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Bi-Gyrogroup: The Group-Like Structure Induced by Bi-Decomposition of Groups

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Abstract

The decomposition $\Gamma = BH$ of a group Γ into a subset B and a subgroup H of Γ induces, under general conditions, a group-like structure for B, known as a gyrogroup. The famous concrete realization of a gyrogroup, which motivated the emergence of gyrogroups into the mainstream, is the space of all relativistically admissible velocities along with a binary operation given by the Einstein velocity addition law of special relativity theory. The latter leads to the Lorentz transformation group $\mathrm{SO}(1, n)$, $n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature $(1, n)$. The study in this article is motivated by generalized Lorentz groups $\mathrm{SO}(m, n)$, $m, n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature (m, n). Accordingly, this article explores the bi-decomposition $\Gamma = H_L B H_R$ of a group Γ into a subset B and subgroups H_L and H_R of Γ, along with the novel bi-gyrogroup structure of B induced by the bi-decomposition of Γ. As an example, we show by methods of Clifford algebras that the quotient group of the spin group $\mathrm{Spin}\,(m, n)$ possesses the bi-decomposition structure.

Keywords: Bi-decomposition of group, bi-gyrogroup, gyrogroup, spin group, pseudo-orthogonal group.

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1. Introduction

Lorentz transformation groups $\Gamma = \mathrm{SO}(1, n)$, $n \in \mathbb{N}$, possess the decomposition structure $\Gamma = BH$, where B is a subset of Γ and H is a subgroup of Γ [26]. The decomposition structure of Γ induces a group-like structure for B. This group-like structure was discovered in 1988 [26] and became known as a *gyrogroup* [27, 28].

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Subsequently, gyrogroups turned out to play a universal computational role that extends far beyond the domain of Lorentz groups $\mathrm{SO}(1, n)$ [32, 33], as noted by Chatelin in [4, p. 523] and in references therein. In fact, gyrogroups are special loops that, according to [17], are placed centrally in loop theory.

The use of Clifford algebras to employ gyrogroups as a computational tool in harmonic analysis is presented by Ferreira in the seminal papers [9, 10]. The use of Clifford algebras to obtain a better understanding of gyrogroups is found, for instance, in [7, 8, 11, 20, 24].

Generalized Lorentz transformation groups $\Gamma = \mathrm{SO}(m, n)$, $m, n \in \mathbb{N}$, possess the so-called *bi-decomposition* structure $\Gamma = H_L B H_R$, where B is a subset of Γ and H_L and H_R are subgroups of Γ. The bi-decomposition structure of Γ induces a group-like structure for B, called a *bi-gyrogroup* [34]. The use of Clifford algebras that may improve our understanding of bi-gyrogroups is found in [12]. Clearly, the notion of bi-gyrogroups extends the notion of gyrogroups. Accordingly, "gyrolanguage", the algebraic language crafted for gyrogroup theory is extended to "bi-gyro-language" for bi-gyrogroup theory.

As a first step towards demonstrating that bi-gyrogroups play a universal computational role that extends far beyond the domain of generalized Lorentz groups $\mathrm{SO}(m, n)$, the aim of the present article is to approach the study of bi-gyrogroups from the abstract viewpoint.

The article is organized as follows. In Section 2 we give the definition of a bi-gyrogroupoid. In Section 3 we show that the bi-transversal decomposition of a group with additional properties yields a highly structured type of bi-gyrogroupoids. In Section 4 we introduce the notion of bi-gyrodecomposition of groups and prove that any bi-gyrodecomposition of a group gives rise to a bi-gyrogroup. Finally, in Sections 5 and 6 we demonstrate that the pseudo-orthogonal group $\mathrm{SO}(m, n)$ and the quotient group of the spin group $\mathrm{Spin}\,(m, n)$ possess the bi-gyrodecomposition structure.

2. Bi-gyrogroupoids

We begin with the abstract definition of a bi-gyrogroupoid, which is modeled on the groupoid $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices with bi-gyroaddition studied in detail in [34]. We recall that a groupoid $(B, \oplus_b)$ is a non-empty set B with a binary operation $\oplus_b$. An automorphism of a groupoid $(B, \oplus_b)$ is a bijection from B to itself that preserves the groupoid operation. The group of all automorphisms of $(B, \oplus_b)$ is denoted by $\mathrm{Aut}\,(B, \oplus_b)$ or simply $\mathrm{Aut}\,(B)$.

Definition 2.1 (Bi-gyrogroupoid). A groupoid $(B, \oplus_b)$ is a *bi-gyrogroupoid* if its binary operation satisfies the following axioms.
(BG1) There is an element $0 \in B$ such that $0 \oplus_b a = a \oplus_b 0 = a$ for all $a \in B$.
(BG2) For each $a \in B$, there is an element $b \in B$ such that $b \oplus_b a = 0$.
(BG3) Each pair of a and b in B corresponds to a left automorphism $\mathrm{lgyr}[a, b]$ and

a right automorphism $\mathrm{rgyr}[a,b]$ in $\mathrm{Aut}\,(B,\oplus_b)$ such that for all $c \in B$,

$$(a \oplus_b b) \oplus_b \mathrm{lgyr}[a,b]c = \mathrm{rgyr}[b,c]a \oplus_b (b \oplus_b c). \tag{1}$$

(BG4) For all $a, b \in B$,

(a) $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \oplus_b b]$, and

(b) $\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{lgyr}[a,b]a, a \oplus_b b]$.

(BG5) For all $a \in B$, $\mathrm{lgyr}[a,0]$ and $\mathrm{rgyr}[a,0]$ are the identity automorphism of B.

A concrete realization of Axioms (BG1) through (BG5) will be presented in Section 5.

Roughly speaking, any bi-gyrogroupoid is a groupoid that comes with two families of automorphisms, called left and right automorphisms or, collectively, bi-automorphisms. Note that if bi-automorphisms of a bi-gyrogroupoid $(B,\oplus_b)$ reduce to the identity automorphism of B, then $(B,\oplus_b)$ forms a group.

Let $\mathrm{lgyr}^{-1}[a,b]$ and $\mathrm{rgyr}^{-1}[a,b]$ be the inverse map of $\mathrm{lgyr}[a,b]$ and $\mathrm{rgyr}[a,b]$, respectively. Let $\circ$ denote *function composition* and let id_X denote the identity map on a non-empty set X. The following theorem asserts that bi-gyrogroupoids satisfy a generalized associative law.

Theorem 2.2. *Any bi-gyrogroupoid B satisfies the left bi-gyroassociative law*

$$a \oplus_b (b \oplus_b c) = (\mathrm{rgyr}^{-1}[b,c]a \oplus_b b) \oplus_b \mathrm{lgyr}[\mathrm{rgyr}^{-1}[b,c]a, b]c \tag{2}$$

and the right bi-gyroassociative law

$$(a \oplus_b b) \oplus_b c = \mathrm{rgyr}[b, \mathrm{lgyr}^{-1}[a,b]c]a \oplus_b (b \oplus_b \mathrm{lgyr}^{-1}[a,b]c) \tag{3}$$

for all $a, b, c \in B$.

Proof. Let $a, b, c \in B$ be arbitrary. Since $\mathrm{rgyr}[b,c]$ is surjective, there is an element $d \in B$ for which $\mathrm{rgyr}[b,c]d = a$. By (BG3),

$$a \oplus_b (b \oplus_b c) = \mathrm{rgyr}[b,c]d \oplus_b (b \oplus_b c) = (d \oplus_b b) \oplus_b \mathrm{lgyr}[d,b]c.$$

Since $d = \mathrm{rgyr}^{-1}[b,c]a$, (2) is obtained. One obtains (3) in a similar way. □

Lemma 2.3. *Any bi-gyrogroupoid B has a unique two-sided identity element.*

Proof. By Definition 2.1, B has a two-sided identity element. Suppose that e and f are two-sided identity elements of B. As e is a left identity, $e \oplus_b f = f$. As f is a right identity, $e \oplus_b f = e$. Hence, $e = e \oplus_b f = f$. □

Following Lemma 2.3, the unique two-sided identity of a bi-gyrogroupoid will be denoted by 0. Let B be a bi-gyrogroupoid and let $a \in B$. We say that $b \in B$ is a *left inverse* of a if $b \oplus_b a = 0$ and that $c \in B$ is a *right inverse* of a if $a \oplus_b c = 0$. To see that each element of a bi-gyrogroupoid has a unique two-sided inverse, we investigate some basic properties of a bi-gyrogroupoid.

Theorem 2.4. *Let B be a bi-gyrogroupoid. The following properties are true.*

1. *For all $a, b \in B$,* $\mathrm{lgyr}[a,b]0 = 0$ *and* $\mathrm{rgyr}[a,b]0 = 0$.
2. *For all $a \in B$,* $\mathrm{lgyr}[a,a] = \mathrm{id}_B$ *and* $\mathrm{rgyr}[a,a] = \mathrm{id}_B$.
3. *If a is a left inverse of b, then* $\mathrm{lgyr}[a,b] = \mathrm{id}_B$ *and* $\mathrm{rgyr}[a,b] = \mathrm{id}_B$.
4. *For all $b, c \in B$, if a is a left inverse of b, then* $\mathrm{rgyr}[b,c]a \oplus_b (b \oplus_b c) = c$.
5. *For all $a \in B$, if b is a left inverse of a, then b is a right inverse of a.*

Proof. (1) Let $a, b \in B$. Let $c \in B$ be arbitrary. Since $\mathrm{lgyr}[a,b]$ is surjective, $c = \mathrm{lgyr}[a,b]d$ for some $d \in B$. Then

$$c \oplus_b \mathrm{lgyr}[a,b]0 = \mathrm{lgyr}[a,b]d \oplus_b \mathrm{lgyr}[a,b]0 = \mathrm{lgyr}[a,b](d \oplus_b 0) = \mathrm{lgyr}[a,b]d = c.$$

Similarly, $(\mathrm{lgyr}[a,b]0) \oplus_b c = c$. Hence, $\mathrm{lgyr}[a,b]0$ is a two-sided identity of B. By Lemma 2.3, $\mathrm{lgyr}[a,b]0 = 0$. Similarly, one can prove that $\mathrm{rgyr}[a,b]0 = 0$.

(2) Setting $b = 0$ in (BG4a) gives $\mathrm{rgyr}[a,a] = \mathrm{rgyr}[a,0] = \mathrm{id}_B$ by (BG5). Similarly, setting $b = 0$ in (BG4b) gives $\mathrm{lgyr}[a,a] = \mathrm{id}_B$.

(3) Let $b \in B$ and let a be a left inverse of b. By (BG4a) and (BG5),

$$\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \oplus_b b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, 0] = \mathrm{id}_B.$$

Similarly, $\mathrm{lgyr}[a,b] = \mathrm{id}_B$ by (BG4b) and (BG5).

(4) Let $b, c \in B$ and let a be a left inverse of b. From Identity (1) and Item (3), we have $\mathrm{rgyr}[b,c]a \oplus_b (b \oplus_b c) = (a \oplus_b b) \oplus_b \mathrm{lgyr}[a,b]c = 0 \oplus_b c = c$.

(5). Let $a \in B$ and let b be a left inverse of a. By (BG2), b has a left inverse, say $\tilde{b}$. From Items (4) and (3), we have

$$a = \mathrm{rgyr}[b,a]\tilde{b} \oplus_b (b \oplus_b a) = \mathrm{rgyr}[b,a]\tilde{b} \oplus_b 0 = \mathrm{rgyr}[b,a]\tilde{b} = \tilde{b}.$$

It follows that $a \oplus_b b = \tilde{b} \oplus_b b = 0$, which proves b is a right inverse of a. □

Theorem 2.5. *Any element of a bi-gyrogroupoid B has a unique two-sided inverse in B.*

Proof. Let $a \in B$. By (BG2), a has a left inverse b in B. By Theorem 2.4 (5), b is also a right inverse of a. Hence, b is a two-sided inverse of a. Suppose that c is a two-sided inverse of a. Then a is a left inverse of c. By Theorem 2.4 (3)–(4), $c = \mathrm{rgyr}[a,c]b \oplus_b (a \oplus_b c) = \mathrm{rgyr}[a,c]b \oplus_b 0 = \mathrm{rgyr}[a,c]b = b$, which proves the uniqueness of b. □

Following Theorem 2.5, if a is an element of a bi-gyrogroupoid, then the unique two-sided inverse of a will be denoted by $\ominus_b a$. We also write $a \ominus_b b$ instead of $a \oplus_b (\ominus_b b)$. As a consequence of Theorems 2.4 and 2.5, we derive the following theorem.

Theorem 2.6. *Let B be a bi-gyrogroupoid. The following properties are true for all $a, b, c \in B$:*

1. $\ominus_b(\ominus_b a) = a$;
2. $\mathrm{lgyr}[a, b](\ominus_b c) = \ominus_b \mathrm{lgyr}[a, b]c$ *and* $\mathrm{rgyr}[a, b](\ominus_b c) = \ominus_b \mathrm{rgyr}[a, b]c$;
3. $\mathrm{lgyr}[a, \ominus_b a] = \mathrm{lgyr}[\ominus_b a, a] = \mathrm{rgyr}[a, \ominus_b a] = \mathrm{rgyr}[\ominus_b a, a] = \mathrm{id}_B$.

Any bi-gyrogroupoid satisfies a generalized cancellation law, as shown in the following theorem.

Theorem 2.7. *Any bi-gyrogroupoid B satisfies the left cancellation law*

$$\ominus_b \mathrm{rgyr}[a, b]a \oplus_b (a \oplus_b b) = b \tag{4}$$

and the right cancellation law

$$(a \oplus_b b) \ominus_b \mathrm{lgyr}[a, b]b = a \tag{5}$$

for all $a, b \in B$.

Proof. Identity (4) follows from Theorem 2.4 (4) and Theorem 2.6 (2). Identity (5) follows from (BG3) with $c = \ominus_b b$. □

Definition 2.8 (Bi-gyrocommutative bi-gyrogroupoid). A bi-gyrogroupoid B is *bi-gyrocommutative* if it satisfies the bi-gyrocommutative law

$$a \oplus_b b = (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b])(b \oplus_b a) \tag{6}$$

for all $a, b \in B$.

Definition 2.9 (Automorphic inverse property). A bi-gyrogroupoid B has the *automorphic inverse property* if

$$\ominus_b(a \oplus_b b) = (\ominus_b a) \oplus_b (\ominus_b b)$$

for all $a, b \in B$.

Definition 2.10 (Bi-gyration inversion law). A bi-gyrogroupoid B satisfies the *bi-gyration inversion law* if

$$\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b, a] \quad \text{and} \quad \mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b, a]$$

for all $a, b \in B$.

Under certain conditions, the bi-gyrocommutative property and the automorphic inverse property are equivalent, as the following theorem asserts.

Theorem 2.11. *Let B be a bi-gyrogroupoid such that*

1. $\operatorname{lgyr}[a,b]\circ\operatorname{rgyr}[a,b]=\operatorname{rgyr}[a,b]\circ\operatorname{lgyr}[a,b]$;
2. $\operatorname{lgyr}^{-1}[a,b]=\operatorname{lgyr}[\ominus_b b,\ominus_b a]$ *and* $\operatorname{rgyr}^{-1}[a,b]=\operatorname{rgyr}[\ominus_b b,\ominus_b a]$;
3. $\ominus_b(a\oplus_b b)=(\operatorname{lgyr}[a,b]\circ\operatorname{rgyr}[a,b])(\ominus_b b\ominus_b a)$

for all $a,b\in B$. If B is bi-gyrocommutative, then B has the automorphic inverse property. The converse is true if B satisfies the bi-gyration inversion law.

Proof. Suppose that B is bi-gyrocommutative and let $a,b\in B$. Then $b\oplus_b a=(\operatorname{lgyr}[b,a]\circ\operatorname{rgyr}[b,a])(a\oplus_b b)$ and hence

$$\begin{aligned}a\oplus_b b&=(\operatorname{lgyr}[b,a]\circ\operatorname{rgyr}[b,a])^{-1}(b\oplus_b a)\\&=(\operatorname{rgyr}^{-1}[b,a]\circ\operatorname{lgyr}^{-1}[b,a])(b\oplus_b a)\\&=(\operatorname{rgyr}[\ominus_b a,\ominus_b b]\circ\operatorname{lgyr}[\ominus_b a,\ominus_b b])(b\oplus_b a)\\&=(\operatorname{lgyr}[\ominus_b a,\ominus_b b]\circ\operatorname{rgyr}[\ominus_b a,\ominus_b b])(b\oplus_b a)\\&=\ominus_b(\ominus_b a\ominus_b b).\end{aligned}\tag{7}$$

The extreme sides of (7) imply $\ominus_b(a\oplus_b b)=\ominus_b a\ominus_b b$ and so B has the automorphic inverse property. Suppose that B satisfies the bi-gyration inversion law and let $a,b\in B$. As in (7), we have

$$(\operatorname{lgyr}[a,b]\circ\operatorname{rgyr}[a,b])(b\oplus_b a)=\ominus_b(\ominus_b a\ominus_b b)=a\oplus_b b.$$

Hence, B is bi-gyrocommutative. □

3. Bi-Transversal Decomposition

In this section we study the bi-decomposition $\Gamma=H_LBH_R$ of a group Γ into a subset B and subgroups H_L and H_R of Γ. The bi-decomposition $\Gamma=H_LBH_R$ leads to a bi-gyrogroupoid B, and under certain conditions, a group-like structure for B, called a *bi-gyrogroup*. Further, in the special case when H_L is the trivial subgroup of Γ, the bi-decomposition $\Gamma=H_LBH_R$ descends to the decomposition studied in [14]. It turns out that the bi-gyrogroup B induced by the bi-decomposition of Γ forms a gyrogroup, a rich algebraic structure extensively studied, for instance, in [7, 9–11, 18, 22–25, 28–31].

Definition 3.1 (Bi-transversal). A subset B of a group Γ is said to be a *bi-transversal* of subgroups H_L and H_R of Γ if every element g of Γ can be written uniquely as $g=h_\ell bh_r$, where $h_\ell\in H_L$, $b\in B$, and $h_r\in H_R$.

Let B be a bi-transversal of subgroups H_L and H_R in a group Γ. For each pair of elements b_1 and b_2 in B, the product b_1b_2 gives unique elements $h_\ell(b_1,b_2)\in H_L$, $b_1\odot b_2\in B$, and $h_r(b_1,b_2)\in H_R$ such that

$$b_1b_2=h_\ell(b_1,b_2)(b_1\odot b_2)h_r(b_1,b_2).\tag{8}$$

Hence, any bi-transversal B of H_L and H_R gives rise to

1. a binary operation $\odot$ in B, called the *bi-transversal operation*;

2. a map $h_\ell\colon B \times B \to H_L$, called *the left transversal map*;

3. a map $h_r\colon B \times B \to H_R$, called *the right transversal map.*

The pair $(B, \odot)$ is called the *bi-transversal groupoid of H_L and H_R.*

We will see shortly that the left and right transversal maps of the bi-transversal groupoid $(B, \odot)$ generate automorphisms of $(B, \odot)$, called *left* and *right gyrations* or, collectively, *bi-gyrations*. Accordingly, left and right gyrations are also called *left* and *right gyroautomorphisms.*

Definition 3.2 (Bi-gyration). Let B be a bi-transversal of subgroups H_L and H_R in a group Γ. Let h_ℓ and h_r be the left and right transversal maps, respectively. The *left gyration* $\mathrm{lgyr}[b_1, b_2]$ *of B generated by* $b_1, b_2 \in B$ is defined by

$$\mathrm{lgyr}[b_1, b_2]b = h_r(b_1, b_2)bh_r(b_1, b_2)^{-1}, \quad b \in B. \tag{9}$$

The *right gyration* $\mathrm{rgyr}[b_1, b_2]$ *of B generated by* $b_1, b_2 \in B$ is defined by

$$\mathrm{rgyr}[b_1, b_2]b = h_\ell(b_1, b_2)^{-1}bh_\ell(b_1, b_2), \quad b \in B. \tag{10}$$

Remark 1. In Definition 3.2, left gyrations are associated with the right transversal map h_r, and right gyrations are associated with the left transversal map h_ℓ.

We use the convenient notation $x^h = hxh^{-1}$ and denote *conjugation by h* by α_h. That is, $\alpha_h(x) = x^h = hxh^{-1}$. With this notation, the left and right gyrations in Definition 3.2 read

$$\mathrm{lgyr}[a, b] = \alpha_{h_r(a,b)} \quad \text{and} \quad \mathrm{rgyr}[a, b] = \alpha_{h_\ell(a,b)^{-1}} \tag{11}$$

for all $a, b \in B$. Let B be a non-empty subset of a group Γ. We say that a subgroup H of Γ *normalizes* B if $hBh^{-1} \subseteq B$ for all $h \in H$.

Definition 3.3 (Bi-gyrotransversal). A bi-transversal B of subgroups H_L and H_R in a group Γ is a *bi-gyrotransversal* if

1. H_L and H_R normalize B, and

2. $h_\ell h_r = h_r h_\ell$ for all $h_\ell \in H_L$, $h_r \in H_R$.

Proposition 3.4. *If B is a bi-gyrotransversal of subgroups H_L and H_R in a group Γ, then H_LH_R is a subgroup of Γ with normal subgroups H_L and H_R. If B contains the identity* 1 *of Γ, then $H_L \cap H_R = \{1\}$. In this case, H_LH_R is isomorphic to the direct product $H_L \times H_R$ as groups.*

Proof. Since $H_LH_R = H_RH_L$, H_LH_R forms a subgroup of Γ by Proposition 14 of [5, Chapter 3]. If $g \in H_LH_R$, then $g = h_\ell h_r$ for some $h_\ell \in H_L$ and $h_r \in H_R$. For any $h \in H_L$, $h_r h = h h_r$ implies $ghg^{-1} = h_\ell h h_\ell^{-1} \in H_L$. Hence, $gH_Lg^{-1} \subseteq H_L$. This proves $H_L \trianglelefteq H_LH_R$. Similarly, $H_R \trianglelefteq H_LH_R$.

Suppose that $1 \in B$ and let $h \in H_L \cap H_R$. The unique decomposition of 1, $1 = hh^{-1} = h1h^{-1}$, implies $h = 1$. Hence, $H_L \cap H_R = \{1\}$. It follows from Theorem 9 of [5, Chapter 5] that $H_LH_R \cong H_L \times H_R$ as groups. □

Theorem 3.5. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If $h \in H_LH_R$, then conjugation by h is an automorphism of $(B, \odot)$.*

Proof. Note first that H_LH_R normalizes B. In fact, if $h = h_\ell h_r$ with h_ℓ in H_L and h_r in H_R, then $hBh^{-1} = h_\ell(h_r B h_r^{-1})h_\ell^{-1} \subseteq B$ for H_R and H_L normalize B.

Let $h \in H_LH_R$. Since H_LH_R normalizes B, α_h is a bijection from B to itself. Next, we will show that $(x \odot y)^h = x^h \odot y^h$ for all $x, y \in B$. Employing (8), we have

$$(xy)^h = (h_\ell(x, y)(x \odot y)h_r(x, y))^h = h_\ell(x, y)^h(x \odot y)^h h_r(x, y)^h.$$

Since $x^h, y^h \in B$, we also have

$$x^h y^h = h_\ell(x^h, y^h)(x^h \odot y^h)h_r(x^h, y^h).$$

Note that $h_\ell(x, y)^h \in H_L$ and $h_r(x, y)^h \in H_R$ because H_L and H_R are normal in H_LH_R. Thus, $(xy)^h = x^h y^h$ implies

$$h_\ell(x, y)^h = h_\ell(x^h, y^h), \quad (x \odot y)^h = x^h \odot y^h, \quad \text{and} \quad h_r(x, y)^h = h_r(x^h, y^h),$$

which completes the proof. □

Corollary 3.6. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. Then $\mathrm{lgyr}[a, b]$ and $\mathrm{rgyr}[a, b]$ are automorphisms of $(B, \odot)$ for all $a, b \in B$.*

Proof. This is because $\mathrm{lgyr}[a, b] = \alpha_{h_r(a,b)}$ and $\mathrm{rgyr}[a, b] = \alpha_{h_\ell(a,b)^{-1}}$. □

The next theorem provides us with *commuting relations* between conjugation automorphisms of the bi-transversal groupoid $(B, \odot)$ and its bi-gyrations.

Theorem 3.7. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. The following commuting relations hold.*

1. $\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[c, d] = \mathrm{rgyr}[c, d] \circ \mathrm{lgyr}[a, b]$ *for all* $a, b, c, d \in B$.
2. $\alpha_h \circ \mathrm{lgyr}[a, b] = \mathrm{lgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h$ *for all* $h \in H_LH_R$ *and* $a, b \in B$.
3. $\alpha_h \circ \mathrm{rgyr}[a, b] = \mathrm{rgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h$ *for all* $h \in H_LH_R$ *and* $a, b \in B$.

Proof. Item (1) follows from the fact that $h_\ell h_r = h_r h_\ell$ for all $h_\ell \in H_L$ and $h_r \in H_R$ and that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $g, h \in \Gamma$.

Let $h \in H_L H_R$ and let $a, b \in B$. As in the proof of Theorem 3.5, $h_r(a,b)^h = h_r(a^h, b^h)$. Hence, $\alpha_h \circ \mathrm{lgyr}[a,b] \circ \alpha_h^{-1} = \mathrm{lgyr}[a^h, b^h]$ and Item (2) follows. Similarly, $h_\ell(a,b)^h = h_\ell(a^h, b^h)$ implies Item (3). □

As a consequence of Theorem 3.7, left gyrations are invariant under right gyrations, and vice versa. In fact, we have the following two theorems.

Theorem 3.8. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If ρ is a finite composition of right gyrations of B, then*

$$\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\rho(a), \rho(b)] \tag{12}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\lambda(a), \lambda(b)] \tag{13}$$

for all $a, b \in B$.

Proof. By assumption, $\rho = \mathrm{rgyr}[a_1, b_1] \circ \mathrm{rgyr}[a_2, b_2] \circ \cdots \circ \mathrm{rgyr}[a_n, b_n]$ for some $a_i, b_i \in B$. Since $\mathrm{rgyr}[a_i, b_i] = \alpha_{h_\ell(a_i,b_i)^{-1}}$ for all i, it follows that $\rho = \alpha_h$, where $h = h_\ell(a_1,b_1)^{-1} h_\ell(a_2,b_2)^{-1} \cdots h_\ell(a_n,b_n)^{-1}$. As $\rho = \alpha_h$ and $h \in H_L$, Theorem 3.7 (2) implies $\rho \circ \mathrm{lgyr}[a,b] = \mathrm{lgyr}[\rho(a), \rho(b)] \circ \rho$. Since ρ and $\mathrm{lgyr}[a,b]$ commute, we have (12). One obtains similarly that $\lambda = \alpha_h$ for some $h \in H_R$, which implies (13) by Theorem 3.7 (3). □

Theorem 3.9. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If ρ is a finite composition of right gyrations of B, then*

$$\rho \circ \mathrm{rgyr}[a,b] = \mathrm{rgyr}[\rho(a), \rho(b)] \circ \rho \tag{14}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\lambda \circ \mathrm{lgyr}[a,b] = \mathrm{lgyr}[\lambda(a), \lambda(b)] \circ \lambda \tag{15}$$

for all $a, b \in B$.

Proof. As in the proof of Theorem 3.8, $\rho = \alpha_h$ for some $h \in H_L$. Hence, (14) is an application of Theorem 3.7 (3). Similarly, (15) is an application of Theorem 3.7 (2). □

The associativity of Γ is reflected in its bi-gyrotransversal decomposition $\Gamma = H_L B H_R$, as shown in the following theorem.

Theorem 3.10. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. For all $a, b, c \in B$,*

$$(a \odot b) \odot \mathrm{lgyr}[a,b]c = \mathrm{rgyr}[b,c]a \odot (b \odot c).$$

Proof. Let $a, b, c \in B$. Set $a_r = \mathrm{rgyr}[b, c]a$ and $c_l = \mathrm{lgyr}[a, b]c$. Then $a_r \in B$ and $c_l \in B$. By employing (8),

$$\begin{aligned} a(bc) &= a(h_\ell(b, c)(b \odot c)h_r(b, c)) \\ &= h_\ell(b, c)(h_\ell(b, c)^{-1} a h_\ell(b, c))(b \odot c)h_r(b, c) \\ &= h_\ell(b, c)a_r(b \odot c)h_r(b, c) \\ &= [h_\ell(b, c)h_\ell(a_r, b \odot c)][a_r \odot (b \odot c)][h_r(a_r, b \odot c)h_r(b, c)] \end{aligned}$$

and, similarly, $(ab)c = [h_\ell(a, b)h_\ell(a \odot b, c_l)][(a \odot b) \odot c_l][h_r(a \odot b, c_l)h_r(a, b)]$. Since $a(bc) = (ab)c$, it follows that $(a \odot b) \odot c_l = a_r \odot (b \odot c)$, which was to be proved. □

Proposition 3.11. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. For all $a, b, c \in B$,*

1. $\mathrm{rgyr}[\mathrm{rgyr}[b, c]a, b \odot c] \circ \mathrm{rgyr}[b, c] = \mathrm{rgyr}[a \odot b, \mathrm{lgyr}[a, b]c] \circ \mathrm{rgyr}[a, b]$*, and*
2. $\mathrm{lgyr}[a \odot b, \mathrm{lgyr}[a, b]c] \circ \mathrm{lgyr}[a, b] = \mathrm{lgyr}[\mathrm{rgyr}[b, c]a, b \odot c] \circ \mathrm{lgyr}[b, c]$.

Proof. As we have computed in the proof of Theorem 3.10,

$$h_\ell(b, c)h_\ell(a_r, b \odot c) = h_\ell(a, b)h_\ell(a \odot b, c_l),$$

where $a_r = \mathrm{rgyr}[b, c]a$ and $c_l = \mathrm{lgyr}[a, b]c$. Thus, Item (1) is obtained. Similarly, $h_r(a_r, b \odot c)h_r(b, c) = h_r(a \odot b, c_l)h_r(a, b)$ gives Item (2). □

Twisted Subgroups

Twisted subgroups abound in group theory, gyrogroup theory, and loop theory, as evidenced, for instance, from [1–3, 6, 13, 14, 18]. Here, we demonstrate that a bi-gyrotransversal decomposition $\Gamma = H_L B H_R$ in which B is a twisted subgroup gives rise to a highly structured type of bi-gyrogroupoids and, eventually, a bi-gyrogroup. We follow Aschbacher for the definition of a twisted subgroup.

Definition 3.12 (Twisted subgroup). A subset B of a group Γ is a *twisted subgroup* of Γ if the following conditions hold:

1. $1 \in B$, 1 being the identity of Γ;
2. if $b \in B$, then $b^{-1} \in B$;
3. if $a, b \in B$, then $aba \in B$.

Theorem 3.13. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ, then the following properties are true for all $a, b \in B$.*

1. $1 \odot b = b \odot 1 = b$.
2. $b^{-1} \in B$ *and* $b^{-1} \odot b = b \odot b^{-1} = 1$.
3. $\mathrm{lgyr}[1, b] = \mathrm{lgyr}[b, 1] = \mathrm{rgyr}[1, b] = \mathrm{rgyr}[b, 1] = \mathrm{id}_B$.
4. $\mathrm{lgyr}[b^{-1}, b] = \mathrm{lgyr}[b, b^{-1}] = \mathrm{rgyr}[b^{-1}, b] = \mathrm{rgyr}[b, b^{-1}] = \mathrm{id}_B$.
5. $\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b^{-1}, a^{-1}]$ *and* $\mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b^{-1}, a^{-1}]$.
6. $(a \odot b)^{-1} = (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b])(b^{-1} \odot a^{-1})$.

Proof. (1) As $b = 1b = h_\ell(1, b)(1 \odot b)h_r(1, b)$, we have $h_\ell(1, b) = 1$, $1 \odot b = b$, and $h_r(1, b) = 1$. Similarly, $b = b1$ implies $b \odot 1 = b$.

(2) Let $b \in B$. Since B is a twisted subgroup, $b^{-1} \in B$. Further,

$$1 = b^{-1}b = h_\ell(b^{-1}, b)(b^{-1} \odot b)h_r(b^{-1}, b)$$

implies $h_\ell(b^{-1}, b) = 1$, $b^{-1} \odot b = 1$, and $h_r(b^{-1}, b) = 1$. Similarly, $bb^{-1} = 1$ implies $b \odot b^{-1} = 1$.

(3) We have $h_\ell(1, b) = h_\ell(b, 1) = h_r(1, b) = h_r(b, 1) = 1$, as computed in Item (1). Hence, Item (3) follows.

(4) We have $h_\ell(b^{-1}, b) = h_\ell(b, b^{-1}) = h_r(b^{-1}, b) = h_r(b, b^{-1}) = 1$, as computed in Item (2). Hence, Item (4) follows.

(5) Let $a, b \in B$. Then $a^{-1}, b^{-1} \in B$. On the one hand, we have

$$(ab)^{-1} = (h_\ell(a, b)(a \odot b)h_r(a, b))^{-1} = h_r(a, b)^{-1}(a \odot b)^{-1}h_\ell(a, b)^{-1},$$

and on the other hand we have $b^{-1}a^{-1} = h_\ell(b^{-1}, a^{-1})(b^{-1} \odot a^{-1})h_r(b^{-1}, a^{-1})$. Since $(ab)^{-1} = b^{-1}a^{-1}$, it follows that

$$\begin{aligned}(a \odot b)^{-1} &= h_r(a, b)h_\ell(b^{-1}, a^{-1})(b^{-1} \odot a^{-1})h_r(b^{-1}, a^{-1})h_\ell(a, b)\\ &= h_\ell(b^{-1}, a^{-1})h_r(a, b)(b^{-1} \odot a^{-1})h_\ell(a, b)h_r(b^{-1}, a^{-1})\\ &= h_\ell(b^{-1}, a^{-1})h_\ell(a, b)\tilde{b}h_r(a, b)h_r(b^{-1}, a^{-1}),\end{aligned} \tag{16}$$

where $\tilde{b} = \mathrm{lgyr}[a, b](\mathrm{rgyr}[a, b](b^{-1} \odot a^{-1}))$. Because $(a \odot b)^{-1}$ and $\tilde{b}$ belong to B, we have from the extreme sides of (16) that

$$h_r(a, b)h_r(b^{-1}, a^{-1}) = 1 \quad \text{and} \quad h_\ell(b^{-1}, a^{-1})h_\ell(a, b) = 1.$$

Hence, $h_r(a, b)^{-1} = h_r(b^{-1}, a^{-1})$, which implies $\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b^{-1}, a^{-1}]$. Likewise, $h_\ell(a, b) = h_\ell(b^{-1}, a^{-1})^{-1}$ implies $\mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b^{-1}, a^{-1}]$.

(6) As in Item (5), $(a \odot b)^{-1} = \tilde{b} = \mathrm{lgyr}[a, b](\mathrm{rgyr}[a, b](b^{-1} \odot a^{-1}))$. □

Remark 2. Note that we do not invoke the third defining property of a twisted subgroup in proving Theorem 3.13.

At this point, we have shown that any bi-gyrotransversal decomposition $\Gamma = H_L B H_R$ in which B is a twisted subgroup of Γ gives the bi-transversal groupoid B that satisfies all the axioms of a bi-gyrogroupoid except for (BG4). In order to complete this, we have to impose additional conditions on the left and right transversal maps, as the following lemma indicates.

Lemma 3.14. *If B is a bi-transversal of subgroups H_L and H_R in a group Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a, b \in B$, then*

$$\mathrm{lgyr}^{-1}[a,b] = \mathrm{lgyr}[b,a] \quad \textit{and} \quad \mathrm{rgyr}^{-1}[a,b] = \mathrm{rgyr}[b,a]$$

for all $a, b \in B$.

Proof. Note first that $\alpha_h^{-1} = \alpha_{h^{-1}}$ for all $h \in \Gamma$. From this we have $\mathrm{lgyr}[b,a] = \alpha_{h_r(b,a)} = \alpha_{h_r(a,b)^{-1}} = \alpha_{h_r(a,b)}^{-1} = \mathrm{lgyr}^{-1}[a,b]$. One can prove in a similar way that $\mathrm{rgyr}^{-1}[a,b] = \mathrm{rgyr}[b,a]$. □

Theorem 3.15. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a, b \in B$, then the following relations hold for all $a, b \in B$:*

1. $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$;
2. $\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$;
3. $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{rgyr}[b,a]a, b \odot a]$;
4. $\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{rgyr}[b,a]a, b \odot a]$.

Proof. Let $a, b \in B$. Set $a_l = \mathrm{lgyr}[a,b]a$. Employing (8), we obtain

$$\begin{aligned}(ab)a &= (h_\ell(a,b)(a \odot b)h_r(a,b))a \\ &= h_\ell(a,b)(a \odot b)a_l h_r(a,b) \\ &= [h_\ell(a,b)h_\ell(a \odot b, a_l)][(a \odot b) \odot a_l][h_r(a \odot b, a_l)h_r(a,b)].\end{aligned} \tag{17}$$

Since $(ab)a \in B$, the extreme sides of (17) imply

$$h_\ell(a,b)h_\ell(a \odot b, a_l) = 1 \quad \text{and} \quad h_r(a \odot b, a_l)h_r(a,b) = 1. \tag{18}$$

The first equation of (18) implies $h_\ell(a \odot b, \mathrm{lgyr}[a,b]a) = h_\ell(a,b)^{-1}$. Hence,

$$\mathrm{rgyr}^{-1}[a \odot b, \mathrm{lgyr}[a,b]a] = \mathrm{rgyr}[a,b].$$

From Lemma 3.14, we have $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$. The second equation of (18) implies $h_r(a,b) = h_r(a \odot b, \mathrm{lgyr}[a,b]a)^{-1}$. Hence,

$$\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{lgyr}[a,b]a, a \odot b].$$

This proves Items (1) and (2). Items (3) and (4) can be proved in a similar way by computing the product $a(ba)$. □

Theorem 3.16. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a,b \in B$, then left and right gyrations of B are even in the sense that*

$$\operatorname{lgyr}[a^{-1}, b^{-1}] = \operatorname{lgyr}[a,b] \quad \textit{and} \quad \operatorname{rgyr}[a^{-1}, b^{-1}] = \operatorname{rgyr}[a,b]$$

for all $a, b \in B$.

Proof. This theorem follows directly from Theorem 3.13 (5) and Lemma 3.14. □

4. Bi-Gyrodecomposition and Bi-Gyrogroups

Taking the key features of bi-gyrotransversal decomposition of a group given in Section 3, we formulate the definition of bi-gyrodecomposition and show that any bi-gyrodecomposition leads to a bi-gyrogroup, which in turn is a gyrogroup. Most of the results in Section 3 are directly translated into results in this section with appropriate modifications.

Definition 4.1 (Bi-gyrodecomposition). Let Γ be a group, let B be a subset of Γ, and let H_L and H_R be subgroups of Γ. A decomposition $\Gamma = H_L B H_R$ is a *bi-gyrodecomposition* if

1. B is a bi-gyrotransversal of H_L and H_R in Γ;
2. B is a twisted subgroup of Γ; and
3. $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a, b \in B$,

where h_ℓ and h_r are the bi-transversal maps given below Definition 3.1.

Theorem 4.2. *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then B equipped with the bi-transversal operation forms a bi-gyrogroupoid.*

Proof. Axiom (BG1) holds by Theorem 3.13 (1), where the identity 1 of Γ acts as the identity of B. Axiom (BG2) holds by Theorem 3.13 (2), where b^{-1} acts as a left inverse of $b \in B$ with respect to the bi-transversal operation. Axiom (BG3) holds by Corollary 3.6 and Theorem 3.10. Axiom (BG4) holds by Theorem 3.15. Axiom (BG5) holds by Theorem 3.13 (3). □

It is shown in Section 3 that any bi-transversal decomposition $\Gamma = H_L B H_R$ gives rise to a bi-transversal groupoid $(B, \odot)$. Theorem 4.2 asserts that in the special case when the decomposition is a bi-gyrodecomposition, the bi-transversal groupoid $(B, \odot)$ becomes the bi-gyrogroupoid $(B, \oplus_b)$ described in Definition 2.1. Hence, in particular, the binary operations $\oplus_b$ and $\odot$ share the same algebraic properties. Further, the identity of the bi-gyrogroupoid B coincides with the group identity of Γ and $\ominus_b b = b^{-1}$ for all $b \in B$.

Theorem 4.3 (Bi-gyration invariant relation). *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition. If ρ is a finite composition of right gyrations of B, then*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[\rho(a), \rho(b)] \tag{19}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[\lambda(a), \lambda(b)] \tag{20}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.8. □

Theorem 4.4 (Bi-gyration commuting relation). *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition. If ρ is a finite composition of right gyrations of B, then*

$$\rho \circ \mathrm{rgyr}[a, b] = \mathrm{rgyr}[\rho(a), \rho(b)] \circ \rho \tag{21}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\lambda \circ \mathrm{lgyr}[a, b] = \mathrm{lgyr}[\lambda(a), \lambda(b)] \circ \lambda \tag{22}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.9. □

Theorem 4.5 (Trivial bi-gyration). *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then for all $a \in B$,*

$$\begin{aligned} \mathrm{lgyr}[0, a] = \mathrm{lgyr}[a, 0] &= \mathrm{id}_B \\ \mathrm{lgyr}[a, \ominus_b a] = \mathrm{lgyr}[\ominus_b a, a] &= \mathrm{id}_B \\ \mathrm{rgyr}[0, a] = \mathrm{rgyr}[a, 0] &= \mathrm{id}_B \\ \mathrm{rgyr}[a, \ominus_b a] = \mathrm{rgyr}[\ominus_b a, a] &= \mathrm{id}_B \\ \mathrm{lgyr}[a, a] = \mathrm{rgyr}[a, a] &= \mathrm{id}_B. \end{aligned} \tag{23}$$

Proof. The theorem follows from Theorem 2.4 (2) and Theorem 3.13 (3)–(4). □

Theorem 4.6 (Bi-gyration inversion law). *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then*

$$\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b, a] \quad \textit{and} \quad \mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b, a]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Lemma 3.14. □

Theorem 4.7 (Even bi-gyration). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then left and right gyrations of* B *are even:*

$$\mathrm{lgyr}[\ominus_b a, \ominus_b b] = \mathrm{lgyr}[a, b] \quad \textit{and} \quad \mathrm{rgyr}[\ominus_b a, \ominus_b b] = \mathrm{rgyr}[a, b]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.16. □

Theorem 4.8 (Left and right cancellation laws). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then* B *satisfies the left cancellation law*

$$\ominus_b \mathrm{rgyr}[a, b]a \oplus_b (a \oplus_b b) = b \tag{24}$$

and the right cancellation law

$$(a \oplus_b b) \ominus_b \mathrm{lgyr}[a, b]b = a \tag{25}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 2.7. □

Theorem 4.9 (Left and right bi-gyroassociative laws). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then* B *satisfies the left bi-gyroassociative law*

$$a \oplus_b (b \oplus_b c) = (\mathrm{rgyr}[c, b]a \oplus_b b) \oplus_b \mathrm{lgyr}[\mathrm{rgyr}[c, b]a, b]c \tag{26}$$

and the right bi-gyroassociative law

$$(a \oplus_b b) \oplus_b c = \mathrm{rgyr}[b, \mathrm{lgyr}[b, a]c]a \oplus_b (b \oplus_b \mathrm{lgyr}[b, a]c) \tag{27}$$

for all $a, b, c \in B$.

Proof. The theorem follows from Theorems 2.2 and 4.6. □

Theorem 4.10 (Left gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[\mathrm{rgyr}[b, a]a, b \oplus_b a] \tag{28}$$

and

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b] \tag{29}$$

for all $a, b \in B$.

Proof. Identity (28) follows from Theorem 3.15 (4). Identity (29) is obtained from (28) by applying the bi-gyration inversion law (Theorem 4.6) followed by interchanging a and b. □

Theorem 4.11 (Right gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[\mathrm{lgyr}[a, b]a, a \oplus_b b] \tag{30}$$

and

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[b \oplus_b a, \mathrm{lgyr}[b, a]b] \tag{31}$$

for all $a, b \in B$.

Proof. Identity (30) follows from Theorem 3.15 (1). Identity (31) is obtained from (30) by applying the bi-gyration inversion law followed by interchanging a and b. □

Theorem 4.12 (Bi-gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[\mathrm{lgyr}[a, b]a, a \oplus_b b] \tag{32}$$

and

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b] \tag{33}$$

for all $a, b \in B$.

Proof. Identity (32) follows from Theorem 3.15 (2). Identity (33) is obtained from Theorem 3.15 (3) by applying the bi-gyration inversion law followed by interchanging a and b. □

Theorem 4.13 (Left and right gyration reduction properties). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\begin{aligned}\mathrm{rgyr}[a, b] &= \mathrm{rgyr}[\ominus_b \mathrm{lgyr}[a, b]b, a \oplus_b b]\\ \mathrm{lgyr}[a, b] &= \mathrm{lgyr}[\ominus_b \mathrm{lgyr}[a, b]b, a \oplus_b b]\end{aligned} \tag{34}$$

and

$$\begin{aligned}\mathrm{rgyr}[a, b] &= \mathrm{rgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a]\\ \mathrm{lgyr}[a, b] &= \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a]\end{aligned} \tag{35}$$

for all $a, b \in B$.

Proof. Setting $c = \ominus_b b$ in Proposition 3.11 (1)–(2) followed by using the bi-gyration inversion law gives (34). Setting $a = \ominus_b b$ in the same proposition followed by using the bi-gyration inversion law gives

$$\begin{aligned}\mathrm{rgyr}[b, c] &= \mathrm{rgyr}[b \oplus_b c, \ominus_b \mathrm{rgyr}[b, c]b]\\ \mathrm{lgyr}[b, c] &= \mathrm{lgyr}[b \oplus_b c, \ominus_b \mathrm{rgyr}[b, c]b].\end{aligned}$$

Replacing b by a and c by b, we obtain (35). □

Theorem 4.14 (Left and right gyration reduction properties). *If* $\Gamma = H_LBH_R$ *is a bi-gyrodecomposition, then*

$$\begin{aligned} \mathrm{lgyr}[a,b] &= \mathrm{lgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \ominus_b a] \\ \mathrm{rgyr}[a,b] &= \mathrm{rgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \ominus_b a] \end{aligned} \tag{36}$$

for all $a, b \in B$.

Proof. From the second equation of (35), we have

$$\mathrm{lgyr}[a,b] = \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a,b]a].$$

Applying Theorem 4.3 to the previous equation with $\rho = \mathrm{rgyr}[b,a]$ gives

$$\begin{aligned} \mathrm{lgyr}[a,b] &= \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a,b]a] \\ &= \mathrm{lgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \mathrm{rgyr}[b,a](\ominus_b \mathrm{rgyr}[a,b]a)] \\ &= \mathrm{lgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \ominus_b a]. \end{aligned}$$

We obtain the last equation since $\mathrm{rgyr}[b,a] = \mathrm{rgyr}^{-1}[a,b]$. Similarly, the first equation of (35) and Identity (21) together imply

$$\begin{aligned} \mathrm{id}_B &= \mathrm{rgyr}^{-1}[a,b] \circ \mathrm{rgyr}[a \oplus_b, \ominus_b \mathrm{rgyr}[a,b]a] \\ &= \mathrm{rgyr}[b,a] \circ \mathrm{rgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a,b]a] \\ &= \mathrm{rgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \mathrm{rgyr}[b,a](\ominus_b \mathrm{rgyr}[a,b]a)] \circ \mathrm{rgyr}[b,a] \\ &= \mathrm{rgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \ominus_b a] \circ \mathrm{rgyr}[b,a]. \end{aligned} \tag{37}$$

The extreme sides of (37) imply $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{rgyr}[b,a](a \oplus_b b), \ominus_b a]$. □

Bi-Gyrogroups

We are now in a position to present the formal definition of a bi-gyrogroup.

Definition 4.15 (Bi-gyrogroup). Let $\Gamma = H_LBH_R$ be a bi-gyrodecomposition. The *bi-gyrogroup operation* $\oplus$ in B is defined by

$$a \oplus b = \mathrm{rgyr}[b,a](a \oplus_b b), \quad a, b \in B. \tag{38}$$

Here, $\oplus_b$ is the bi-transversal operation induced by the decomposition $\Gamma = H_LBH_R$. The groupoid $(B, \oplus)$ consisting of the set B and the bi-gyrogroup operation $\oplus$ is called a *bi-gyrogroup*.

Throughout the remaining of this section, we assume that $\Gamma = H_LBH_R$ is a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup.

Proposition 4.16. *The unique two-sided identity element of* $(B, \oplus)$ *is* 0. *For each* $a \in B$, $\ominus_b a$ *is the unique two-sided inverse of* a *in* $(B, \oplus)$.

Proof. Let $a \in B$. Since $\operatorname{rgyr}[a, 0] = \operatorname{rgyr}[0, a] = \operatorname{id}_B$, we have

$$a \oplus 0 = \operatorname{rgyr}[0, a](a \oplus_b 0) = a = \operatorname{rgyr}[a, 0](0 \oplus_b a) = (0 \oplus a).$$

Hence, 0 is a two-sided identity of $(B, \oplus)$. The uniqueness of 0 follows, as in the proof of Lemma 2.3. Since $\operatorname{rgyr}[a, \ominus_b a] = \operatorname{rgyr}[\ominus_b a, a] = \operatorname{id}_B$, we have

$$a \oplus (\ominus_b a) = \operatorname{rgyr}[\ominus_b a, a](a \ominus_b a) = 0 = \operatorname{rgyr}[a, \ominus_b a](\ominus_b a \oplus_b a) = (\ominus_b a) \oplus a.$$

Hence, $\ominus_b a$ acts as a two-sided inverse of a with respect to $\oplus$. Suppose that b is a two-sided inverse of a with respect to $\oplus$. Then $0 = a \oplus b = \operatorname{rgyr}[b, a](a \oplus_b b)$, which implies $a \oplus_b b = 0$. Similarly, $b \oplus a = 0$ implies $b \oplus_b a = 0$. This proves that b is a two-sided inverse of a with respect to $\oplus_b$. Hence, $b = \ominus_b a$ by Theorem 2.5. □

Following Proposition 4.16, if a is an element of B, then the unique two-sided inverse of a with respect to $\oplus$ will be denoted by $\ominus a$. Further,

$$\ominus a = \ominus_b a$$

for all $a \in B$. We also write $a \ominus b$ instead of $a \oplus (\ominus b)$. The following theorem asserts that left and right gyrations of the bi-transversal groupoid $(B, \oplus_b)$ ascend to automorphisms of the bi-gyrogroup $(B, \oplus)$.

Theorem 4.17. *If λ is a finite composition of left gyrations of $(B, \oplus_b)$, then*

$$\lambda(a \oplus b) = \lambda(a) \oplus \lambda(b) \tag{39}$$

for all $a, b \in B$. If ρ is a finite composition of right gyrations of $(B, \oplus_b)$, then

$$\rho(a \oplus b) = \rho(a) \oplus \rho(b) \tag{40}$$

for all $a, b \in B$.

Proof. Let $a, b \in B$. By Theorem 3.7 (1), λ and $\operatorname{rgyr}[b, a]$ commute. Hence,

$$\begin{aligned}\lambda(a \oplus b) &= (\lambda \circ \operatorname{rgyr}[b, a])(a \oplus_b b)\\ &= (\operatorname{rgyr}[b, a] \circ \lambda)(a \oplus_b b)\\ &= \operatorname{rgyr}[b, a](\lambda(a) \oplus_b \lambda(b))\\ &= \operatorname{rgyr}[\lambda(b), \lambda(a)](\lambda(a) \oplus_b \lambda(b))\\ &= \lambda(a) \oplus \lambda(b).\end{aligned}$$

We have the third equation since λ is a finite composition of left gyrations; the forth equation from (20); and the last equation from Definition 4.15. Similarly, (40) is obtained from (21). □

Lemma 4.18. *In the bi-gyrogroup B,*

$$\text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a] = \text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b]$$

for all $a, b, c \in B$.

Proof. By Theorem 4.6 and Proposition 3.11 (1),

$$\begin{aligned} &\text{rgyr}[b, a] \circ \text{rgyr}[\text{lgyr}[a, b]c, a \oplus_b b] \\ &\quad = (\text{rgyr}[a \oplus_b b, \text{lgyr}[a, b]c] \circ \text{rgyr}[a, b])^{-1} \\ &\quad = (\text{rgyr}[\text{rgyr}[b, c]a, b \oplus_b c] \circ \text{rgyr}[b, c])^{-1} \\ &\quad = \text{rgyr}[c, b] \circ \text{rgyr}[b \oplus_b c, \text{rgyr}[b, c]a]. \end{aligned} \tag{41}$$

By Identity (21) and Theorem 4.6, the extreme sides of (41) imply

$$\text{rgyr}[c, \text{rgyr}[b, a](a \oplus_b b)] \circ \text{rgyr}[b, a] = \text{rgyr}[\text{rgyr}[c, b](b \oplus_b c), a] \circ \text{rgyr}[c, b].$$

According to Definition 4.15, the previous equation reads

$$\text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a] = \text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b],$$

which completes the proof. □

Theorem 4.19 (Bi-gyroassociative law in bi-gyrogroups). *The bi-gyrogroup B satisfies the left bi-gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(c) \tag{42}$$

and the right bi-gyroassociative law

$$(a \oplus b) \oplus c = a \oplus (b \oplus (\text{lgyr}[b, a] \circ \text{rgyr}[a, b])(c)) \tag{43}$$

for all $a, b, c \in B$.

Proof. From Theorem 3.10, we have

$$(a \oplus_b b) \oplus_b \text{lgyr}[a, b]c = \text{rgyr}[b, c]a \oplus_b (b \oplus_b c).$$

Applying $\text{rgyr}[c, b]$ followed by applying $\text{rgyr}[b \oplus c, a]$ to the previous equation gives

$$(\text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b])((a \oplus_b b) \oplus_b \text{lgyr}[a, b]c) = a \oplus (b \oplus c). \tag{44}$$

On the other hand, we compute

$$\begin{aligned} &(a \oplus b) \oplus (\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(c) \\ &\quad = (a \oplus b) \oplus (\text{rgyr}[b, a] \circ \text{lgyr}[a, b])(c) \\ &\quad = [\text{rgyr}[b, a](a \oplus_b b)] \oplus [\text{rgyr}[b, a](\text{lgyr}[a, b]c)] \\ &\quad = \text{rgyr}[b, a]((a \oplus_b b) \oplus \text{lgyr}[a, b]c) \\ &\quad = (\text{rgyr}[b, a] \circ \text{rgyr}[\text{lgyr}[a, b]c, a \oplus_b b])((a \oplus_b b) \oplus_b \text{lgyr}[a, b]c) \\ &\quad = (\text{rgyr}[c, \text{rgyr}[b, a](a \oplus_b b)] \circ \text{rgyr}[b, a])((a \oplus_b b) \oplus_b \text{lgyr}[a, b]c) \\ &\quad = (\text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a])((a \oplus_b b) \oplus_b \text{lgyr}[a, b]c). \end{aligned} \tag{45}$$

We obtain the first equation from Theorem 3.7 (1); the third equation from (40); the fifth equation from Identity (21) and Theorem 4.6.

By the lemma, $\mathrm{rgyr}[b \oplus c, a] \circ \mathrm{rgyr}[c, b] = \mathrm{rgyr}[c, a \oplus b] \circ \mathrm{rgyr}[b, a]$. Hence, (44) and (45) together imply $a \oplus (b \oplus c) = (a \oplus b) \oplus (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a])(c)$. Replacing c by $(\mathrm{lgyr}[b, a] \circ \mathrm{rgyr}[a, b])(c)$ in (42) followed by commuting $\mathrm{lgyr}[b, a]$ and $\mathrm{rgyr}[a, b]$ gives (43). □

Theorem 4.20 (Left gyration reduction property of bi-gyrogroups). *The bi-gyrogroup B has the left gyration left reduction property*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus b, b] \tag{46}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a, b \oplus a] \tag{47}$$

for all $a, b \in B$.

Proof. From (29), (19) with $\rho = \mathrm{rgyr}[b, a]$, and Theorem 4.6, we have the following series of equations

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b] \\ &= \mathrm{lgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\mathrm{rgyr}[a, b]b)] \\ &= \mathrm{lgyr}[a \oplus b, b],\end{aligned}$$

thus proving (46). One obtains similarly that

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[\mathrm{rgyr}[b, a]a, b \oplus_b a] \\ &= \mathrm{lgyr}[\mathrm{rgyr}[a, b](\mathrm{rgyr}[b, a]a), \mathrm{rgyr}[a, b](b \oplus_b a)] \\ &= \mathrm{lgyr}[a, b \oplus a].\end{aligned}$$

□

Theorem 4.21 (Right gyration reduction property of bi-gyrogroups). *The bi-gyrogroup B satisfies the right gyration left reduction property*

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus b, b] \tag{48}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a, b \oplus a] \tag{49}$$

for all $a, b \in B$.

Proof. From (33), (21) with $\rho = \mathrm{rgyr}[b, a]$, and Theorem 4.6, we have the following series of equations

$$\begin{aligned}\mathrm{id}_B &= \mathrm{rgyr}[b, a] \circ \mathrm{rgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b] \\ &= \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\mathrm{rgyr}[a, b]b)] \circ \mathrm{rgyr}[b, a] \\ &= \mathrm{rgyr}[a \oplus b, b] \circ \mathrm{rgyr}[b, a].\end{aligned} \tag{50}$$

Hence, the extreme sides of (50) imply $\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus b, b]$. Applying the bi-gyration inversion law to (48) followed by interchanging a and b gives (49). □

Let $(B, \oplus)$ be the corresponding bi-gyrogroup of a bi-gyrodecomposition $\Gamma = H_L B H_R$. By Theorem 4.17, left and right gyrations of $(B, \oplus_b)$ preserve the bi-gyrogroup operation. This result and Theorem 4.19 motivate the following definition.

Definition 4.22 (Gyration of bi-gyrogroups). Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup. The *gyrator* is the map

$$\mathrm{gyr}\colon B \times B \to \mathrm{Aut}\,(B, \oplus)$$

defined by

$$\mathrm{gyr}[a, b] = \mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a] \tag{51}$$

for all $a, b \in B$.

Theorem 4.23. *For all $a, b \in B$, $\mathrm{gyr}[a, b]$ is an automorphism of the bi-gyrogroup B.*

Proof. The theorem follows from Theorem 4.17. □

Theorem 4.24 (Gyroassociative law in bi-gyrogroups). *The bi-gyrogroup B satisfies the left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \tag{52}$$

and the right gyroassociative law

$$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c) \tag{53}$$

for all $a, b, c \in B$.

Proof. The theorem follows directly from Theorem 4.19 and Definition 4.22. □

Theorem 4.25 (Gyration reduction property in bi-gyrogroups). *The bi-gyrogroup B has the left reduction property*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b] \tag{54}$$

and the right reduction property

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a] \tag{55}$$

for all $a, b \in B$.

Proof. From (46) and (49), we have the following series of equations

$$\begin{aligned} \mathrm{gyr}[a \oplus b, b] &= \mathrm{lgyr}[a \oplus b, b] \circ \mathrm{rgyr}[b, a \oplus b] \\ &= \mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a] \\ &= \mathrm{gyr}[a, b], \end{aligned}$$

thus proving (54). Similarly, (47) and (48) together imply (55). □

Theorems 4.24 and 4.25 indicate that any bi-gyrogroup is indeed a gyrogroup. Therefore, we recall the following definition of a gyrogroup.

Definition 4.26 (Gyrogroup, [29]). A groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.

(G3) For all a, b in G, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$$

for all $c \in G$.

(G4) For all a, b in G, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

Definition 4.27 (Gyrocommutative gyrogroup, [29]). A gyrogroup $(G, \oplus)$ is *gyrocommutative* if it satisfies the gyrocommutative law

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$.

Theorem 4.28. *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup. Then B equipped with the bi-gyrogroup operation is a gyrogroup.*

Proof. Axioms (G1) and (G2) are validated in Proposition 4.16. Axiom (G3) is validated in Theorems 4.23 and 4.24. Axiom (G4) is validated in Theorem 4.25. □

Definition 4.29. A bi-gyrodecomposition $\Gamma = H_L B H_R$ is *bi-gyrocommutative* if its bi-transversal groupoid is bi-gyrocommutative in the sense of Definition 2.8.

Theorem 4.30. *If $\Gamma = H_L B H_R$ is a bi-gyrocommutative bi-gyrodecomposition, then B equipped with the bi-gyrogroup operation is a gyrocommutative gyrogroup.*

Proof. Let $a, b \in B$. We compute

$$\begin{aligned} a \oplus b &= \mathrm{rgyr}[b, a](a \oplus_b b) \\ &= \mathrm{rgyr}[b, a](\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b](b \oplus_b a)) \\ &= (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a])(\mathrm{rgyr}[a, b](b \oplus_b a)) \\ &= \mathrm{gyr}[a, b](b \oplus a), \end{aligned}$$

thus proving that B satisfies the gyrocommutative law. □

We close this section by proving that having a bi-gyrodecomposition is an invariant property of groups.

Theorem 4.31. *Let* Γ_1 *and* Γ_2 *be isomorphic groups via an isomorphism* ϕ*. If* $\Gamma_1 = H_L B H_R$ *is a bi-gyrodecomposition, then so is* $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_r)$*.*

Proof. The proof of this theorem is straightforward, using the fact that ϕ is a group isomorphism from Γ_1 to Γ_2. □

Theorem 4.32. *Let* Γ_1 *and* Γ_2 *be isomorphic groups via an isomorphism* ϕ*. If* $\Gamma_1 = H_L B H_R$ *is a bi-gyrocommutative bi-gyrodecomposition, then so is* $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_r)$*.*

Proof. This theorem follows from the fact that

$$\begin{aligned}\mathrm{rgyr}[\phi(b_1), \phi(b_2)]\phi(b) &= \phi(\mathrm{rgyr}[b_1, b_2]b)\\ \mathrm{lgyr}[\phi(b_1), \phi(b_2)]\phi(b) &= \phi(\mathrm{lgyr}[b_1, b_2]b)\end{aligned}$$

for all $b_1, b_2 \in B$. □

Theorem 4.33. *Let* Γ_1 *and* Γ_2 *be isomorphic groups via an isomorphism* ϕ *and let* $\Gamma_1 = H_L B H_R$ *be a bi-gyrodecomposition. Then the bi-gyrogroups* B *and* $\phi(B)$ *are isomorphic as gyrogroups via* ϕ*.*

Proof. By Theorem 4.28, B forms a gyrogroup whose gyrogroup operation is given by $a \oplus b = \mathrm{rgyr}[b, a](a \odot_1 b)$ for all $a, b \in B$, and $\phi(B)$ forms a gyrogroup whose gyrogroup operation is given by $c \oplus d = \mathrm{rgyr}[d, c](c \odot_2 d)$ for all $c, d \in \phi(B)$. Let $a, b \in B$. We compute

$$\begin{aligned}\phi(a \oplus b) &= \phi(\mathrm{rgyr}[b, a](a \odot_1 b))\\ &= \mathrm{rgyr}[\phi(b), \phi(a)]\phi(a \odot_1 b)\\ &= \mathrm{rgyr}[\phi(b), \phi(a)](\phi(a) \odot_2 \phi(b))\\ &= \phi(a) \oplus \phi(b).\end{aligned}$$

Hence, the restriction of ϕ to B acts as a gyrogroup isomorphism from B to $\phi(B)$. □

5. Special Pseudo-Orthogonal Groups

In this section, we provide a concrete realization of a bi-gyrocommutative bi-gyrodecomposition.

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature $(m, n), m, n \in \mathbb{N}$, is an $(m + n)$-dimensional linear space with the pseudo-Euclidean inner product of signature (m, n). The *special pseudo-orthogonal group*, denoted by $\mathrm{SO}(m, n)$, consists of all the Lorentz transformations of order (m, n) that leave the pseudo-Euclidean inner product invariant and that can be reached continuously from the identity transformation in $\mathbb{R}^{m,n}$. Denote by $\mathrm{SO}(m)$ the group of $m \times m$ special orthogonal matrices and by $\mathrm{SO}(n)$ the group of $n \times n$ special orthogonal matrices.

Following [34], $\mathrm{SO}(m)$ and $\mathrm{SO}(n)$ can be embedded into $\mathrm{SO}(m,n)$ as subgroups by defining

$$\rho\colon O_m \mapsto \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix}, \quad O_m \in \mathrm{SO}(m), \tag{56}$$

$$\lambda\colon O_n \mapsto \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix}, \quad O_n \in \mathrm{SO}(n). \tag{57}$$

Let β be the map defined on the space $\mathbb{R}^{n\times m}$ of all $n\times m$ real matrices by

$$\beta\colon P \mapsto \begin{pmatrix} \sqrt{I_m + P^{\mathrm{t}}P} & P^{\mathrm{t}} \\ P & \sqrt{I_n + PP^{\mathrm{t}}} \end{pmatrix}, \quad P \in \mathbb{R}^{n\times m}. \tag{58}$$

It is easy to see that β is a bijection from $\mathbb{R}^{n\times m}$ to $\beta(\mathbb{R}^{n\times m})$.

Note that

$$\rho(\mathrm{SO}(m)) = \left\{ \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} : O_m \in \mathrm{SO}(m) \right\}$$

$$\lambda(\mathrm{SO}(n)) = \left\{ \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} : O_n \in \mathrm{SO}(n) \right\}$$

$$\beta(\mathbb{R}^{n\times m}) = \left\{ \begin{pmatrix} \sqrt{I_m + P^{\mathrm{t}}P} & P^{\mathrm{t}} \\ P & \sqrt{I_n + PP^{\mathrm{t}}} \end{pmatrix} : P \in \mathbb{R}^{n\times m} \right\}.$$

It follows from Examples 22 and 23 of [34] that $\lambda(\mathrm{SO}(n))$ and $\rho(\mathrm{SO}(m))$ are subgroups of $\mathrm{SO}(m,n)$. Further, $\mathrm{SO}(m)$ and $\rho(\mathrm{SO}(m))$ are isomorphic as groups via ρ, and $\mathrm{SO}(n)$ and $\lambda(\mathrm{SO}(n))$ are isomorphic as groups via λ.

We will see shortly that

$$\mathrm{SO}(m,n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n))$$

is a bi-gyrocommutative bi-gyrodecomposition.

By Theorem 8 of [34], $\beta(\mathbb{R}^{n\times m})$ is a bi-transversal of subgroups $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ in the pseudo-orthogonal group $\mathrm{SO}(m,n)$. From Lemma 6 of [34], we have

$$\rho(O_m)\beta(P)\rho(O_m)^{-1} = \beta(PO_m^{-1})$$
$$\lambda(O_n)\beta(P)\lambda(O_n)^{-1} = \beta(O_nP)$$

for all $O_m \in \mathrm{SO}(m)$, $O_n \in \mathrm{SO}(n)$, and $P \in \mathbb{R}^{n\times m}$. Hence, $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ normalize $\beta(\mathbb{R}^{n\times m})$. Setting $P = 0_{n,m}$ in the third identity of (77) of [34], we have

$$\lambda(O_n)\rho(O_m) = \rho(O_m)\lambda(O_n)$$

for all $O_m \in \mathrm{SO}(m), O_n \in \mathrm{SO}(n)$ because $\beta(P) = \beta(0_{n,m}) = I_{m+n}$. Thus, $\beta(\mathbb{R}^{n\times m})$ is a bi-gyrotransversal of $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ in $\mathrm{SO}(m,n)$.

In Theorem 13 of [34], the bi-gyroaddition, $\oplus_U$, and bi-gyrations in the parameter bi-gyrogroupoid $\mathbb{R}^{n\times m}$ are given by

$$\begin{aligned}
P_1 \oplus_U P_2 &= P_1\sqrt{I_m + P_2^{\mathrm{t}}P_2} + \sqrt{I_n + P_1P_1^{\mathrm{t}}}P_2 \\
\mathrm{lgyr}[P_1,P_2] &= \sqrt{I_n + P_{1,2}P_{1,2}^{\mathrm{t}}}^{\,-1}\left\{P_1P_2^{\mathrm{t}} + \sqrt{I_n + P_1P_1^{\mathrm{t}}}\sqrt{I_n + P_2P_2^{\mathrm{t}}}\right\} \\
\mathrm{rgyr}[P_1,P_2] &= \left\{P_1^{\mathrm{t}}P_2 + \sqrt{I_m + P_1^{\mathrm{t}}P_1}\sqrt{I_m + P_2^{\mathrm{t}}P_2}\right\}\sqrt{I_m + P_{1,2}^{\mathrm{t}}P_{1,2}}^{\,-1}
\end{aligned}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$ and $P_{1,2} = P_1 \oplus_U P_2$.

From (74) of [34], we have $I_{m+n} = B(0_{n,m}) \in \beta(\mathbb{R}^{n\times m})$. From Theorem 10 of [34], we have $\beta(P)^{-1} = \beta(-P) \in \beta(\mathbb{R}^{n\times m})$ for all $P \in \mathbb{R}^{n\times m}$. From Equations (179) and (184) of [34], we have

$$\beta(P_1)\beta(P_2)\beta(P_1) = \beta((P_1 \oplus_U P_2) \oplus_U \mathrm{lgyr}[P_1,P_2]P_1).$$

Hence, $\beta(P_1)\beta(P_2)\beta(P_1) \in \beta(\mathbb{R}^{n\times m})$ for all $P_1, P_2 \in \mathbb{R}^{n\times m}$. This proves that $\beta(\mathbb{R}^{n\times m})$ is a twisted subgroup of $\mathrm{SO}(m,n)$.

By (104) of [34],

$$\beta(P_1)\beta(P_2) = \rho(\mathrm{rgyr}[P_1,P_2])\beta(P_1 \oplus_U P_2)\lambda(\mathrm{lgyr}[P_1,P_2]) \tag{59}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$. Hence, the left and right transversal maps induced by the decomposition $\mathrm{SO}(m,n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times})\lambda(\mathrm{SO}(n))$ are given by

$$h_\ell(\beta(P_1),\beta(P_2)) = \rho(\mathrm{rgyr}[P_1,P_2]) \tag{60}$$

and

$$h_r(\beta(P_1),\beta(P_2)) = \lambda(\mathrm{lgyr}[P_1,P_2]) \tag{61}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

By (162b) of [34], $\mathrm{rgyr}^{-1}[P_1,P_2] = \mathrm{rgyr}[P_2,P_1]$. Hence,

$$h_\ell(\beta(P_1),\beta(P_2))^{-1} = \rho(\mathrm{rgyr}^{-1}[P_1,P_2]) = \rho(\mathrm{rgyr}[P_2,P_1]) = h_\ell(\beta(P_2),\beta(P_1)).$$

Similarly, (162a) of [34] implies $h_r(\beta(P_1),\beta(P_2))^{-1} = h_r(\beta(P_2),\beta(P_1))$. Combining these results gives

Theorem 5.1. *The decomposition*

$$\mathrm{SO}(m,n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n)) \tag{62}$$

is a bi-gyrodecomposition.

By (59), the bi-transversal operation induced by the decomposition (62) is given by

$$\beta(P_1) \oplus_b \beta(P_2) = \beta(P_1 \oplus_U P_2) \tag{63}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

Note that $\mathrm{rgyr}[P_1, P_2]$ is an $m \times m$ matrix and $\mathrm{lgyr}[P_1, P_2]$ is an $n \times n$ matrix, while $\mathrm{rgyr}[\beta(P_1), \beta(P_2)]$ and $\mathrm{lgyr}[\beta(P_1), \beta(P_2)]$ are maps. By (11), the action of left and right gyrations on $\beta(\mathbb{R}^{n\times n})$ is given by

$$\mathrm{lgyr}[\beta(P_1), \beta(P_2)]\beta(P) = \beta(\mathrm{lgyr}[P_1, P_2]P) \tag{64}$$

and

$$\mathrm{rgyr}[\beta(P_1), \beta(P_2)]\beta(P) = \beta(P\mathrm{rgyr}[P_1, P_2]) \tag{65}$$

for all $P_1, P_2, P \in \mathbb{R}^{n\times m}$. Using (64) and (65), together with Theorem 25 of [34], we have

Theorem 5.2. *The bi-gyrodecomposition*

$$\mathrm{SO}(m, n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n))$$

is bi-gyrocommutative.

By Theorem 52 of [34], the space $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices forms a gyrocommutative gyrogroup under the operation $\oplus'_U$ given by

$$P_1 \oplus'_U P_2 = (P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1], \quad P_1, P_2 \in \mathbb{R}^{n\times m}. \tag{66}$$

Theorem 5.3. *The set*

$$\beta(\mathbb{R}^{n\times m}) = \left\{ \begin{pmatrix} \sqrt{I_m + P^tP} & P^t \\ P & \sqrt{I_n + PP^t} \end{pmatrix} : P \in \mathbb{R}^{n\times m} \right\}$$

together with the bi-gyrogroup operation $\oplus$ given by

$$\beta(P_1) \oplus \beta(P_2) = \beta((P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1])$$

is a gyrocommutative gyrogroup isomorphic to $(\mathbb{R}^{n\times m}, \oplus'_U)$.

Proof. The theorem follows from Theorems 5.1, 5.2, 4.28, and 4.30. Further, the bi-gyrogroup operation $\oplus$ is given by

$$\begin{aligned} \beta(P_1) \oplus \beta(P_2) &= \mathrm{rgyr}[\beta(P_2), \beta(P_1)](\beta(P_1) \oplus_b \beta(P_2)) \\ &= \mathrm{rgyr}[\beta(P_2), \beta(P_1)]\beta(P_1 \oplus_U P_2) \\ &= \beta((P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1]). \end{aligned}$$

From (66), we have $\beta(P_1) \oplus \beta(P_2) = \beta(P_1 \oplus'_U P_2)$. Hence, β acts as a gyrogroup isomorphism from $\mathbb{R}^{n\times m}$ to $\beta(\mathbb{R}^{n\times m})$. □

6. Spin Groups

We establish that the spin group of the Clifford algebra of pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n) has a bi-gyrocommutative bi-gyrodecomposition. For basic knowledge of Clifford algebras, the reader is referred to [15, 16, 19, 21].

Let (V, B) be a real quadratic space. That is, V is a linear space over $\mathbb{R}$, together with a non-degenerate symmetric bilinear form B. Let Q be the associated quadratic form given by $Q(v) = B(v, v)$ for $v \in V$. Denote by $\mathrm{C}\ell(V, Q)$ the *Clifford algebra* of (V, B). Set

$$\Gamma(V, Q) = \{g \in \mathrm{C}\ell^{\times}(V, Q) \colon \forall v \in V,\ \hat{g}vg^{-1} \in V\}. \tag{67}$$

Here, $\hat{}$ stands for the unique involutive automorphism of $\mathrm{C}\ell(V, Q)$ such that $\hat{v} = -v$ for all $v \in V$, known as the *grade involution*. If V is *finite* dimensional, then $\Gamma(V, Q)$ is indeed a subgroup of the group of units of $\mathrm{C}\ell(V, Q)$, called the *Clifford group of* $\mathrm{C}\ell(V, Q)$. In this case, any element g of $\Gamma(V, Q)$ induces the linear automorphism T_g of V given by

$$T_g(v) = \hat{g}vg^{-1}, \quad v \in V. \tag{68}$$

Since $T_g \circ T_h = T_{gh}$ for all $g, h \in \Gamma(V, Q)$, the map $\pi \colon g \mapsto T_g$ defines a group homomorphism from $\Gamma(V, Q)$ to the general linear group $\mathrm{GL}\,(V)$, known as the *twisted adjoint representation of* $\Gamma(V, Q)$. The kernel of π equals $\mathbb{R}^{\times}1 := \{\lambda 1 \colon \lambda \in \mathbb{R}, \lambda \neq 0\}$. By the Cartan-Dieudonné theorem, π maps $\Gamma(V, Q)$ onto the orthogonal group $\mathrm{O}(V, Q)$.

Recall that, in the Clifford algebra $\mathrm{C}\ell(V, Q)$, we have $v^2 = Q(v)$ for all $v \in V$. Hence, if $v \in V$ and $Q(v) \neq 0$, then v is invertible whose inverse is $v/Q(v)$. Further, we have an important identity $uv + vu = 2B(u, v)1$ for all $u, v \in V$. Using this identity, we obtain

$$-vuv^{-1} = u - (uv + vu)v^{-1} = u - (2B(u, v)1)\left(\frac{v}{Q(v)}\right) = u - \frac{2B(u, v)}{Q(v)}v,$$

which implies $\hat{v}uv^{-1} = -vuv^{-1} \in V$ for all $u \in V$. Hence, if $v \in V$ and $Q(v) \neq 0$, then $v \in \Gamma(V, Q)$. In fact, T_v is *the reflection about the hyperplane orthogonal to* v. We also have the following important subgroup of the Clifford group of $\mathrm{C}\ell(V, Q)$:

$$\mathrm{Spin}\,(V, Q) = \{v_1 v_2 \cdots v_r \colon r \text{ is even},\ v_i \in V,\ \text{and } Q(v_i) = \pm 1\}, \tag{69}$$

known as the *spin group of* $\mathrm{C}\ell(V, Q)$.

The following theorem is well known in the literature. Its proof can be found, for instance, in Theorem 2.9 of [19].

Theorem 6.1. *The restriction of the twisted adjoint representation to the spin group of* $\mathrm{C}\ell(V, Q)$ *is a surjective group homomorphism from* $\mathrm{Spin}\,(V, Q)$ *to the special orthogonal group* $\mathrm{SO}(V, Q)$ *of* V. *Its kernel is* $\{1, -1\}$.

Corollary 6.2. *The quotient group* $\mathrm{Spin}\,(V,Q)/\{1,-1\}$ *and the special orthogonal group* $\mathrm{SO}(V,Q)$ *are isomorphic.*

As V is a linear space over $\mathbb{R}$, we can choose an ordered basis for V so that

$$Q(v) = v_1^2 + v_2^2 + \cdots + v_m^2 - v_{m+1}^2 - v_{m+2}^2 - \cdots - v_{m+n}^2$$

for all $v = (v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}) \in \mathbb{R}^{m+n}$, [16, Theorem 4.5]. Hence, $\mathrm{SO}(V,Q) \equiv \mathrm{SO}(m,n)$ and $\mathrm{Spin}\,(V,Q) \equiv \mathrm{Spin}\,(m,n)$. Corollary 6.2 implies that

$$\mathrm{Spin}\,(m,n)/\{1,-1\} \cong \mathrm{SO}(m,n). \tag{70}$$

Hence, we have the following theorem.

Theorem 6.3. *The quotient group*

$$\mathrm{Spin}\,(m,n)/\{1,-1\}$$

has a bi-gyrocommutative bi-gyrodecomposition.

Proof. This theorem follows directly from (70) and Theorems 4.32 and 5.2. □

7. Conclusion

A gyrogroup is a non-associative group-like structure in which the non-associativity is controlled by a special family of automorphisms called gyrations. Gyrations, in turn, result from the extension by abstraction of the relativistic effect known as *Thomas precession*. In this paper we generalize the notion of gyrogroups, which involves a single family of gyrations, to that of bi-gyrogroups, which involves two distinct families of gyrations, collectively called bi-gyrations.

The bi-transversal decomposition $\Gamma = H_L B H_R$, studied in Section 3, naturally leads to a groupoid $(B, \odot)$ that comes with two families of automorphisms, left and right ones. This groupoid is related to the bi-gyrogroupoid $(B, \oplus_b)$, studied earlier in Section 2. Bi-gyrogroupoids $(B, \oplus_b)$ form an intermediate structure that suggestively leads to the desired bi-gyrogroup structure $(B, \oplus)$. The bi-transversal operation $\odot$ arises naturally from the bi-transversal decomposition (8). Under the natural conditions of Definition 4.1, the bi-transversal operation $\odot$ becomes the bi-gyrogroupoid operation $\oplus_b$. The latter operation leads to the desired bi-gyrogroup operation $\oplus$ by means of (38).

As we have shown in Section 4, any bi-gyrodecomposition $\Gamma = H_L B H_R$ of a group Γ induces the bi-gyrogroup structure on B, giving rise to a bi-gyrogroup $(B, \oplus)$ along with left gyrations $\mathrm{lgyr}[a, b]$ and right gyrations $\mathrm{rgyr}[a, b]$, $a, b \in B$. Further, in the case where H_L is the trivial subgroup of Γ, the bi-gyrodecomposition reduces to the decomposition $\Gamma = BH$ studied in [14]. The bi-gyrogroup $(B, \oplus)$ induced by a bi-gyrodecomposition of a group is indeed an abstract version of the bi-gyrogroup $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices studied in [34].

Bi-gyrogroups are group-like structures. For instance, they satisfy the bi-gyroassociative law (Theorem 4.19), which descends to the associative law if their left and right gyrations are the identity automorphism. A concrete realization of a bi-gyrogroup is found in the special pseudo-orthogonal group $\mathrm{SO}(m, n)$ of the pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n), as shown in [34] and in Section 5. Moreover, bi-gyrogroups arise in the group counterpart of Clifford algebras as we establish in Section 6 that the quotient group $\mathrm{Spin}\,(m, n)/\{1, -1\}$ of the spin group possesses a bi-gyrodecomposition.

By Theorem 4.28, any bi-gyrogroup is a gyrogroup. Yet, in general, the bi-gyrostructure of a bi-gyrogroup is richer than the gyrostructure of a gyrogroup. To see this clearly, we note that gyrations $\mathrm{gyr}[a, b]$ of a gyrogroup $(B, \oplus)$, $a, b \in B$, are completely determined by the gyrogroup operation according to the *gyrator identity* in Theorem 2.10 (10) of [29]:

$$\mathrm{gyr}[a, b]x = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) \tag{71}$$

for all a, b, x in the gyrogroup $(B, \oplus)$. In contrast, the *bi-gyrator identity* analogous to (71) is

$$(\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a])(x) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) \tag{72}$$

for all a, b, x in a bi-gyrogroup $(B, \oplus)$. Here, the bi-gyrogroup operation completely determines the composite automorphism $\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a]$. However, it does not determine straightforwardly each of the two automorphisms $\mathrm{lgyr}[a, b]$ and $\mathrm{rgyr}[a, b]$. Thus, the presence of two families of gyrations in a bi-gyrogroup, as opposed to the presence of a single family of gyrations in a gyrogroup, significantly enriches the bi-gyrostructure of bi-gyrogroups.

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