $\mathrm{M}_1(C_3) = 12$, $\mathrm{M}_1(P_2) = 2$, $t_{C_3} = 1$ and $t_{P_2} = 0$ in relation (1), $\mathrm{PI}_v(\Gamma) = 48$.

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