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On $L(d, 1)$-labelling of Trees

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Abstract

Given a graph G and a positive integer d, an $L(d, 1)$-labelling of G is a function f that assigns to each vertex of G a non-negative integer such that if two vertices u and v are adjacent, then $|f(u) - f(v)| \geq d$ and if u and v are at distance two, then $|f(u) - f(v)| \geq 1$. The $L(d, 1)$-number of G, $\lambda_d(G)$, is the minimum m such that there is an $L(d, 1)$-labelling of G with $f(V) \subseteq \{0, 1, \ldots, m\}$. A tree T is of type 1 if $\lambda_d(T) = \Delta + d - 1$ and is of type 2 if $\lambda_d(T) \geq \Delta + d$. This paper provides sufficient conditions for $\lambda_d(T) = \Delta + d - 1$ generalizing the results of Wang [11] and Zhai, Lu, and Shu [12] for $L(2, 1)$-labelling.

Keywords: $L(d, 1)$-labelling, Tree, Distance, Δ-vertex.

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1. Introduction

For given positive integers $h \geq k$, an $L(h, k)$-labelling of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq h$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ when $d_G(u, v) = 2$. It has been shown that $\lambda_{h,k}(G) \geq h + (\Delta - 1)k$ for any graph G, where Δ is the maximum degree of G. The graphs achieving this bound are called $\lambda_{h,k}$-minimal. For convenience, we usually write λ_d for $\lambda_{d,1}$. In [3], it was also proved that if T is a tree, then $\Delta + d - 1 \leq \lambda_d(T) \leq \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$. The upper bound has been improved in [6]. In [9], alternative upper and lower bounds for $\lambda_{h,k}(T)$ are provided by introducing a new

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relevant parameter, $\mathcal{M}(T)$, the maximum ordering-degree. For further results and references we refer to the survey [1] and to more recent publications, c.f. [13].

Trees with minimal $\lambda_{h,k}$ are said to be of type 1, and all other trees are of type 2. For a decade, characterization of type 1 and type 2 trees has been an open problem [7], until a characterization for the general $L(h,k)$-labelling was provided by Chang and Lu [4] who showed that any type 1 tree must be a subtree of certain constructed (usually) infinite tree $\mathcal{T}_\Delta$ which is defined using the so called Δ-sequences. Chang and Lu [4] also provided a polynomial algorithm for computing λ_d thus deciding whether a tree is of type 1. The approach has been further generalized in [8], resulting in a classification of c class trees, i.e. trees with $\lambda_d(T) = \Delta + d + c - 2$ ($c = 1, 2, \ldots, \min\{d, \Delta\}$), by proving that these trees are subtrees of certain structures. However, as observed by Jonck et al. [8], the classifications still do not tell us nicely how these trees "look like" and therefore, they use their techniques to find a few necessary conditions for a tree to be $\lambda_{2,1}$-minimal. In this paper, we provide some new sufficient conditions that may shed some light to the structure of type 1 trees from other perspective. We focus on $L(d,1)$-labelling of trees. Our theorems are extending the known results for $L(2,1)$-labeling from [11, 12].

The rest of the paper is organized as follows. In the next section we provide the basic definitions and some preliminary observations that are needed for the outline of our results that follow in Section 3. The proofs of the theorems given in Section 3 are provided in Section 4 (case $d \leq \Delta - 1$, Theorem 1) and in Section 5 (case $d \geq \Delta$, Theorems 2 and 3).

2. Preliminaries

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. A *uv-path* is a path between vertices u and v. The *distance* $d_G(u,v)$, or briefly $d(u,v)$, between vertices u and v is the length of a shortest uv-path in G. For a vertex $v \in V(G)$, $N_G^k(v) = \{u \mid u \in V(G)$ and $d_G(u,v) = k\}$ denote the vertices at distance k from v. As usual, closed neighborhoods are denoted by $N_G^k[v] = \{u \mid u \in V(G)$ and $d_G(u,v) \leq k\}$. So, for example $N_G^1[v] = N_G^1(v) \cup \{v\}$. Furthermore, $d_G(v)$ stands for the degree of v in G. A vertex of degree k is called a *k-vertex*.

Trees are connected graphs without cycles. In a tree, a *leaf* is a 1-vertex. For convenience, where no confusion is possible, we will write $\Delta, \lambda_d, N^k(v), N^k[v], d(v)$ and $d(u,v)$ for $\Delta(T), \lambda_d(T), N_T^k(v), N_T^k[v], d_T(v)$ and $d_T(u,v)$, respectively. A *major handle* is a Δ-vertex adjacent to exactly one vertex of degree greater than one. A *weak major handle* is a Δ-vertex adjacent to exactly two vertices of degree greater than one. A subtree T_1 of T is called a Δ-*subtree* if $\Delta(T_1) = \Delta(T)$.

A *star* S_Δ, or briefly S, with center x is a tree that consists of a Δ-vertex x and Δ leaves. A *proper double star* dS_Δ, or briefly dS, with center $\{x, y\}$ is a tree with exactly two adjacent Δ-vertices (x and y) and all other vertices are leaves.

Note [4] that a proper double star can be obtained by identifying a leaf of one star with the center of another star. A *double star* is a tree which is either a star or a proper double star. Let S_Δ be a star with center x. Join at most $\Delta - 2$ leaves to each $u \in N^1(x)$. The resulting tree we call a *generalized star* gS_Δ (briefly gS) with center x. Similarly, a *generalized double star* gdS_Δ (briefly gdS) with center $\{x, y\}$ is defined as follows: join at most $\Delta - 2$ leaves to each $u \in N^1(x) \setminus \{y\}$ and at most $\Delta - 2$ leaves to each $u \in N^1(y) \setminus \{x\}$. We will consider the generalized stars and generalized double stars as induced subgraphs of a tree. Examples of a generalized star and a generalized double star are shown in Figure 1(a,b). In both examples, the number of leaves drawn is maximal.

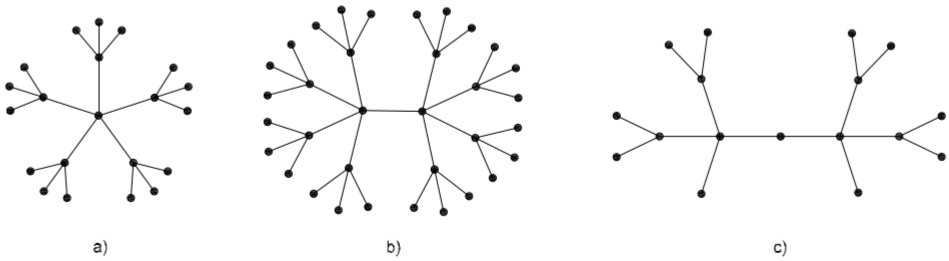

Figure 1: (**a**) A generalized star. (**b**) A generalized double star. (**c**) A tree with the path between Δ-vertices of length 2.

In particular, we will be interested in distances among the Δ-vertices and will denote $D_\Delta(T) = \{d_T(u, v) \mid u, v \text{ are two distinct } \Delta\text{-vertices}\}$, briefly D_Δ.

Let T be a tree with $\Delta \geq 4$ and $2 \notin D_\Delta$. Let x and y be adjacent Δ-vertices. The subtree of T induced on vertices $N^2[x] \cup N^2(y)$ is isomorphic to the gdS with $\{x, y\}$ as its center. If x is not adjacent to any Δ-vertex, then the subtree of T induced on vertices $N^2[x]$ is isomorphic to the gS with center x.

Furthermore, we will need the notion of distance between generalized double stars which is defined as the distance between their closest centers, i.e. the distance between the nearest Δ-vertices, each of them belonging to one of the two generalized double stars. If the path between such Δ-vertices does not contain a Δ-vertex, then the generalized double stars are *adjacent*. (Recall that in a tree, any path between a pair of vertices is also the unique shortest path between them.)

We conclude the section with several easily proven facts. For completeness, we provide brief arguments. First note that it is easy to construct a $L(1, 1)$-labelling of a tree with $\Delta + 1$ labels $\{0, 1, \ldots, \Delta\}$. Choose any vertex as a root and proceed in, say, the breadth first search order. At each vertex v, there is only one label used for its father u, and one label for the grandfather of v. There are at most $\Delta - 2$ other neighbors of u, and hence all sons of u can be properly labelled (with distinct labels). Thus we have

Fact 2.1. *Let T be a tree with $\Delta \geq 2$. There is a $L(1,1)$-labelling of T with $\Delta+1$ labels, so T is of type 1.*

Similarly, it is also easy to construct a $L(0,1)$-labelling of a tree with Δ labels $\{0, 1, \ldots, \Delta-1\}$, just using the fact that, when needed, one of the sons may use the label of the father.

Fact 2.2. *Let T be a tree with $\Delta \geq 2$. There is a $L(0,1)$-labelling of T with Δ labels, so T is of type 1.*

Therefore, the case $d \in \{0,1\}$ is trivial in our context, and we may restrict our attention only to labelling trees with $d > 1$.

We conclude the section with two examples, showing that in a type 1 tree (1) there may be some restrictions of degrees for vertices that are not of maximum degree, and also (2) the vertices of maximum degree cannot be at certain distances.

Before writing these two examples, let us recall the following lemma which appeared in [3].

Lemma 2.3. [3] If G is a graph of maximum degree $\Delta \geq 1$, then $\lambda_d(G) \geq \Delta+d-1$. Moreover, if $\lambda_d(G) = \Delta + d - 1$ and $d \geq 2$, then $f(x) = 0$ or $f(x) = \Delta + d - 1$ for any λ_d-labelling of G and any Δ-vertex x; consequently, it is impossible to have a set of three Δ-vertices such that any two of them are of distance at most two apart.

For convenience, we denote $\mathcal{B} = \{0, 1, 2, \ldots, \Delta+d-1\}$ and use this set of labels in $(\Delta + d - 1)$-labellings unless stated explicitly otherwise.

Example 2.4. Consider the generalized star of degree 5 from Figure 1a) and let $d = 3$. By Lemma 2.3, in a type 1 tree, in any $L(d,1)$-labelling with $\lambda_d(T) + 1$ labels, the Δ-vertices must be assigned either label 0 or $d + \Delta - 1$. If we assign the label 0 to the center, then the neighbors will be assigned labels $3, 4, 5, 6, 7$. However, there are not enough labels for labelling all the leaves. In particular, consider the leaves adjacent to the vertex just labelled with 3, and observe that we can use only 6 and 7, thus we can not label all its three leaves. Analogous reasoning applies when using $d + \Delta - 1$ at the central vertex. We conclude that a generalized star is of type 1 only subject to some conditions on the number (and distribution) of leaves.

Example 2.5. Consider the graph on Figure 1c). Maximum degree is $\Delta = 4$ and there are two Δ-vertices, at distance two. Let $d = 4$. Clearly, the two Δ-vertices may not receive the same label as they are at distance two, and, on the other hand they can not get 0 and 7, because then their common neighbor can not be properly labelled. Thus, the tree is not of type 1 in this case.

Consideration of similar examples in which the path between the Δ-vertices is longer shows that the labelings can be constructed when the distance between the Δ-vertices is at least five. We conclude that certain conditions may apply on allowed distances among the Δ vertices in type 1 trees.

3. Our Results

Before stating the main results of the paper, let us recall briefly the previously known results for $L(2,1)$-labelling. Wang [11] has shown that $\lambda_2(T) = \Delta + 1$ for a tree T with $\Delta \geq 3$ and $1, 2, 4 \notin D_\Delta$. An improved condition was provided in [12]. Zhai, Lu, and Shu have shown that $\lambda_2(T) = \Delta + 1$ for (1) trees with $\Delta \geq 5$ and $2, 4 \notin D_\Delta$ and (2) trees with $\Delta = 3, 4$ and $2, 4 \notin D_\Delta$ with some additional conditions. They also construct trees with $\Delta = 3, 4$ and $2, 4 \notin D_\Delta$ that are not of type 1.

It appears that a generalization of these results to $L(d,1)$-labelling is far from trivial. In this communication, we give sufficient conditions for a tree T to be of type 1. We prove the next theorems:

Theorem 3.1. Let T be a tree for which $2, 4 \notin D_\Delta$ and $d(x) \leq \Delta - d + 1$ if x is not a Δ-vertex of T.

1. If $\Delta \geq 5$ and $d \leq \Delta - 2$, then $\lambda_d = \Delta + d - 1$.
2. If $\Delta \geq 4$ and $d = \Delta - 1$, then $\lambda_d = \Delta + d - 1$.

Theorem 3.2. Let T be a tree with $\Delta \geq 4$ and $d \geq \Delta$. Let each induced Δ-subtree of T that is isomorphic to gdS, be of type 1 and let each pair of such subtrees be at distance more than 7. Let $d(x) \leq 2$ when x is not a vertex of gdS. Then $\lambda_d = \Delta + d - 1$.

Theorem 3.3. Let T be a tree with $\Delta \geq 5$ and $d \geq \Delta$. If $2, 3 \notin D_\Delta$, $d(x) \leq 2$ when x is not a Δ-vertex and each Δ-vertex has at least one leaf, then $\lambda_d = \Delta + d - 1$.

4. Trees with $d \leq \Delta - 1$

Zhai, Lu, and Shu [12] improved the result of Wang [11] about the $L(2,1)$-labelling of trees. As a partial result in their work the next Lemma was given which will also be used in the study of $L(d,1)$-labelling.

Lemma 4.1. [12] Let T be a tree with $\Delta \geq 4$ and $2, 4 \notin D_\Delta$. If T is not a double star, then T contains one of the following configurations:
(C1) A leaf v adjacent to a vertex u with $d(u) < \Delta$.
(C2) A path $x_1x_2x_3x_4$ such that $d(x_2) = d(x_3) = 2$ and x_1 is a major handle.
(C3) A path $x_1x_2x_3x_4x_5$ such that $d(x_2) = d(x_4) = 2, d(x_3) = 3$, x_1 and another neighbor y of x_3 are major handles.
(C4) A path $x_1x_2x_3x_4x_5$ such that $d(x_3) = d(x_4) = 2$, x_1 is a major handle and x_2 is a weak major handle.

Proof of the next Lemma is analogous to the proof of Lemma 10 from [12].

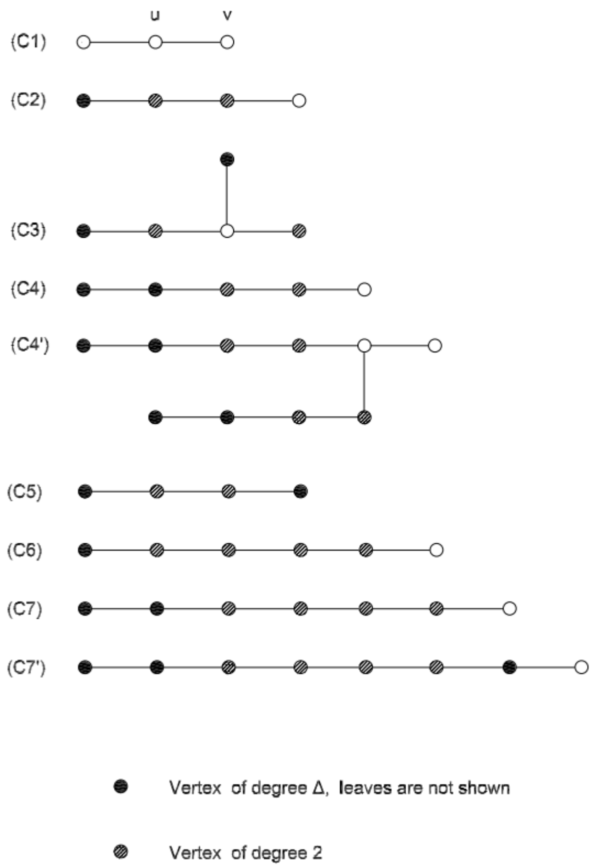

Figure 2: Configurations (C1) - (C7').

Lemma 4.2. Let T be a tree with $\Delta \geq 3$, $2, 4 \notin D_\Delta$ and $d(x) \leq 2$ if x is not a Δ-vertex. If T is not a double star and does not contain (C1), then T contains one of the following configurations:
(C5) A path $x_1x_2x_3x_4$ such that $d(x_2) = d(x_3) = 2, d(x_4) = \Delta$ and x_1 is a major handle.
(C6) A path $x_1x_2 \dots x_6$ such that $d(x_i) = 2$ for $i = 2, 3, 4, 5$ and x_1 is a major handle.
(C7) A path $x_1x_2 \dots x_7$ such that $d(x_i) = 2$ for $i = 3, 4, 5, 6$, x_1 is a major handle and x_2 is a weak major handle.

Proof. Suppose that T does not contain (C1), (C5) and (C6). We will prove that T contains (C7). Let $x_0x_1 \dots x_n$ be a longest path in T. Because T does not

contain (C1), x_1 and x_{n-1} are both major handles. Since T is not a double star and $2,4 \notin D_\Delta$, then either $n=5$ or $n \geq 7$.
If $n=5$, then $d(x_4)=\Delta$ and thus $d(x_2)=d(x_3)=2$. However, now T contains (C5). If $n=7$, then $d(x_6)=\Delta$ and thus $d(x_i)=2$ for $i=2,3,4,5$. Hence, T contains (C6), a contradiction. Thus $n \geq 8$. Since $2,4 \notin D_\Delta$ and $d(x_1)=\Delta$, then $d(x_3)=d(x_5)=2$. If $d(x_2)=2$, then T contains (C5) or (C6). Thus $d(x_2)=\Delta$. Since $2,4 \notin D_\Delta$ and $d(x_2)=\Delta$, then $d(x_4)=d(x_6)=2$. □

Lemma 4.3. Let T be a tree with $\Delta \geq 3$ that is not a double star. Let v be a vertex of T with $d(v) > 1$ and let v be adjacent to exactly one vertex u of degree greater than one. If the tree $T \setminus (N(v) \setminus \{u\})$ has an $L(d,1)$-labelling f using the label set $\mathcal{B}$ with $f(v) \in \{0, \Delta+d-1\}$, then f can be extended to an $L(d,1)$-labelling of T.

Proof. Let $f(v)=0$ and $\mathcal{B}' = \mathcal{B} \setminus \{0,1,2,\dots,d-1,f(u)\}$. Then $|\mathcal{B}'| = \Delta+d-(d+1) = \Delta-1$ (for $f(v)=\Delta+d-1$, let $\mathcal{B}' = \mathcal{B} \setminus \{\Delta+d-1, \Delta+d-2, \dots, \Delta, f(u)\}$). Since $|N(v)\setminus\{u\}| = d(v)-1 \leq \Delta-1$, the vertices from $N(v)\setminus\{u\}$ can be labelled with mutually different labels from $\mathcal{B}'$ and f is extended to an $L(d,1)$-labelling of T accordingly. □

Lemma 4.4. Let T be a tree that is not a double star and $d(x) \leq \Delta-d+1$ if x is not a Δ-vertex.

1. Let $\Delta \geq 3$. Assume T contains configuration (C1). If $T\setminus\{v\}$ has a $(\Delta+d-1)$-labelling, so does T.

2. Let $\Delta \geq 4, d \leq \Delta-2$. Assume T contains configuration (C2) or (C3). If $T \setminus N[x_1]$ has a $(\Delta+d-1)$-labelling, so does T.

3. Let $\Delta \geq 5, d \leq \Delta-2$. Assume T contains configuration (C4). If $T\setminus(N(x_1) \cup N(x_2))$ has a $(\Delta+d-1)$-labelling, so does T.

Proof. 1. Assume that T contains (C1), hence there ia s leaf v adjacent to vertex u with $d(u) < \Delta$. Let $V(T)\setminus\{v\}$ have an $L(d,1)$-labelling f using the label set $\mathcal{B}$. Since $d(u) \leq \Delta-d+1$, at most $\Delta-d$ different labels are needed for labelling the neighbors of u in $T\setminus\{v\}$. Observe that the labels in $\{f(u) \pm k \mid k=0,1,2,\dots,d-1\} \cap \mathcal{B}$ are forbidden labels for v. This is altogether at most $(\Delta-d)+1+2(d-1) = \Delta+d-1$ labels. As $|\mathcal{B}| = \Delta+d$, we can label v with some label from $\mathcal{B}$.

2. Assume that T contains (C2) and x_i are defined as in Lemma 4.1. Let $T\setminus N[x_1]$ has a $(\Delta+d-1)$-labelling f using the label set $\mathcal{B}$. Denote $c_i = f(x_i)$. We consider the next cases according to the label of c_3:

   (i) Assume $c_3 = 0$, we can define $c_1 = \Delta+d-1$ or vice versa if $c_3 = \Delta+d-1$, then $c_1 = 0$. In both cases, we can label x_2 with some label from nonempty set $\{d, d+1, \dots, \Delta-1\} \setminus \{c_4\}$, since $\Delta > d+1$.

(ii) Assume $0 < c_3 \leq \lfloor \frac{\Delta+d-1}{2} \rfloor$.Then we can define $c_1 = 0$ and $c_2 \in \{c_3 + d, c_3 + d + 1\} \setminus \{c_4\}$.
As $d \leq \Delta - 2$, $\frac{\Delta+d-1}{2} + d + 1 = \frac{\Delta+3d+1}{2} \leq \frac{2\Delta+2d-1}{2} = \Delta + d - \frac{1}{2}$, therefore, $c_3 + d + 1 \leq \lfloor \frac{\Delta+d-1}{2} \rfloor + d + 1 \leq \lfloor \Delta + d - \frac{1}{2} \rfloor = \Delta + d - 1$. Hence x_2 can be labelled properly.

(iii) Assume $\lfloor \frac{\Delta+d-1}{2} \rfloor < c_3 < \Delta + d - 1$, we define $c_1 = \Delta + d - 1, c_2 \in \{c_3 - d, c_3 - d - 1\} \setminus \{c_4\}$.
As $d \leq \Delta - 2, c_3 - d - 1 \geq \lfloor \frac{\Delta+d-1}{2} \rfloor - d \geq \frac{\Delta+d-2}{2} - d = \frac{\Delta-d-2}{2} \geq 0$. Hence x_2 can be labelled properly.

By Lemma 4.3, the vertices in $N(x_1) \setminus \{x_2\}$ can be labelled properly.

3. Let T contains (C3) and let $T \setminus N[x_1]$ has a $(\Delta + d - 1)$-labelling f using the label set $\mathcal{B}$. Since y is major handle in $T \setminus N[x_1]$, without loss of generality, say $f(y) = 0$ and hence $c_3 \in \{d, d + 1, \ldots, \Delta + d - 1\}$.

   We distinguish several cases according to c_3. Note that as $d \leq \Delta - 2$, we have $2d + 1 \leq \Delta + d - 1$.

   (i) Let $c_3 = d$, we define $c_1 = 0$ and $c_2 \in \{2d, 2d + 1\} \setminus \{c_4\}$.

   (ii) Let $c_3 = d + 1$, then we set $c_2 \in \{1, 2d + 1\} \setminus \{c_4\}$. If $c_2 = 1$ then $c_1 = 2d + 1$, if $c_2 = 2d + 1$ then $c_1 = 0$.

   (iii) Let $c_3 \in \{d + 2, d + 3, \ldots, \Delta + d - 2\}$. We define $c_1 = \Delta + d - 1, c_2 \in \{1, 2\} \setminus \{c_4\}$.

   (iv) Let $c_3 = \Delta + d - 1$. We define $c_1 = 0, c_2 \in \{d, d + 1\} \setminus \{c_4\}$.

   By Lemma 4.3, we can provide a proper labelling for the vertices in $N(x_1) \setminus \{x_2\}$.

4. Let T be a tree that contains (C4) and assume $T \setminus (N(x_1) \cup N(x_2))$ has a $(\Delta + d - 1)$-labelling f. We consider the next cases according to the label of x_4.

   (i) If $c_4 = 0$, we define $c_2 = \Delta + d - 1, c_1 = 0$ or if $c_4 = \Delta + d - 1$, we define $c_2 = 0, c_1 = \Delta + d - 1$. Since $|\{d, \ldots, \Delta - 1\}| \geq 2$ $(d \leq \Delta - 2)$, we can label x_3 with some label from $\{d, \ldots, \Delta - 1\} \setminus \{c_5\}$.

   (ii) If $c_4 \in \{1, \ldots, d-1\}$, we define $c_2 = 0, c_1 = \Delta+d-1, c_3 \in \{\Delta+d-3, \Delta+d-2\} \setminus \{c_5\}$. (Since $(\Delta+d-3)-c_4 \geq (\Delta+d-3)-(d-1) = \Delta-2 \geq d$, x_3 can be labelled properly.)

   (iii) If $c_4 \in \{\Delta, \ldots, \Delta + d - 2\}$, then we define $c_2 = \Delta + d - 1, c_1 = 0, c_3 \in \{1, 2\} \setminus \{c_5\}$. (Since $c_4 - 2 \geq \Delta - 2 \geq d$, x_3 can be labelled properly.)

(iv) Finally $c_4 \in \{d, \dots, \Delta - 1\}$. We consider two cases $d < \Delta - 2$ and $d = \Delta - 2$.

a) Let $d < \Delta - 2$.

- If $c_4 = d$, then we define $c_1 = \Delta + d - 1, c_2 = 0, c_3 \in \{2d, 2d + 1\} \setminus \{c_5\}$.
- If $c_4 = d+1$, then we can label x_3 with some label from $\{1, 2d+1\} \setminus \{c_5\}$. Let $c_1 = 0, c_2 = \Delta + d - 1$ when $c_3 = 1$, and let $c_1 = \Delta + d - 1, c_2 = 0$ when $c_3 = 2d + 1$.
- If $c_4 \in \{d+2, \dots, \Delta - 1\}$, then we can label x_3 with some label from $\{1, 2\} \setminus \{c_5\}$, and let $c_1 = 0, c_2 = \Delta + d - 1$.

b) Let $d = \Delta - 2$, then $c_4 \in \{d, d + 1\}$.

- If $c_4 = d$, then we know that x_5 is labelled with some label from $\{0, 2d, 2d + 1\}$. If $c_5 = 0$ or $c_5 = 2d + 1$, then we define $c_3 = 2d, c_2 = 0, c_1 = 2d + 1$. The argument in the case $c_5 = 2d$ is somewhat more tricky. Vertices in $N(x_5)$ can be labelled with labels from $\{0, 1, \dots, d\}$. Since $d \geq 3$ (if $d = 2$ then $\Delta = 4$), these are at least 4 labels. But $d(x_5) \leq 3$, so there is at least one unused label in $\{0, 1, \dots, d - 1\}$, say a. Replace the label of x_4 with a and further deal with two cases $a = 0$ and $a \in \{1, \dots, d - 1\}$. In the first case we can label vertices x_3, x_2, x_1 with $d, 2d+1, 0$ and in the second with $2d-1, 0, 2d+1$, respectively.
- For $c_4 = d + 1$ the proof is similar as the above. In this case, x_5 is labelled with some label in $\{0, 1, 2d + 1\}$. Let $c_5 = 0$ or $c_5 = 2d + 1$. Thus we can define $c_3 = 1, c_2 = 2d + 1, c_1 = 0$. If $c_5 = 1$ then replace the label of x_4 with an unused label $a \in \{d + 2, d + 3, \dots, 2d + 1\}$ as above. We consider two cases according to the label of a:
  $a \in \{d+2, d+3, \dots, 2d\}$, we label vertices x_3, x_2, x_1 with $2, 2d+1, 0$, respectively. If $a = 2d + 1$, then we can label vertices x_3, x_2, x_1 with $d, 0, 2d + 1$, respectively.

In all cases, vertices in $(N(x_1) \cup N(x_2)) \setminus \{x_1, x_2, x_3\}$ are leaves, so we can label them easily.

□

Proof of Theorem 3.1. The proof is proceeded by induction on the number of vertices of T. It is easy to construct an $L(d,1)$-labelling of T using the label set $\mathcal{B}$, if T is a double star (note that $\Delta + 1 \leq |V(T)| \leq 2\Delta$). Two adjacent vertices x and y must be labelled with 0 and $\Delta + d - 1$, respectively, vertices from $N(x) \setminus \{y\}$ with mutually different labels from $\{d, d + 1, \dots, \Delta + d - 2\}$ and vertices from $N(y) \setminus \{x\}$ with mutually different labels from $\{1, 2, \dots, \Delta - 1\}$. Assume that T is not a double star.

1. Let T be a tree with $\Delta \geq 5, d \leq \Delta - 2$ and $2, 4 \notin D_\Delta$. Let $d(x) \leq \Delta - d + 1$ if x is not a Δ-vertex. If T contains (C1), then $T \setminus \{v\}$ has an $L(d, 1)$-labelling f using the label set $\mathcal{B}$ by the induction hypothesis. Hence $\lambda_d(T) = \Delta + d - 1$ by Lemma 4.4.

   Thus we may suppose, that T contains no (C1). Suppose that T contains one of the configurations (C2) or (C3). If $T \setminus N[x_1]$ is a Δ-subtree of T, by the induction hypothesis, $T \setminus N[x_1]$ has a $(\Delta + d - 1)$-labeling. Hence $\lambda_d(T) = \Delta + d - 1$ by Lemma 4.4. Let $\Delta(T \setminus N[x_1])$ be strictly less than $\Delta(T)$. Then $\Delta(T \setminus N[x_1]) \leq \Delta - d + 1$. Since $\Delta + d - 1 \leq \lambda_d(T) \leq \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$ for any tree T of maximum degree Δ [3] and since $\min\{\Delta + 2d - 2, 2\Delta + d - 2\} = \Delta + 2d - 2$ for $d < \Delta$, we get $\lambda_d(T \setminus N[x_1]) \leq \Delta(T \setminus N[x_1]) + 2d - 2 \leq \Delta - d + 1 + 2d - 2 = \Delta + d - 1$. By Lemma 4.4, we also have $\lambda_d(T) = \Delta + d - 1$.

   We only need to consider the case when T contains (C4). Zhai, Lu, and Shu [12] have shown that $T \setminus (N(x_1) \cup N(x_2))$ is a Δ-subtree of T. Therefore, by the induction hypothesis, $T \setminus (N(x_1) \cup N(x_2))$ has a $(\Delta + d - 1)$-labelling. Hence $\lambda_d(T) = \Delta + d - 1$ by Lemma 4.4.

2. Let T be a tree with $\Delta \geq 4, d = \Delta - 1$ and $2, 4 \notin D_\Delta$. Let $d(x) \leq 2$ if x is not a Δ-vertex.

   If T contains (C1), then $\lambda_d(T) = \Delta + d - 1 = 2d$ by Lemma 4.4.

   By Lemma 4.2, T contains one of the configurations (C5), (C6) and (C7).

   Let T contains (C5). By the induction hypothesis, $T \setminus N[x_1]$ has a $L(d, 1)$-labelling f using the label set $\{0, 1, \ldots, 2d\}$. Denote $c_i = f(x_i)$. Since $d(x_4) = \Delta$, without loss of generality, we may suppose $c_4 = 0$. Then clearly $c_3 \in \{d, d + 1, \ldots, 2d\}$. Check all possibilities for labelling x_3:

   - Let $c_3 = d$, we define $c_2 = 2d, c_1 = 0$.
   - Let $c_3 = 2d$, we define $c_2 = d, c_1 = 0$.
   - Let $c_3 \in \{d + 1, \ldots, 2d - 1\}$, we define $c_2 = 1, c_1 = 2d$.

   By Lemma 4.3, we can label all vertices from $N(x_1) \setminus \{x_2\}$.

   Let T contains (C6) and again, by the induction hypothesis $T \setminus (N[x_1] \cup \{x_3, x_4\})$ has a $2d$-labelling f. In view of simmetry of the labels, we can assuming $c_6 \in \{0, 1, \ldots, d\}$ to extend f into a $2d$-labelling of T:

   (i) Assume $c_6 = 0 \Rightarrow c_5 \in \{d, \ldots, 2d\}$. Consider the following three subcases:

      - Let $c_5 = d$, we can label x_4, x_3, x_2, x_1 by $2d, 1, 2d - 1, 0$.
      - Let $c_5 = 2d$, we can label x_4, x_3, x_2, x_1 by $1, d + 1, 0, 2d$.
      - Let $c_5 \in \{d + 1, d + 2, \ldots, 2d - 1\}$, we can label x_4, x_3, x_2, x_1 by $1, 2d, d, 0$.

(ii) Assume $c_6 \in \{1, \ldots, d\}$, then $c_5 \in \{c_6 + d, c_6 + d + 1, \ldots, 2d\}$, we can label x_4, x_3, x_2, x_1 by $0, d, 2d, 0$.

By Lemma 4.3 we can give a proper labelling for the vertices from $N(x_1) \setminus \{x_2\}$.

Finally, we assume that T contains (C7). We know that $T \setminus (N(x_1) \cup N(x_2) \cup \{x_4, x_5, x_6\})$ is a Δ-subtree of T. By the induction hypothesis, $T \setminus (N(x_1) \cup N(x_2) \cup \{x_4, x_5, x_6\})$ has a $2d$-labeling f. Again, for x_8 it is enough to check labels from the set $\{0, 1, \ldots, d\}$. See Table 1.

By Lemma 4.3, the vertices from $(N(x_1) \cup N(x_2)) \setminus \{x_1, x_2, x_3\}$ can be labelled properly.

Table 1: T contains (C7).

c_8	0			$k, 1 \leq k \leq d-1$		d	
c_7	d	$d+1, \ldots, 2d-1$	$2d$	$k+d, \ldots, 2d-1$	$2d$	0	$2d$
c_6	$2d$	1	d	0	d	$2d-1$	0
c_5	0	$2d$	0	$2d$	0	1	$2d-1$
c_4	$2d-1$	0	$2d$	1	$2d$	$2d$	1
c_3	1	d	d	$d+1$	d	d	$d+1$
c_2	$2d$	$2d$	0	0	0	0	0
c_1	0	0	$2d$	$2d$	$2d$	$2d$	$2d$

This completes the proof of the theorem. □

In [12], two trees of maximum degree 3 and 4 with $2, 4 \notin D_\Delta$ were constructed, such that for $d = 2$ these trees are not of type 1. It was also shown, that if $\Delta(T) = 4$, then $\lambda_d = \Delta + d - 1$ if any Δ-subtree of T contains no (C4') and if $\Delta(T) = 3$, then $\lambda_d = \Delta + d - 1$ if any Δ-subtree of T contains no (C7') (see Figure 2).

5. Trees with $d \geq \Delta$

First, we observe that when $d > \Delta$ some labels in the set $\{0, 1, 2, \ldots, \Delta, \Delta + 1, \ldots, d, d+1, \ldots, \Delta + d - 1\}$ can only be used for labeling isolated vertices, and are thus useless.

Lemma 5.1. Let $d \geq \Delta$. Only 2Δ labels are useful in the label set $\mathcal{B} = \{0, 1, 2, \ldots, \Delta, \Delta + 1, \ldots, d, d + 1, \ldots, \Delta + d - 1\}$. These are labels $\{0, 1, \ldots, \Delta - 1, d, d + 1, \ldots, \Delta + d - 1\}$.

Proof. For label $c \in \{\Delta, \Delta+1, \ldots, d-1\}$, we find $c+d > \Delta+d-1$ and $c-d < 0$. Hence if a vertex is labelled with such c there is no available label for any of its neighbors. □

Lemma 5.2. Let $\Delta \geq 3$ and $d \geq \Delta$. Let gdS be a generalized double star with $\{x, y\}$ as its center and the other neighbors of x and y be $x_{\Delta-1}, x_{\Delta-2}, \ldots, x_2, x_1 \in N_1(x) \setminus \{y\}$, and $y_{\Delta-1}, y_{\Delta-2}, \ldots, y_2, y_1 \in N_1(y) \setminus \{x\}$ where $d(x_i) \geq d(x_j)$ and $d(y_i) \geq d(y_j)$ for any $i \geq j$. Then gdS is of type 1 if and only if $d(x_i) \leq i$ and $d(y_i) \leq i$ for any $1 \leq i \leq \Delta - 1$.

Proof. By Lemma 2.3, Δ vertices must get labels 0 and $\Delta + d - 1$. Without loss of generality, label x with 0 and y with $\Delta + d - 1$. Then we can label $x_k \in N_1(x) \setminus \{y\}$ with $d + k - 1$, $y_k \in N_1(y) \setminus \{x\}$ with $\Delta - k$, vertices in $N_1(x_k) \setminus \{x\}$ with labels from $\{1, 2, \ldots, k-1\}$ and vertices in $N_1(y_k) \setminus \{y\}$ with labels from $\{\Delta + d - k, \Delta + d - k + 1, \ldots, \Delta + d - 2\}$, where $1 \leq k \leq \Delta - 1$.

On the other hand, assume that gdS is of type 1. Therefore, there exists a labeling in which, say x has label 0 and y has label $\Delta + d - 1$. Consider the neighbor of y that is labelled by k, $1 \leq k \leq \Delta - 1$, and denote it by y_k. Clearly, y_k has at most k neighbors including y, otherwise the labeling is not well defined. Similar argument implies the conditions on degrees of x_k. □

The generalized double star with $\Delta = 5$ and $d = 6$ is shown in Figure 3.

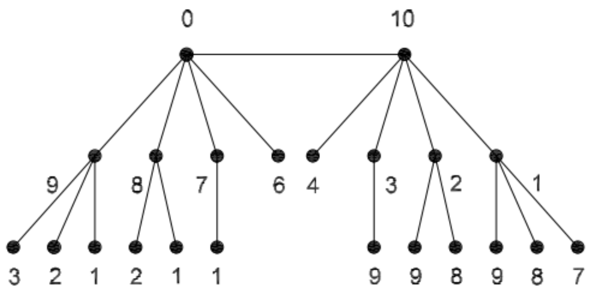

Figure 3: Generalized double star of type 1 with $\Delta = 5$ in $d = 6$.

Similarly, we can label a generalized star of type 1.

Lemma 5.3. Let $\Delta \geq 3$ and $d \geq \Delta$. Let gS be a generalized star with x as its center. Let the neighbours of x be $x_{\Delta-1}, x_{\Delta-1}, x_{\Delta-2}, \ldots, x_2, x_1 \in N_1(x)$ where $d(x_i) \geq d(x_j)$ for any $i \geq j$. Then gS is of type 1 if and only if $d(x_i) \leq i$ for any $1 \leq i \leq \Delta - 1$.

Proof. Analogous to the proof of Lemma 5.2. □

Lemma 5.4. Let T be a tree with $\Delta \geq 3$ and $d \geq \Delta$. Let T contain at most two Δ-vertices x and y and assume they are adjacent. If the subtree T' of T induced on vertices $N^2[x] \cup N^2(y)$ is isomorphic to the gdS of type 1 and if $d(x) \leq 2$ for $x \notin V(T')$, then T is of type 1.

Proof. Assume that there are two adjacent Δ-vertices in T. We first label gdS as in the proof of Lemma 5.2. Observe that the unlabelled vertices induce a forest in which the connected components are paths and isolated vertices. Hence any of the components gives rise to a path $x_0x_1 \cdots x_n$, where x_0 is the center of gdS, $d(x_1) < \Delta$ and x_0, x_1, x_2 are already labelled. Since x_2 is not a leaf, the method used in the proof of Lemma 5.2 assures that it is labeled with a label from $\{0, 1, 2, \ldots, \Delta - 2, d+1, d+2, \ldots, \Delta + d - 1\}$. Hence, we can label $x_3, x_4, x_5 \ldots$ with labels from $\{0, 1, \Delta + d - 2, \Delta + d - 1\}$, in all these cases.

The proof is similar when there is only one Δ-vertex in T. □

A special case of Lemma 5.4 may be worth stating explicitly.

Lemma 5.5. Let T be a tree with $\Delta \geq 3$ and a unique Δ-vertex x. Let $d \geq \Delta$. If the subtree of T induced on vertices $N^2[x]$ is isomorphic to the gS of type 1 and if $d(x) \leq 2$ for $x \notin gS$, then T is of type 1.

Proof of Theorem 3.2. First, observe that we can define the labelling of the subgraph induced on Δ-vertices and the paths among them, and then it is easy to complete the labeling for the remaining vertices of degree one and two, in the same way as in the proof of Lemma 5.4.

Therefore, it is enough to consider the subgraph of T induced on Δ-vertices, and the paths among them. Start with arbitrary Δ-vertex, say x. It is by assumption either a center of a gS or a gdS. Label the vertices of $N^2[x]$ (or, the vertices of $N^2[x] \cup N^2(y)$), as in the proof of Lemma 5.5 (respectively, Lemma 5.4).

Now we proceed by induction. If all Δ-vertices has been labelled already, we stop. Otherwise, choose a Δ-vertex z that has not been labelled such that the associated gS or gdS is adjacent to one of the gS or gdS that have already been labelled. Without loss of generality assume that the shortest path between centers of these two generalized double stars is the path $x_0x_1 \cdots x_n$, where $x_n = z$ and x_0 is the nearest already labelled Δ-vertex. The labelling of the path to z and the generalized star (or, double star) centered at z is defined as follows, depending on the parity of the path.

If its length is even, then we label x_n with the same label as labelled x_0, without loss of generality, we say with 0. If the length is odd, without loss of generality, we label x_0 with 0 and x_n with $\Delta + d - 1$. The other vertices of the generalized star or generalized double star centered by z is labelled as in Lemma 5.4. The vertices $x_3, x_4, \ldots, x_{n-3}$ are labelled as follows. Denote the labels shortly by $c_i = f(x_i)$, $i = 3, 4, \ldots, n-3$.

1. Suppose that $n = 8 + 2k, k = 0, 1, 2, \ldots$.

Vertices $x_3, x_5 \dots x_{n-3}$ we can label from left to right: $c_3 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_1\}, c_5 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_3\}, \dots, c_{n-5} \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_{n-7}\}, c_{n-3} \in \{\Delta + d - 1, \Delta + d - 2, \Delta + d - 3\} \setminus \{c_{n-5}, c_{n-1}\}$ and vertices $x_{n-4}, x_{n-6} \dots x_4$ from right to left: $c_{n-4} = 0, c_{n-6} = 1, \dots, c_6 \in \{0, 1\} \setminus \{c_8\}, c_4 \in \{0, 1, 2\} \setminus \{c_2, c_6\}$.

If $c_{n-2} = \Delta - 2$ and if $c_{n-3} = \Delta + d - 3$ is obtained in this way, we must to redefine the label of x_{n-2}. We know that $c_{n-1} \geq \Delta + d - 2$. If label 1 has not been used for labelling vertices from $N(x_{n-1}) \setminus \{x_{n-2}, x_n\}$, then we label x_{n-2} with it. If it is select for a vertex $y \in N(x_{n-1}) \setminus \{x_{n-2}, x_n\}$, then we can replace label of x_{n-2} with label of y, since $d(y) \leq 2$. We redefine $f(x_{n-2}) = 1$ and $f(y) = \Delta - 2$.

2. Suppose that $n = 9 + 2k, k = 0, 1, 2, \dots$.

   We may label as follows: $c_3 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_1\}, c_5 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_3\}, \dots, c_{n-4} \in \{\Delta + d - 1, \Delta + d - 2, \Delta + d - 3\} \setminus \{c_{n-6}, c_{n-2}\}$ and $c_{n-3} \in \{0, 1\} \setminus \{c_{n-1}\}, c_{n-5} \in \{0, 1\} \setminus \{c_{n-3}\}, \dots, c_6 \in \{0, 1\} \setminus \{c_8\}$ and $c_4 \in \{0, 1, 2\} \setminus \{c_2, c_6\}$.
   If $\Delta = 4$ and $n = 9$, it can be obtained in this way that $c_4 = 2$ and $c_5 = d+1$. This is possible if $c_2 = c_8 = 1$ and $c_1 = c_7 \in \{d + 2, d + 3\}$. In this case, we redefine $c_3 = d + 1, c_4 = 0, c_5 \in \{d + 2, d + 3\} \setminus \{c_7\}$ and $c_6 = 2$.

□

Proof of Theorem 3.3. The proof is similar to the proof of Theorem 3.2. Start with an arbitrary Δ-vertex x. Choose the labelling of the corresponding generalized star or double star centered at x as in the proofs of Lemma 5.1 or Lemma 5.2. We proceed by induction. Repeat, while there are unlabelled Δ-vertices, by choosing a double star (or star) with unlabelled center that is adjacent to one of the double stars (or stars) that have already been labelled. Let z be the Δ-vertex of the chosen star such that the shortest path between the centers of these two double stars is the path $x_0 x_1 \cdots x_n$, where $x_n = z$ and x_0 is the nearest already labelled Δ-vertex.

Suppose that $n \geq 8$. In view of symmetry of the labels, we only need to consider the case when x_0 is labelled with 0. As before, let c_i be the label of x_i. We can assume that $c_0 = 0$ and $c_2 = 1$, because clearly $c_2 \neq 0$ and if $c_2 > 1$, then we can relabel x_2 and set $c_2 = 1$. The labeling of the path can be extended as follows (as in the proof of Theorem 3.2): when n is even, we set $c_n = 0$, $c_{n-1} = \Delta + d - 2$ and $c_{n-2} = 1$. When n is odd, then $c_n = \Delta + d - 1$, $c_{n-1} = 1$ and $c_{n-2} = \Delta + d - 2$.

The remaining cases, $4 \leq n < 8$, are considered below. Again, w.l.o.g assume that $c_0 = 0$, and observe that x_2 can be relabelled if needed. It is straightforward to see that the following are partial labellings of the path $x_0 x_1 \dots x_n$.

(i) $n = 4 : c_2 = 1, c_3 \in \{\Delta + d - 2, \Delta + d - 3\} \setminus \{c_1\}$, $c_4 = 0$.

(ii) $n = 5 : c_2 = 1, c_3 \in \{\Delta + d - 2, \Delta + d - 3\} \setminus \{c_1\}, c_4 = 2$ and $c_5 = \Delta + d - 1$.

(iii) $n = 6 : c_6 = 0$

- $c_1 \neq d + 1 \Rightarrow c_2 = 2, c_3 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_1\}, c_4 = 1, c_5 = \Delta + d - 3$.
- $c_1 = d + 1 \Rightarrow c_2 = 1, c_3 = \Delta + d - 1, c_4 = 2, c_5 = \Delta + d - 2$.

(iv) $n = 7 : c_2 = 1, c_3 \in \{\Delta + d - 1, \Delta + d - 2\} \setminus \{c_1\}, c_4 = 0, c_5 = \Delta + d - 3, c_6 = 1$ and $c_7 = \Delta + d - 1$.

After all Δ-vertices are labelled, the unlabelled vertices form a set of paths and isolated vertices. The labelling is thus easily completed (as in the proofs before). □

In the case $\Delta = 4, n = 5$ and $c_1 = \Delta + d - 2 = d + 2$, similarly as in the proof above gives $c_2 = 1, c_3 = d + 1$ and thus necessarily $c_4 = 0$. But then x_5 can not be adjacent to any Δ-vertex. This shows the following consequence.

Corollary 5.6. Let T be a tree with $\Delta = 4$ and $d \geq 4$. Then $\lambda_d = d + 3$ if each 4-vertex has at least one leaf, $d(x) \leq 2$ if x is not a 4-vertex and $1, 2, 3 \notin D_4$ or $2, 3, 5 \notin D_4$.

Let now T be a tree with $\Delta = 3$ and let $x_0x_1x_2x_3x_4$ be a path between two 3-vertices. Let x_0 and x_4 get label 0 and vertices from $N^1(x_0)$ labels $d + 1, d + 2$. Define $c_2 = 1, c_3 \in \{d + 2, d + 1\} \setminus \{c_1\}$. Consequently, we have:

Corollary 5.7. Let T be a tree with $\Delta = 3$ and $d \geq 3$. Then $\lambda_d = d + 2$ if each 3-vertex has at least one leaf, $d(x) \leq 2$ if x is not a 3-vertex and $D_3 = \{4k \mid k \in \mathbb{N}\}$.

Confilcts of Interest. The authors declare that they have no conflicts of interest.

References

[1] T. Calamoneri, The L(h,k)-labeling problem: An updated survey and annotated bibliography, *Comput. J.* **54** (8) (2011) 1344 – 1371.

[2] G. J. Chang and D. Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM J. Discrete Math.* **9** (2) (1996) 309 – 316.

[3] G. J. Chang, W. -T. Ke, D. Kuo, D. D. -F. Liu and R. K. Yeh, On $L(d,1)$-labeling of graphs, *Discrete Math.* **220** (2000) 57 – 66.

[4] G. J. Chang and C. Lu, Distance-two labelings of graphs, *European J. Combin.* **24** (2003) 53 – 58.

[5] J. Georges, D. Mauro and M. Whittlesey, Relating path covering to vertex labelings with a condition at distance two, *Discrete Math.* **135** (1994) 103 – 111.

[6] D. Gonçalves, On the $L(p,1)$-labelling of graphs, *Discrete Math.* **308** (2008) 1405 − 1414.

[7] J. R. Grigs and R. K. Yeh, Labeling graphs with a condition at distance 2, *SIAM J. Discrete Math.* **5** (1992) 586 − 595.

[8] E. Jonck, J. H. Hattingh and C. J. Ras, A characterization of $\lambda_{d,1}$-minimal trees and other attainable classes, *Discrete Math.* **309** (2009) 2340 − 2348.

[9] J. S. -T. Juan, D. D. -F. Liu and L. -Y. Chen, $L(j,k)$-labelling and maximum ordering-degrees for trees, *Discrete Appl. Math.* **158** (2010) 692 − 698.

[10] D. Liu and R. K. Yeh, On distance two labelings of graphs, *Ars. Combin.* **47** (1997) 13 − 22.

[11] W. Wang, The $L(2,1)$-labeling of trees, *Discrete Appl. Math.* **154** (2006) 598 − 603.

[12] M. Zhai, C. Lu and J. Shu, A note on $L(2,1)$-labeling of Trees, *Acta. Math. Appl. Sin.* **28** (2012) 395 − 400.

[13] J. Zhu, Z. Bu, M. P. Pardalos, H. Du, H. Wang and B. Liu, Optimal channel assignment and $L(p,1)$-labeling, *J. Global Optim.* **72** (2018) 539 − 552.

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