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k-Intersection Graph of a Finite Set

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Abstract

For any non-empty set Ω and k-subset Λ, the k-intersection graph denoted by $\Gamma_m(\Omega, \Lambda)$ is undirected simple graph whose vertices are all m-subsets of Ω and two distinct vertices A and B are adjacent if and only if $A \cap B \nsubseteq \Lambda$. In this paper, we determine diameter, girth, some numerical invariants and planarity, hamiltonian and perfect matching of these graphs. Moreover adjacency matrix is considered at the end.

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1. Introduction

In [3], Csàkèany and Pollàk defined the intersection graph of nontrivial proper subgroups of groups. This study was continued with definition of intersection graph of nontrivial proper subsemigroups of semigroups by Bosák (see [2]). Then Zelinka investigated the intersection graph of subgroups of finite abelian groups [9]. Also, Charabarty et al. studied the intersection graph of ideals of rings [4]. In this paper, we are going to define the intersection graph on sets which is a generalization of the intersection graphs as mentioned above.

Let n, m, k be positive integers and let Ω be a non-empty set and Λ be a subset of Ω with $|\Omega| = n$ and $|\Lambda| = k$. The k-intersection graph $\Gamma_m(\Omega, \Lambda)$ is the undirected graph with vertex set consisting of all m-subsets of Ω and two vertices V and W are adjacent, whenever $V \cap W$ is not contained in Λ. The importance of this graph comes from various kinds of examples in many fields of

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science such as economics, social system modeling, financial matters and so on. For example, consider n people who want to invest for some companies. These people consist of k foreigners and $n-k$ natives. Each company has m investors. This situation can be modelled with a k-intersection graph with parameters n, m and k. Each company with m investors is considered as a vertex and two vertices are connected if the corresponding companies have at least one native investor in common. The independence number and click number of $\Gamma_m(\Omega, \Lambda)$ have important interpretations in this situation.

Our aim is to study k-intersection graph of finite sets and state some important graph theoretical properties of them. Sections 2 and 3 deal with basic results and numerical invariants of $\Gamma_m(\Omega, \Lambda)$. In section 4, we find some Hamiltonian cycles of $\Gamma_m(\Omega, \Lambda)$ and determine some conditions under which the graph has a perfect matching. In section 5 we determine adjacency matrix of the graph and attempt on the eigenvalues of this matrix. The final section is devoted to the investigation of planarity and 1-planarity of k-intersection graph. We note that all the notations and terminologies about the graphs are standard throughout this paper (for instance see [7]).

2. Some Basic Results

We start with definition of graph $\Gamma_m(\Omega, \Lambda)$ as the following.

Definition 2.1. For positive integers n, m, k, let Ω be a non-empty set and Λ be a subset of Ω with $|\Omega| = n$ and $|\Lambda| = k$. Then $\Gamma_m(\Omega, \Lambda)$ is a graph whose vertices are the m-subsets of Ω and two vertices V and W are adjacent whenever $V \cap W \nsubseteq \Lambda$.

If $m = 1$, then $\Gamma_m(\Omega, \Lambda)$ is an empty graph with $\binom{|\Omega|}{m}$ vertices and if $m = |\Omega|$, then $\Gamma_m(\Omega, \Lambda)$ is a single vertex. Hence, all over of this paper, we always assume that $m \neq 1$, $m \neq |\Omega|$ and also $|\Omega| \neq 1$. Hence, we must have $|\Omega| \geq 3$. For convenience, we let $|\Omega| = n$ and $|\Lambda| = k$.

First, let us consider the connectivity of this graph.

Proposition 2.2. The graph $\Gamma_m(\Omega, \Lambda)$ is connected if and only if $m > k$.

Proof. First suppose that $m > k$ and let V and W be two non-adjacent vertices, then there exists $v_1 \in V \setminus \Lambda$ and $w_1 \in W \setminus \Lambda$. Let U be a vertex containing $v_1$and v_2. Then $V \sim U \sim W$, which implies that $\Gamma_m(\Omega, \Lambda)$ is connected. The converse is obvious. □

Theorem 2.3. The graph $\Gamma_m(\Omega, \Lambda)$ contains

(i) $\binom{k}{m}$ isolated vertices,

(ii) $\binom{n-k}{m}$ vertices of the form $\{x_1, \dots, x_m\}$, where $x_i \in \Omega \setminus \Lambda$ $(1 \le i \le m)$ each of which has degree

$$\sum_{i=1}^{m} (-1)^{i+1} \binom{n-i}{m-i} \binom{m}{i} - 1.$$

(iii) $\binom{k}{l}\binom{n-k}{m-l}$ vertices of the form $\{x_1, \dots, x_l, x_{l+1}, \dots, x_m\}$, that $x_1, \dots, x_l \in \Lambda$, $x_{l+1}, \dots, x_m \in \Omega \setminus \Lambda$. The degrees of these vertices are

$$\sum_{i=1}^{m-l} (-1)^{i+1} \binom{n-i}{m-i} \binom{m-l}{i} - 1, \quad 1 \le l \le k.$$

Proof. Since any isolated vertex should contain elements which are all belong to Λ, we have $\binom{k}{m}$ such vertices. Now, assume $\{x_1, \dots, x_m\}$ is a vertex such that $x_i \in \Omega \setminus \Lambda$ for $1 \le i \le m$. Using the inclusion-exclusion principle, the degree of such a vertex equals

$$\binom{n-1}{m-1}\binom{m}{1} - \binom{n-2}{m-2}\binom{m}{2} + \cdots + (-1)^{m+1}\binom{n-m}{0}\binom{m}{m} - 1$$
$$= \sum_{i=1}^{m} (-1)^{i+1} \binom{n-i}{m-i} \binom{m}{i} - 1.$$

Now, suppose that $\{x_1, \dots, x_l, x_{l+1}, \dots, x_m\}$ is an arbitrary vertex of the graph, where $x_1, \dots, x_l \in \Lambda$ and $x_{l+1}, \dots, x_m \in \Omega \setminus \Lambda$ for some $1 \le l \le k$. Again, by using the inclusion-exclusion principle, the degree of such a vertex equals

$$\binom{n-1}{m-1}\binom{m-l}{1} - \binom{n-2}{m-2}\binom{m-l}{2} + \cdots + (-1)^{m-l+1}\binom{n-m+l}{l}\binom{m-l}{m-l} - 1$$
$$= \sum_{i=1}^{m-l} (-1)^{i+1} \binom{n-i}{m-i} \binom{m-l}{i} - 1.$$

□

The above theorem has some important consequences. For the graph $\Gamma_m(\Omega, \Lambda)$ we have

(i) $\Delta(\Gamma_m(\Omega, \Lambda)) = \sum_{i=1}^{m-1} (-1)^{i+1} \binom{n-i}{m-i} \binom{m}{i}$.

(ii) $\delta(\Gamma_m(\Omega, \Lambda)) = \sum_{i=1}^{m-k} (-1)^{i+1} \binom{n-i}{m-i} \binom{m-k}{i} - 1$ when $\Gamma_m(\Omega, \Lambda)$ is connected.

(iii) $|E(\Gamma_m(\Omega,\Lambda))|$ equals

$$\frac{1}{2}\binom{n-k}{m}\left(\sum_{i=1}^{m-1}(-1)^{i+1}\binom{n-i}{m-i}\binom{m}{i}\right) + \frac{1}{2}\sum_{l=1}^{k}\left[\binom{k}{l}\binom{n-k}{m-l}\left(\sum_{i=1}^{m-l}(-1)^{i+1}\binom{n-i}{m-i}\binom{m-l}{i}-1\right)\right].$$

(iv) $\Gamma_m(\Omega,\Lambda)$ is $\left(\sum_{i=1}^{m}\binom{n-i}{m-i}\binom{m}{i}(-1)^{i+1}-1\right)$-regular when $\Lambda=\emptyset$.

(v) $\Gamma_m(\Omega,\Lambda)$ is regular if and only if $\Gamma_m(\Omega,\Lambda)$ is complete or $\Lambda=\emptyset$.

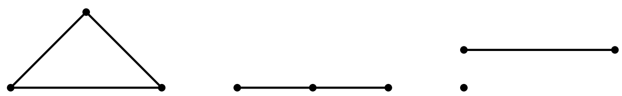

Figure 1: $\Gamma_m(\Omega,\Lambda)$ for $|\Omega|=3$, $|\Lambda|=0$; $|\Omega|=3$, $|\Lambda|=1$ and $|\Omega|=3$, $|\Lambda|=2$.

One can easily check that the graph $\Gamma_m(\Omega,\Lambda)$ is not empty (it has at least one edge) if and only if $\Omega\neq\Lambda$ and $2\leq m$. Moreover, it would be a cycle if and only if $|\Omega|=3$ and $|\Lambda|=0$. We note that if $|\Omega|=3$, then we have the following cases according to $|\Lambda|=0,1,2$ (see Figure 1). If $|\Omega|=4$, then one can easily see that either $\Gamma_m(\Omega,\Lambda)$ is disconnected or it has a vertex of degree ≥ 3. If $|\Omega|\geq 5$, then there are at least four vertices which are pairwise adjacent. As we have mentioned, $\Gamma_m(\Omega,\Lambda)$ is acyclic when $|\Omega|=3$ and $|\Lambda|=1$ or $|\Lambda|=2$. Also, $\Gamma_m(\Omega,\Lambda)\cong K_3$ when $|\Omega|=3$ and $|\Lambda|=0$, hence $\mathrm{girth}(\Gamma_m(\Omega,\Lambda))=3$ in this case. The following lemma helps us to show that $\mathrm{girth}(\Gamma_m(\Omega,\Lambda))=3$ whenever $|\Omega|\geq 4$.

Theorem 2.4. If $|\Omega|\geq 4$ and $\Omega\neq\Lambda$, then every edge of $\Gamma_m(\Omega,\Lambda)$ is contained in a clique of size $\binom{|\Omega|-1}{m-1}$.

Proof. Let V and W be two adjacent vertices. Then $V\cap W\nsubseteq\Lambda$ and there exists $x\in V\cap W$ such that $x\notin\Lambda$. We have $\binom{|\Omega|-1}{m-1}$ vertices which they are all adjacent. Hence, we get a clique of size $\binom{|\Omega|-1}{m-1}$. □

Thus, if $|\Omega|\geq 4$ and $\Omega\neq\Lambda$ then $girth(\Gamma_m(\Omega,\Lambda))=3$. Also the graph $\Gamma_m(\Omega,\Lambda)$ is a tree if and only if $|\Omega|=3$ and $|\Lambda|=1$.

Theorem 2.5. Let $\Gamma_m(\Omega,\Lambda)$ be connected. Then $\mathrm{diam}(\Gamma_m(\Omega,\Lambda))\leq 2$.

Proof. If $\Gamma_m(\Omega, \Lambda)$ is either a complete graph or an empty graph, then either $\mathrm{diam}(\Gamma_m(\Omega, \Lambda)) = 1$ or $\mathrm{diam}(\Gamma_m(\Omega, \Lambda)) = 0$. Hence, we may assume that V and W are two nonisolated and nonadjacent vertices of $\Gamma_m(\Omega, \Lambda)$. By the same method as in the proof of Proposition 2.2, we can obtain a vertex U such that $U \neq V$, $U \neq W$ and $V \sim U \sim W$ which imply that $\mathrm{diam}(\Gamma_m(\Omega, \Lambda)) = 2$. □

In the following theorem, we obtain some conditions under which the graph $\Gamma_m(\Omega, \Lambda)$ is complete.

Theorem 2.6. The graph $\Gamma_m(\Omega, \Lambda)$ is complete if and only if

$$2(m - |\Lambda|) > |\Omega| - |\Lambda|.$$

Proof. Suppose $\Gamma_m(\Omega, \Lambda)$ is complete. Let $\Omega = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ and $\Lambda = \{x_1, \ldots, x_k\}$. First note that those vertices in $\Gamma_m(\Omega, \Lambda)$ that contain all members of Λ have the smallest degree, such as $V = \{x_1, \ldots, x_k, v_1, \ldots, v_{m-k}\}$ and $W = \{x_1, \ldots, x_k, w_1, \ldots, w_{m-k}\}$. Since $\Gamma_m(\Omega, \Lambda)$ is a complete graph, we have $2(m - k) > |\Omega| - |\Lambda|$. Therefore $2(m - |\Lambda|) > |\Omega| - |\Lambda|$. The converse is obvious. □

3. Numerical Invariants of $\Gamma_m(\Omega, \Lambda)$ and Planarity

We start with the following result on the clique number of the graph $\Gamma_m(\Omega, \Lambda)$.

Theorem 3.1. Let $m > k$, we have

(i) If $m - k > \frac{n-k}{2}$, then $\omega(\Gamma_m(\Omega, \Lambda)) = \binom{n}{m}$.

(ii) If $\frac{n-2k}{2} < m - k \leq \frac{n-k}{2}$, then $\omega(\Gamma_m(\Omega, \Lambda)) \geq \binom{n-1}{m-1} + \sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$, where $r = 2m - n$.

(iii) If $\frac{n-2k}{2} \geq m - k$, then $\omega(\Gamma_m(\Omega, \Lambda)) \geq \binom{n-1}{m-1}$.

Proof. (i) Since $m - k > \frac{n-k}{2}$, we conclude that $\Gamma_m(\Omega, \Lambda)$ is a complete graph and hence $\omega(\Gamma_m(\Omega, \Lambda)) = \binom{n}{m}$. (ii) Let $\Omega = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ and $\Lambda = \{x_1, \ldots, x_k\}$. The number of vertices containing x_{k+1} equals $\binom{n-1}{m-1}$ each of which are adjacent. Let $r = 2m - n$ and for $0 \leq i \leq r - 1$, consider those vertices of $\Gamma_m(\Omega, \Lambda)$ containing i elements of Λ but do not contain x_{k+1}. The number of such vertices is equal to $\sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$ and they are adjacent to $\binom{n-1}{m-1}$ vertices containing x_{k+1}. Because, if for instance, for some $0 \leq i \leq r - 1$, $V = \{v_1, \ldots, v_i, v_{i+1}, \ldots, v_m\}$, where $v_1, \ldots, v_i \in \Lambda$ and $U = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_m\}$ and they are not adjacent. Then we have $|V \setminus \Lambda| + |U \setminus \Lambda| \leq |\Omega \setminus \Lambda|$ or equivalently $m - i + m - k \leq n - k$, so $r = 2m - n \leq i$ which is a contraction. Therefore $\Gamma_m(\Omega, \Lambda)$ has a clique of size $\binom{n-1}{m-1} + \sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$. (iii) Since $\frac{n-2k}{2} \geq m - k$, therefore there is no other vertex in $\Gamma_m(\Omega, \Lambda)$ that is adjacent to all member of the clique that contains all vertex containing x_{k+1}. Hence $\omega(\Gamma_m(\Omega, \Lambda)) \geq \binom{n-1}{m-1}$. □

The independence number of a graph G is the size of largest set of vertices of G that are pairwise non-adjacent and is shown by $\alpha(G)$.

Theorem 3.2. We have

(i) If $m > k$, then $\alpha(\Gamma_m(\Omega,\Lambda)) = \left[\frac{n-k}{m-k}\right]$.

(ii) If $m \leq k$, then $\alpha(\Gamma_m(\Omega,\Lambda)) = \binom{k}{m} + n - k$.

Proof. (i) It is enough to consider all vertices containing Λ such that their other $m-k$ elements are pairwise disjoint. This is an independent set consisting of $\left[\frac{n-k}{m-k}\right]$ vertices. (ii) It is enough to consider all isolated vertices along with all vertices with $m-1$ common elements of Λ. This is an independent set consisting of $\binom{k}{m} + n - k$ vertices. $\square$

Recall that for a graph G a *dominating set* is a set D of vertices of G, in which every vertex not in D is adjacent to a vertex in D. The size of the smallest dominating set is called *domination number* and is shown by $\gamma(G)$.

Theorem 3.3. Let $m > k$, we have

(i) If $n - k \geq m$, then $\gamma(\Gamma_m(\Omega,\Lambda)) = \lceil \frac{n-k}{m} \rceil$; where $\lceil \frac{n-k}{m} \rceil$ is ceiling function of $\frac{n-k}{m}$.

(ii) If $n - k < m$, then $\gamma(\Gamma_m(\Omega,\Lambda)) = 1$.

Proof. (i) Consider those vertices arising from a partition of $\Omega \setminus \Lambda$ into $\left[\frac{n-k}{m}\right]$ sets of size m along with a vertex containing the r remaining elements ($r < m$), then the result follows. (ii) It is clear, since $n - k < m$ and we may consider the vertex containing all elements of $\Omega \setminus \Lambda$. $\square$

Clearly, $\chi(\Gamma_m(\Omega,\Lambda)) \geq \binom{n-1}{m-1}$. Here we state some more results.

Theorem 3.4. Let $m > k$, if $\frac{n-2k}{2} < m - k \leq \frac{n-k}{2}$, then $\chi(\Gamma_m(\Omega,\Lambda)) = \binom{n-1}{m-1} + \sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$, where $r = 2m - n$.

Proof. Let $\Omega = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ and $\Lambda = \{x_1, \ldots, x_k\}$. By Theorem 3.1 part (ii), when $\frac{n-2k}{2} < m - k \leq \frac{n-k}{2}$, then there exists a clique of size $\binom{n-1}{m-1} + \sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$ in $\Gamma_m(\Omega,\Lambda)$ for $r = 2m - n$. Therefore $\chi(\Gamma_m(\Omega,\Lambda)) \geq \binom{n-1}{m-1} + \sum_{i=0}^{r-1} \binom{k}{i}\binom{n-k-1}{m-i}$. Let $V = \{v_1, \ldots, v_l, v_{l+1} \ldots, v_m\}$ be a vertex outside of the clique mentioned as above, where $v_1, \ldots, v_l \in \Lambda$. Thus $v_i \neq x_{k+1}$ and $l \geq r$, hence for each vertex V out of the mentioned clique there exists a vertex V' in the clique that V and V' are not adjacent. It is enough to take $V' = \{x_1, \ldots, x_k, x_{k+1}, v'_1, \ldots, v'_{m-k-1}\}$ so that $\{v_{l+1}, \ldots, v_m\} \cap \{x_{k+1}, v'_1, \ldots, v'_{m-k-1}\} = \emptyset$ (it is possible, since $l \geq r$). Therefore V can have the colour of V'. Note that since for any two vertices V and W, vertices V' and W' inside the clique are distinct, so there is no problem in the case that V and W are adjacent. $\square$

It is worth noting that when $m-k > \frac{n-k}{2}$, then graph $\Gamma_m(\Omega,\Lambda)$ is complete and hence its chromatic number equals the number of its vertices.

Theorem 3.5. Let $\frac{n-k}{2} = m-k$, then $\chi(\Gamma_m(\Omega,\Lambda)) = \binom{n-1}{m-1}$.

Proof. Since $\frac{n-k}{2} = m-k$, so $n = 2m$ and we have $\binom{n}{m-1} = \binom{n-1}{m-1}$. Therefore the number of vertices out of the clique containing x_{k+1} is equal to the number of vertices in the clique, hence $\chi(\Gamma_m(\Omega,\Lambda)) = \binom{n-1}{m-1}$. $\square$

A graph is called planar if it can be drawn without crossing edges. Also, a graph is 1-planar if it can be drawn on the plan so that each edge is crossed by no more than one other edge. In this section, we discuss conditions under which the graph $\Gamma_m(\Omega,\Lambda)$ is planar and then we investigate 1-planarity of $\Gamma_m(\Omega,\Lambda)$ (see [6] and [8] for more details).

Theorem 3.6. The graph $\Gamma_m(\Omega,\Lambda)$ is planar if and only if one of the following conditions hold:

(i) $|\Omega| = 3$,

(ii) $|\Omega| = 4$,

(iii) $|\Omega| = 5$, $m = 2$ and $|\Lambda| = 1, 2, 3$ or 4,

(iv) $|\Omega| = 5$, $m = 4$ and $|\Lambda| = 3$ or 4.

Proof. Let $x_i \in \Omega \setminus \Lambda$. Then number of vertices of $\Gamma_m(\Omega,\Lambda)$ containing x_i equals $\binom{|\Omega|-1}{m-1}$. If $|\Omega| \geq 6$, then

$$\binom{|\Omega|-1}{m-1} \geq 5.$$

Hence, there exist 5 vertices containing x_i, which induces a graph isomorphic to K_5 so that $\Gamma_m(\Omega,\Lambda)$ is not planar in this case.

If $|\Omega| = 3$, the number of vertices of $\Gamma_m(\Omega,\Lambda)$ is 3 and it is a planar graph. If $|\Omega| = 4$ and $m = 3$, then the number of vertices of $\Gamma_m(\Omega,\Lambda)$ is 4 and it is planar. Also, if $m = 2$ and $|\Lambda| \neq 0$, then $\Gamma_m(\Omega,\Lambda)$ has 6 vertices two of which have degree 2. Hence, $\Gamma_m(\Omega,\Lambda)$ has no subgraphs isomorphic to $K_{3,3}$ or K_5, that is, $\Gamma_m(\Omega,\Lambda)$ is planar. If $m = 2$ and $|\Lambda| = 0$, then $\Gamma_m(\Omega,\Lambda)$ has no subgraphs isomorphic to $K_{3,3}$ or K_5 so that it is planar in this case. Now, consider the case $|\Omega| = 5$. If $m = 2$ and $|\Lambda| = 0$, then $\Gamma_m(\Omega,\Lambda)$ is not planar for the number of its vertices and edges equal 10 and 30, respectively, and $30 \nleq 3 \times 10 - 6$. So, $\Gamma_m(\Omega,\Lambda)$ is not planar. If $m = 2$ and $|\Lambda| \neq 0$, then $\Gamma_m(\Omega,\Lambda)$ is planar for it does not have any subgraph isomorphic to K_5 or $K_{3,3}$. If $m = 3$, the graph $\Gamma_m(\Omega,\Lambda)$ has a subgraph isomorphic to K_5 so that it is not planar. If $m = 4$, then $\Gamma_m(\Omega,\Lambda)$ is isomorphic to K_5 when $|\Lambda| = 0, 1$ or 2, that is, $\Gamma_m(\Omega,\Lambda)$ is not planar. If $|\Lambda| = 3$ or 4, then we have a vertex of degree 3, which implies that $\Gamma_m(\Omega,\Lambda)$ is planar. $\square$

Now, we investigate 1-planarity of the graph $\Gamma_m(\Omega, \Lambda)$.

Theorem 3.7. The graph $\Gamma_m(\Omega, \Lambda)$ is 1-planar if and only if one of the following conditions hold:

(i) $|\Omega| = 5$, $|\Lambda| = 3, 4$ and $m = 3$,

(ii) $|\Omega| = 5$, $|\Lambda| = 0, 1, 2$ and $m = 4$,

(iii) $|\Omega| = 6$, $|\Lambda| = 2, 3, 4, 5$ and $m = 2$,

(iv) $|\Omega| = 6$ and $m = 5$,

(v) $|\Omega| = 7$, $|\Lambda| = 3, 4, 5, 6$ and $m = 2$,

(vi) $|\Omega| = 7$, $|\Lambda| = 6$ and $m = 6$.

Proof. The method of the proof is very similar to the proof of Theorem 3.6 and we omit hear. But one has to consider all cases (i) to (vi) and make sure that there is no subgraph isomorphic to K_7, $K_{4,5}$, $K_{3,7}$ and other forbidden subgraphs (see [6], [8]). □

4. Hamiltonian and Perfect Matching

In this section, we determine conditions such that the graphs $\Gamma_m(\Omega, \Lambda)$ have Hamiltonian cycles or matching. If $|\Omega| = 3$ and $|\Lambda| = 0$ one can easily see that $\Gamma_m(\Omega, \Lambda)$ is Hamiltonian. Now in the following theorem, we show that $\Gamma_m(\Omega, \Lambda)$ is Hamiltonian whenever $|\Omega| \geq 4$.

Theorem 4.1. If $|\Omega| \geq 4$ and $m > k$, then $\Gamma_m(\Omega, \Lambda)$ is Hamiltonian.

Proof. Let $\Omega = \{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ and $\Lambda = \{x_1, \dots, x_k\}$. Consider the most difficult case where $m = k + 1$. Since $m > k$, the graph has no isolated vertex. Partition the vertices of the graph into $n - k$ sets A_i in such a way that A_i is the set of all vertices containing x_{k+i} and excluding $x_{k+i+1}, \dots, x_n$, for $i = 1, \dots, n - k$. Now begin with A_{k+1} with a single vertex. There is a vertex of A_{k+1} adjacent to some vertex of A_{k+2}. Clearly, all vertices of A_{k+2} are adjacent. Also, there exists a vertex of A_{k+2} adjacent to some vertex of A_{k+3}. Again, all vertices of A_{k+3} are adjacent. Continuing this way, we observe that all vertices of A_n are adjacent and there is a vertex of A_n which is adjacent to a vertex of A_1. □

For a graph G a *matching* is a set of edges of G such that no two edges have a vertex common. A matching is called *perfect matching* if all vertices of G are an end point of one of the edges in the matching. A matching that just leaves a single vertex unmatched is called *near perfect matching.*

Theorem 4.2. Suppose $\Gamma_m(\Omega, \Lambda)$ has no isolated vertex, if the number of vertices of $\Gamma_m(\Omega, \Lambda)$ is odd, then $\Gamma_m(\Omega, \Lambda)$ has a near perfect matching, and if the number of vertices of $\Gamma_m(\Omega, \Lambda)$ is even then $\Gamma_m(\Omega, \Lambda)$ has a perfect matching.

Proof. Let $\Omega = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ and $\Lambda = \{x_1, \ldots, x_k\}$. Since the graph has no isolated vertex, we have $m > k$. Partition the vertices of the graph into $n-k$ sets $A_i, k+1 \leq i \leq n$, so that A_i is the set of all vertices including x_i and excluding $x_{i+1}, \cdots, x_n$. Consider A_e with even members. Since the induced sub-graph of $\Gamma_m(\Omega, \Lambda)$ with A_e as the set of vertices, is a clique with even vertices, so A_e has a perfect matching. Now if $\Gamma_m(\Omega, \Lambda)$ has odd number of vertices, so there are odd numbers of A_is with odd members, let say $A_{i_1}, \cdots, A_{i_o}; |A_{i_1}| \leq \cdots \leq |A_{i_o}|$. Begin with A_{i_1}. Since the induced sub-graph with A_{i_1} as the set of vertices is a clique with odd number of vertices so there exists a matching M_1 that matches all vertices of A_{i_1} except one vertex say $V = \{v_1, \cdots, v_{m-1}, x_{i_1}\}$. There exists a vertex in A_{i_2} that is adjacent to V, say $V' = \{v_2, \cdots, v_{m-1}, x_{i_1}, x_{i_2}\}$. There is a matching M_2 for A_{i_2} that matches all vertices of A_{i_2} except V'. $M_1 \cup M_2$ with the edge that joins V and V' is a perfect matching for $A_{i_1} \cup A_{i_2}$. Similarly one can consider the remain A_is tow by tow. So there exists a perfect matching for all A_is except for the last one, namely A_{i_o} that has a near perfect matching. Therefore the union of these matchings is a near perfect matching for $\Gamma_m(\Omega, \Lambda)$.

By a similar argument, one can show that in the case that the graph has even number of vertices, the graph has perfect matching. □

Theorem 4.3. If $|\Lambda| = 1$, then the graph $\Gamma_2(\Omega, \Lambda)$ has a perfect matching if and only if $|\Omega| = 4s + 1$ or $|\Omega| = 4s$.

Proof. First suppose that $\Gamma_2(\Omega, \Lambda)$ has a perfect matching. Then the number of vertices of $\Gamma_2(\Omega, \Lambda)$ is even, which implies that $\frac{n(n-1)}{2}$ is even, in which $n = |\Omega|$. Since n and $n-1$ are consecutive numbers, it follows that one of them is odd and the other is even. Hence, we have the following cases:

Case 1. n is even. Then $n = 2r$ for some $r \in \mathbb{N}$ and so

$$\binom{n}{2} = \frac{2r(2r-1)}{2} = r(2r-1).$$

Since $\binom{n}{2}$ is even it follows that r is even so that $r = 2s$ for some $s \in \mathbb{N}$. Hence, $|\Omega| = n = 4s$.

Case 2. n is odd. Then $n = 2r - 1$ for some $r \in \mathbb{N}$ and so

$$\binom{n}{2} = \frac{(2r+1)2r}{2} = r(2r+1).$$

Since $\binom{n}{2}$ is even, it follows that r is even so that $r = 2s$ for some $s \in \mathbb{N}$. Hence, $|\Omega| = n = 4s + 1$.

Clearly, we must have $|\Lambda| = 1$, if not $|\Lambda| > 1$ and $\Gamma_2(\Omega, \Lambda)$ has isolated vertices contradicting the fact that $\Gamma_2(\Omega, \Lambda)$ has a perfect matching.

Conversely, since $|\Omega| = 4s$ or $4s+1$, the number of vertices of $\Gamma_2(\Omega, \Lambda)$ is even, from which by the previous theorem, $\Gamma_2(\Omega, \Lambda)$ has a perfect matching. □

Corollary 4.4. If $n = 2^r$, $r > 1$ and $m > k$, then $\Gamma_m(\Omega, \Lambda)$ has a perfect matching.

Proof. It is enough to show that for $0 < m < 2^r$, $\binom{2^r}{m}$ is even. We have

$$\begin{aligned}\binom{2^r}{m} &= \frac{2^r!}{m!(r^2-m)!} = \frac{2^r(2^r-1)\cdots(2^r-(m-1))(2^r-m)!}{1\times 2\times\cdots\times m\times(2^r-m)!}\\ &= \frac{2^r(2^r-1)\cdots(2^r-(m-1))}{1\times 2\times\cdots\times m}.\end{aligned}$$

Each even factor like e in the denominator (except when m if even) can simplify with $2^r - e$ in numerator. If m is even, say $m = 2^s m_1$, then since $m < 2^r$ and so $s < r$. Therefore m will simplify by even factor of 2^r. Since $s < r$ there is still an even factor in numerator, so $\binom{2^r}{m}$ is even and hence, $\Gamma_m(\Omega, \Lambda)$ has a perfect matching. □

By a similar proof, one can show that for $n = 2^r + 1$ and $r \in \mathbb{N}$ we have $\binom{n}{m}$ is even when $m \neq 1$ and $m \neq 2^r$. Hence, we have the following result.

Corollary 4.5. If $n = 2^r+1$ and $m > k$, when $m \neq 1$ and $m \neq 2^r$, then $\Gamma_m(\Omega, \Lambda)$ has a perfect matching.

Corollary 4.6. If $n = 2^r - 1$, then $\Gamma_m(\Omega, \Lambda)$ has no perfect matchings.

Proof. We have:

$$\begin{aligned}\binom{2^r-1}{m} &= \frac{(2^r-1)!}{m!(2^r-1-m)!}\\ &= \frac{(2^r-1)(2^r-2)(2^r-3)\cdots(2^r-1-(m-1))(2^r-1-m)!}{m!(2^r-1-m)!}\\ &= \frac{(2^r-1)(2^r-2)(2^r-3)\cdots(2^r-m)}{1\times 2\times\cdots\times m}.\end{aligned}$$

Since for each $2 \leq a \leq m$, all even factors in $(2^r - a)$ in the numerator will simplify with even factors of a in denominator. So $\binom{2^r-1}{m}$ in odd. □

For the last assertion in this section, we remind that by *Lucas Theorem* [5] in the number theory, for each nonnegative integers n, m and prime p we have,

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} (mod\ p),$$

where $n = n_k p^k + \cdots + n_1 p + n_0$ and $m = m_k p^k + \cdots + m_1 p + m_0$. Since $\binom{n}{m} = 0$ when $n < m$, one can easily see that $\binom{2^r+2^s}{m}$ is even when $m \neq 2^s$ and $m \neq 2^r$ and so we can state the following corollary.

Corollary 4.7. If $n = 2^r + 2^s$, $m > k$, $m \neq 2^s$ and $m \neq 2^r$, then $\binom{2^r+2^s}{m}$ is even and $\Gamma_m(\Omega, \Lambda)$ has a perfect matching.

5. Adjacency Matrix

For a graph G with n vertices, the adjacency matrix denoted by $A(G)$ is an $n \times n$ matrix whose rows and columns are indexed by $V(G)$. For $i \neq j$, $a_{ij} = 1$ iff v_i and v_j are adjacent and $a_{ij} = 0$ iff v_i and v_j are nonadjacent. The adjacency matrix of each graph is a symmetric matrix with 0 on it's main diagonal. In this section we state some facts on adjacency matrix and eigenvalues of $\Gamma_m(\Omega, \Lambda)$. Here $A(\Gamma_m(\Omega, \Lambda))$ is a square matrix of order $\binom{n}{m}$.

Proposition 5.1. For non empty sets Ω', Λ' that $\Lambda' \subseteq \Omega'$ and $|\Omega'| \leq |\Omega|$ and $|\Lambda'| = |\Lambda|$, then $\Gamma_m(\Omega', \Lambda')$ is an induced subgraph of $\Gamma_m(\Omega, \Lambda)$.

Proof. The assertion is clear since the adjacency matrix of the graph $\Gamma_m(\Omega', \Lambda')$ is a submatrix of the adjacency matrix of $\Gamma_m(\Omega, \Lambda)$. $\square$

As a consequence of 5.1 one can easily see that the complement of all Kneser graphs $K_{n-k,r}$ that $m - k \leq r \leq m$, or equivalently all Johnson graphs with parameters $J(n-k, r)$ with $m - k \leq r \leq m$ are induced subgraph of $\Gamma_m(\Omega, \Lambda)$.

When $m \leq k$, then $\Gamma_m(\Omega, \Lambda)$ has at least one single vertex and therefore $Det(A(\Gamma_m(\Omega, \Lambda))) = 0$. Note that $Det(B)$ is stand for the determinant of matrix B.

Theorem 5.2. If $m - k = \dfrac{n-k}{2}$, then $Det(A(\Gamma_m(\Omega, \Lambda))) = 0$.

Proof. Let $\Omega = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_m, x_{m+1}, \ldots, x_n\}$ and $\Lambda = \{x_1, \ldots, x_k\}$. Consider two vertices $V = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_m\}$ and $W = \{x_1, \ldots, x_k, x_{m+1}, \ldots, x_n\}$.

Now let $U = \{y_1, \ldots, y_r, y_{r+1}, \ldots, y_m\}$ be an arbitrary vertex of $\Gamma_m(\Omega, \Lambda)$ other than V and W, that $\{y_1, \ldots, y_r\} \in \Lambda$ and $\{y_{r+1}, \ldots, y_m\} \in \Omega \setminus \Lambda$. Then one can see that $\{y_{r+1}, \ldots, y_m\} \cap \{x_{k+1}, \ldots, x_m\} \neq \emptyset$ since $r \leq k$. Similarly $\{y_{r+1}, \ldots, y_m\} \cap \{x_{m+1}, \ldots, x_n\} \neq \emptyset$ and so U is adjacent to both V and W. Moreover, V and W are not adjacent and so the two rows of $A(\Gamma_m(\Omega, \Lambda))$ indexed by V and W are the same. Hence $Det(A(\Gamma_m(\Omega, \Lambda))) = 0$. $\square$

Hear we have some statements about eigenvalues of $\Gamma_m(\Omega, \Lambda)$.

Theorem 5.3. -1 is an eigenvalue of $A(\Gamma_m(\Omega,\Lambda))$ with multiplicity $\sum_{i=1}^{k}(\binom{k}{i}-1)\binom{n-k}{m-i}$.

Proof. Let $\Omega=\{x_1,\dots,x_k,x_{k+1},\dots,x_n\}$ and $\Lambda=\{x_1,\dots,x_k\}$. Consider two vertices V and W so that $V\cap\Omega\setminus\Lambda=W\cap\Omega\setminus\Lambda$. Then for each vertex $U\in V(\Gamma_m(\Omega,\Lambda))$, U is adjacent to V iff U is adjacent to W. Since V and W are adjacent, the two rows assigned to V and W in the adjacency matrix are the same except entries $a_{V,V},a_{V,W}$ and $a_{W,V},a_{W,W}$. Therefore the two corresponding rows in matrix $A+I$ are the same and therefore $Det(A+I)=0$. So -1 is an eigenvalue of $A(\Gamma_m(\Omega,\Lambda))$. Now, let U be a vertex of $\Gamma_m(\Omega,\Lambda)$ such that $|U\cap\Lambda|=r$. Then there are $\binom{k}{r}$ vertices of $\Gamma_m(\Omega,\Lambda)$ that have the same behavior as U. Thus the multiplicity of -1 is $\sum_{i=1}^{k}(\binom{k}{i}-1)\binom{n-k}{m-i}$. □

Theorem 5.4. Let $\lambda_1\geq\lambda_2\ldots\geq\lambda_{\binom{n}{m}}$ be all eigenvalues of $\Gamma_m(\Omega,\Lambda)$. Then

$$\lambda_1\geq\binom{n-1}{m-1}-1.$$

.

Proof. For any induced subgraph H of $\Gamma_m(\Omega,\Lambda)$, the adjacency matrix of H is a principal submatrix of $A(\Gamma_m(\Omega,\Lambda))$. If $\gamma_1\geq\ldots\geq\gamma_p$ are all eigenvalues of H, then by Cauchy interlacing theorem (see page 7 of [1]), we have $\lambda_1\geq\gamma_1\geq\ldots\geq\gamma_p\geq\lambda_{\binom{n}{m}}$. By Theorem 3.1, $\Gamma_m(\Omega,\Lambda)$ has a clique of size $\binom{n-1}{m-1}$. Since the largest eigenvalue of $K_{\binom{n-1}{m-1}}$ is $\binom{n-1}{m-1}-1$, we have $\lambda_1\geq\binom{n-1}{m-1}-1$ as required. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. B. Bapat, *Graphs and Matrices*, Springer-Verlag, London, 2010.

[2] J. Bosák, The graphs of semigroups, *Proc. Symposium Smolenice, Prague* (1964) 119–125.

[3] B. Csákány and G. Pollák, The graph of subgroups of a finite group, *Czechoslovak Math. J.* **19** (1969) 241–247.

[4] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of ring, *Discrete Math.* **309** (2009) 5381–5392.

[5] L. E. Dickson, *History of the Theory of Numbers*, Chelsea, 1952.

[6] I. Fabrici and T. Madaras, The structure of 1-planar graphs, *Discrete Math.* **307** (2007) 854–865.

[7] C. Godsil and G. F. Royle, *Algebraic Graph Theory*, Springer, 2001.

[8] V. P. Korzhik, Minimal non-1-planar graphs, *Discrete Math.* **308** (2008) 1319–1327.

[9] B. Zelinka, Intersection graphs of finite abelian groups, *Czechoslovak Math. J.* **25**(100) (1975) 171–174.

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