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Gyrovector Spaces on the Open Convex Cone of Positive Definite Matrices

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Abstract

In this article we review an algebraic definition of the gyrogroup and a simplified version of the gyrovector space with two fundamental examples on the open ball of finite-dimensional Euclidean spaces, which are the Einstein and Möbius gyrovector spaces. We introduce the structure of gyrovector space and the gyroline on the open convex cone of positive definite matrices and explore its interesting applications on the set of invertible density matrices. Finally we give an example of the gyrovector space on the unit ball of Hermitian matrices.

Keywords: Gyrogroup, gyrovector space, gyroline, gyromidpoint, positive definite matrix, density matrix.

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1. Introduction

In the theory of special relativity founded by Albert Einstein, the velocities are 3-dimensional vectors with speed bounded by the speed of light $s \approx 3 \times 10^8$ m/s, called the *admissible vectors.* The relativistic sum of two admissible vectors $\mathbf{u}$ and $\mathbf{v}$, called the *Einstein vector addition* is given by

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}^T\mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u} \right\}, \tag{1}$$

where $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}}$ is the well-known *Lorentz factor.* We denote as $\mathbf{u}^T\mathbf{v}$ the usual inner product in matrix form. To study abstractly the Einstein vector addi-

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tion in special relativity, A. Ungar [16] has introduced a group-like structure that he called a *gyrogroup* or *gyrocommutative gyrogroup*. Gyrogroups and gyrocommutative gyrogroups are equivalent to Bol-loops and K-loops (Bruck loops) [6, 14], respectively.

As a vector space is used in Euclidean geometry, a *gyrovector space* is a mathematical concept introduced by A. Ungar for studying hyperbolic geometry. We review in Section 2 the definitions of gyrogroup and gyrovector space with two important examples, the Einstein gyrovector space and Möbius gyrovector space. The axioms of gyrovector space in this article are more loose than those proposed by A. Ungar, but they also give a plenty of applications [4, 6]. For instance, the Einstein gyrovector space and Möbius gyrovector space provide the algebraic tool to study the Beltrami-Klein ball model and the Poincaré ball model of hyperbolic geometry, respectively. It has been proved that the Einstein and Möbius gyrovector spaces are isomorphic. See [7, 16] for more details.

In Section 3 we see examples of gyrovector space on the open convex cone of all positive definite matrices and on the set of all invertible density matrices. Furthermore, we show the isomorphism between the gyrovector space of all qubit invertible density matrices and the Einstein gyrovector space on the Bloch sphere, the open unit ball of $\mathbb{R}^3$. It generalizes the result in Theorem 3.4 of [3].

A *gyroline* uniquely determined by given two points on the gyrovector space plays an important role in the concepts of gyrocentroid and gyroparallelogram law. In Section 4 we discuss a gyroline on the open convex cone of all positive definite matrices and on the set of all invertible density matrices. Finally we give a different example of a gyrovector space on the open unit ball of all Hermitian matrices constructed by the exponential and logarithmic maps.

2. Gyrovector Spaces

We review in this section the algebraic structure of a gyrogroup as a natural extension of a group into the regime of the nonassociative algebra. We then introduce a gyrovector space providing the setting for hyperbolic geometry just as a vector space provides the setting for Euclidean geometry. A. A. Ungar has introduced and intensely studied them in a series of papers and books; see [16] and its bibliography.

The binary operation in a gyrogroup is not associative, in general. The breakdown of associativity for gyrogroup operations is salvaged in a modified form, called gyroassociativity. The axioms for a (gyrocommutative) gyrogroup G are reminiscent of those for a (commutative) group.

Definition 2.1. A binary system $(G, \oplus)$ is a *gyrogroup* if it satisfies the following for all $a, b, c \in G$:

(G1) $e \oplus a = a \oplus e = a$ (existence of identity);

(G2) $a \oplus (\ominus a) = (\ominus a) \oplus a = e$ (existence of inverses);

(G3) There is an automorphism $\mathrm{gyr}[a, b] : G \to G$ for each $a, b \in G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \quad \text{(gyroassociativity)};$$

(G4) $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$ (loop property).

A gyrogroup $(G, \oplus)$ is *gyrocommutative* if it satisfies

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \quad \text{(gyrocommutativity)}.$$

A gyrogroup is *uniquely* 2-*divisible* if for every $b \in G$, there exists a unique element $a \in G$ such that $a \oplus a = b$.

In (G3) the automorphism $\mathrm{gyr}[a, b]$ for each $a, b \in G$ is called the *Thomas gyration* or the *gyroautomorphism*, or simply, the *gyration* generated by a and b. From (G2) and (G3) we have

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus [a \oplus (b \oplus c)]$$

for all $a, b, c \in G$. In Euclidean space it plays a role of rotation in the plane spanned by $\{a, b\}$ leaving the orthogonal complement fixed.

It has been shown in [14] that gyrocommutative gyrogroups are equivalent to Bruck loops with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to B-loops. The two approaches have remained quite distinctive in the literature, but we primarily use a notion of gyrogroups rather than a notion of loops.

For arbitrary fixed positive constant s, we let

$$\mathbf{B}_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}$$

be the open s-ball in the n-dimensional vector space $\mathbb{R}^n$. We consider elements in $\mathbb{R}^n$ naturally as column vectors, so that $\mathbf{u}^T\mathbf{v}$ is the usual inner product written in matrix form. We here see two important examples of gyrogroups [16].

Example 2.2. We define the binary operation $\oplus_E$ in $\mathbf{B}_s$ by

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}^T\mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u} \right\} \tag{2}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_s$, where $\gamma_{\mathbf{u}}$ is the well-known *Lorentz factor* such that

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}}.$$

The equation (2) is called the *Einstein addition* of relativistically admissible velocities, introduced by Einstein in his 1905 paper. The binary system $(\mathbf{B}_s, \oplus_E)$ forms a gyrocommutative gyrogroup, called the *standard real relativistic gyrogroup* or the *Einstein gyrogroup*.

Example 2.3. In the open s-ball $\mathbf{B}_s$, we define the binary operation $\oplus_M$ by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{1}{1 + \frac{2}{s^2}\mathbf{u}^T\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\left\{\left(1 + \frac{2\mathbf{u}^T\mathbf{v}}{s^2} + \frac{\|\mathbf{v}\|^2}{s^2}\right)\mathbf{u} + \left(1 - \frac{\|\mathbf{u}\|^2}{s^2}\right)\mathbf{v}\right\} \quad (3)$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_s$. The equation (3) is called the *Möbius addition*, known as Möbius translation on the open s-ball (see formula (4.5.5) of [13]). The binary system $(\mathbf{B}_s, \oplus_M)$ forms also a gyrocommutative gyrogroup, called the *nonstandard real relativistic gyrogroup* or *Möbius gyrogroup*.

Suksumran and Wiboonton [15] have recently shown by using the Clifford algebra that the open ball $\mathbf{B}_s$ equipped with binary operations $\oplus_E$ and $\oplus_M$, respectively, is a uniquely 2-divisible gyrocommutative gyrogroup.

In the same way that vector spaces are commutative groups of vectors that admit scalar multiplication, gyrovector spaces are gyrocommutative gyrogroups of gyrovectors that admit properly scalar multiplication. We give a definition of gyrovector spaces slightly different from Definition 6.2 in [16].

Definition 2.4. A gyrocommutative gyrogroup $(G, \oplus)$ equipped with a scalar multiplication

$$(t, x) \mapsto t \otimes x : \mathbb{R} \times G \to G$$

is called a *gyrovector space* if it satisfies the following for $s, t \in \mathbb{R}$ and $a, b, c \in G$.

(V1) $1 \otimes a = a$, $0 \otimes a = t \otimes e = e$, and $(-1) \otimes a = \ominus a$.

(V2) $(s + t) \otimes a = s \otimes a \oplus t \otimes a$.

(V3) $(st) \otimes a = s \otimes (t \otimes a)$.

(V4) $\mathrm{gyr}[a, b](t \otimes c) = t \otimes \mathrm{gyr}[a, b]c$.

Definition 2.5. A *topological gyrovector space* is a gyrovector space $(G, \oplus, \otimes)$ equipped with Hausdorff topology such that both $\oplus : G \times G \to G$ and $\otimes : \mathbb{R} \times G \to G$ are continuous.

Remark 2.6. In a topological gyrovector space $(G, \oplus, \otimes)$, it has been proved from [4] that

$$\mathrm{gyr}[s \otimes a, t \otimes a] = \mathrm{id}_G$$

for any $s, t \in \mathbb{R}$ and $a \in G$, where id denotes the identity map on G.

We have seen two distinctive examples of gyrocommutative gyrogroups in the open s-ball $\mathbf{B}_s$ of the n-dimensional vector space $\mathbb{R}^n$. Via defining a scalar multiplication we see two common examples of inner product gyrovector spaces, also corresponding to two models of hyperbolic geometry.

Example 2.7. Let $\mathbf{B}_s$ be the Einstein gyrogroup with Einstein addition $\oplus_E$, or the Möbius gyrogroup with Möbius addition $\oplus_M$, respectively. We define a map $\otimes : \mathbb{R} \times \mathbf{B}_s \to \mathbf{B}_s$ by

$$\begin{aligned} t \otimes \mathbf{v} &= s \cdot \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^t - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^t}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^t + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^t} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh\left(t \tanh^{-1} \frac{\|\mathbf{v}\|}{s}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{4}$$

for $t \in \mathbb{R}$ and $\mathbf{v}(\neq \mathbf{0}) \in \mathbf{B}_s$, and define $t \otimes \mathbf{0} := \mathbf{0}$. We call $(\mathbf{B}_s, \oplus_E, \otimes)$ and $(\mathbf{B}_s, \oplus_M, \otimes)$ the *Einstein gyrovector space* and the *Möbius gyrovector space*, respectively.

The Beltrami-Klein ball model of hyperbolic geometry is algebraically regulated by Einstein gyrovector spaces. The geodesics of this model, called gyrolines, are Euclidean straight lines in the open s-ball. On the other hand, the Poincaré ball model of hyperbolic geometry is algebraically regulated by Möbius gyrovector spaces. The geodesics of this model are Euclidean circular arcs in the open s-ball that intersect the boundary of the ball orthogonally.

3. On the Cone of Positive Definite Matrices

We have seen two fundamental examples of a gyrovector space, Einstein gyrovector space and Möbius gyrovector space, on the open s-ball $\mathbf{B}_s$. In this section we give an example of a gyrovector space on the open convex cone $\mathbb{P}$ of all $n \times n$ positive definite matrices.

Example 3.1. [4, Example 2.2, Example 3.2] Let $\mathbb{P}$ be an open convex cone of positive definite Hermitian matrices. Define the binary operation $\oplus$ and a scalar multiplication $\circ$ by

$$\begin{aligned} &\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P},\ A \oplus B = A^{1/2} B A^{1/2}, \\ &\circ : \mathbb{R} \times \mathbb{P} \to \mathbb{P},\ t \circ A = A^t \end{aligned}$$

for any $A, B \in \mathbb{P}$ and $t \in \mathbb{R}$. Then the system $(\mathbb{P}, \oplus, \circ)$ forms a gyrovector space, and the gyroautomorphism generated by A and B is given by

$$\mathrm{gyr}[A, B]C = U(A, B) C U(A, B)^{-1}, \tag{5}$$

where $U(A, B) = (A^{1/2} B A^{1/2})^{-1/2} A^{1/2} B^{1/2}$ is a unitary part of the polar decomposition for $A^{1/2} B^{1/2}$ such that

$$A^{1/2} B^{1/2} = (A \oplus B)^{1/2} U(A, B).$$

Indeed, let us check (V4). For any $A, B, X \in \mathbb{P}$

$$\begin{aligned}\operatorname{gyr}[A,B](t \circ X) &= U(A,B)X^t U(A,B)^{-1} \\ &= U(A,B)\exp(t \log X)U(A,B)^{-1} \\ &= \exp[t \log U(A,B)XU(A,B)^{-1}] \\ &= [U(A,B)XU(A,B)^{-1}]^t = t \circ \operatorname{gyr}[A,B]X.\end{aligned}$$

One can easily see that the binary operation $\oplus$ and the scalar multiplication $\circ$ are both continuous. Thus, the system $(\mathbb{P}, \oplus, \circ)$ is a topological gyrovector space.

Remark 3.2. The inner product on $\mathrm{M}_n(\mathbb{C})$, the vector space of all $n \times n$ matrices with complex entries, is naturally defined as $\langle A, B\rangle = \operatorname{tr}(AB^*)$, where X^* is a complex conjugate transpose of a matrix X. The gyroautomorphism on $\mathbb{P}$ preserves the inner product, and so the norm induced by inner product. Indeed, for any $A, B, X, Y \in \mathbb{P}$

$$\begin{aligned}\langle \operatorname{gyr}[A,B]X, \operatorname{gyr}[A,B]Y\rangle &= \operatorname{tr}[U(A,B)XU(A,B)^{-1}(U(A,B)YU(A,B)^{-1})^*] \\ &= \operatorname{tr}[U(A,B)XY^*U(A,B)^*] \\ &= \operatorname{tr}[XY^*] = \langle X, Y\rangle.\end{aligned}$$

A. Ungar has explained a gyrogroup structure for qubit density matrices in Chapter 9, [16]. We now see an example of gyrovector space for arbitrary dimensional density matrices.

Example 3.3. [3] Let $\mathbb{D}_n$ be a set of all $n \times n$ invertible density matrices, which are positive definite Hermitian matrices of trace 1. We define a binary operation $\odot$ and a scalar multiplication $\star$ given by

$$\begin{aligned}\odot : &\mathbb{D}_n \times \mathbb{D}_n \to \mathbb{D}_n, \ \rho \odot \sigma = \frac{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}{\operatorname{tr}(\rho\sigma)} = \frac{\rho \oplus \sigma}{\operatorname{tr}(\rho \oplus \sigma)} \\ \star : &\mathbb{R} \times \mathbb{D}_n \to \mathbb{D}_n, \ t * \rho = \frac{\rho^t}{\operatorname{tr}(\rho^t)} = \frac{t \circ \rho}{\operatorname{tr}(t \circ \rho)}\end{aligned}$$

for any $\rho, \sigma \in \mathbb{D}_n$ and $t \in \mathbb{R}$. Then $(\mathbb{D}_n, \odot, \star)$ is a gyrovector space. Note that the identity element in $(\mathbb{D}_n, \odot, \star)$ is $\frac{1}{n} I_n$ and the inverse of ρ is $(-1) \star \rho = \frac{1}{\operatorname{tr}(\rho^{-1})}\rho^{-1}$, where I_n denotes the $n \times n$ identity matrix.

In [3, Theorem 3.4] it has been shown the relationship between the Einstein gyrogroup $(\mathbf{B}_{s=1}, \oplus_E)$ and the gyrogroup $(\mathbb{D}_2, \odot)$ of 2×2 invertible density matrices. In other words, the map

$$\rho : (\mathbf{B}_{s=1}, \oplus_E) \to (\mathbb{D}_2, \odot), \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \rho_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix}$$

is a gyrogroup isomorphism. We give an extension of the isomorphism between gyrovector spaces.

Theorem 3.4. *The Einstein gyrovector space* $(\mathbf{B}_{s=1}, \oplus_E, \otimes)$ *and the gyrovector space* $(\mathbb{D}_2, \odot, \star)$ *of* 2×2 *invertible density matrices are isomorphic.*

Proof. It remains to show that

$$\rho_{t\otimes \mathbf{v}} = t \star \rho_{\mathbf{v}} = \frac{1}{\mathrm{tr}(\rho_{\mathbf{v}}^t)}\rho_{\mathbf{v}}^t$$

for any $t \in \mathbb{R}$. Set

$$T = \{t \in \mathbb{R} : \rho_{t\otimes \mathbf{v}} = \frac{1}{\mathrm{tr}(\rho_{\mathbf{v}}^t)}\rho_{\mathbf{v}}^t \text{ for any } \mathbf{v} \in \mathbf{B}_{s=1}\}$$

Our goal is to show that the set T contains all dyadic rational numbers, since this implies by the density of dyadic rational numbers that $T = \mathbb{R}$.

Easily $0, 1 \in T$. Moreover, $\frac{1}{2} \in T$. Indeed,

$$\frac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1}\mathbf{v}.$$

So we obtain from [3, Lemma 3.3] that

$$\begin{aligned}
\rho_{\frac{1}{2}\otimes \mathbf{v}} &= \frac{1}{2}\begin{pmatrix} 1 + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}v_3 & \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}(v_1 - iv_2) \\ \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}(v_1 + iv_2) & 1 - \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}v_3 \end{pmatrix} \\
&= \frac{1}{2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1}\begin{pmatrix} 1 + v_3 + \frac{1}{\gamma_{\mathbf{v}}} & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 + \frac{1}{\gamma_{\mathbf{v}}} \end{pmatrix} \\
&= \frac{1}{\mathrm{tr}(\rho_{\mathbf{v}}^{1/2})}\rho_{\mathbf{v}}^{1/2}.
\end{aligned}$$

This gives us that $\frac{t}{2} \in T$ whenever $t \in T$.

From $\rho_{\mathbf{u}\oplus\mathbf{v}} = \rho_{\mathbf{u}} \odot \rho_{\mathbf{v}}$ we have

$$\rho_{2\otimes \mathbf{v}} = \frac{1}{\mathrm{tr}(\rho_{\mathbf{v}}^2)}\rho_{\mathbf{v}}^2 \text{ and } \rho_{(-1)\otimes \mathbf{v}} = \frac{1}{\mathrm{tr}(\rho_{\mathbf{v}}^{-1})}\rho_{\mathbf{v}}^{-1}.$$

That is, $2t \in T$ and $-t \in T$ whenever $t \in T$. Then for $s, t \in T$

$$\begin{aligned}
\rho_{(2s-t)\otimes \mathbf{v}} &= \rho_{(2s)\otimes \mathbf{v} \oplus (-t)\otimes \mathbf{v}} \\
&= \rho_{(2s)\otimes \mathbf{v}} \odot \rho_{(-t)\otimes \mathbf{v}} \\
&= (2s) \circ \rho_{\mathbf{v}} \odot (-t) \circ \rho_{\mathbf{v}} \\
&= (2s - t) \circ \rho_{\mathbf{v}}.
\end{aligned}$$

In other words, $2s - t \in T$ whenever $s, t \in T$. So the set T contains all dyadic rational numbers in $\mathbb{R}$. ☐

It is still an open question whether or not the Einstein gyrovector space $(\mathbf{B}_{s=1}, \oplus_E, \otimes)$ and the gyrovector space $(\mathbb{D}_n, \odot, \star)$ are isomorphic for $n > 2$.

4. Gyrolines and Gyromidpoints

The *gyroline* passing through the points a and b in the gyrovector space $(G, \oplus, \otimes)$ is defined in Definition 6.19, [16], by

$$L : \mathbb{R} \times G \times G \to G, \ \ L(t; a, b) = a \oplus t \otimes (\ominus a \oplus b). \tag{6}$$

The gyroline is uniquely determined by given points, and a left gyrotranslation of a gyroline is again a gyroline by Theorem 6.21 in [16]. In other words,

$$x \oplus L(t; a, b) = L(t; x \oplus a, x \oplus b)$$

for any $x \in G$.

Example 4.1. From Example 3.1 we obtain the gyroline on $(\mathbb{P}, \oplus, \circ)$ passing through A and B such that

$$L(t; A, B) = A \oplus t \circ ((-1) \circ A \oplus B) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \tag{7}$$

for $t \in [0, 1]$. This is usually called the *weighted geometric mean* of A and B, and denoted by $L(t; A, B) = A\#_t B$. Moreover, it is known in [1, Chapter 6] as a unique geodesic connecting from A to B on $\mathbb{P}$ with respect to the Riemannian trace metric δ:

$$\delta(A, B) = \| \log(A^{-1/2}BA^{-1/2})\|_F,$$

where $\|X\|_F$ denotes the Frobenius norm of X. Note that for any $A, B, C, D \in \mathbb{P}$ and $t \in [0, 1]$

$$\delta(A\#_t B, C\#_t D) \leq (1 - t)\delta(A, C) + t\delta(B, D).$$

It is also satisfied that for any invertible matrix M,

$$M(A\#_t B)M^* = (MAM^*)\#_t(MBM^*).$$

This implies that a left gyrotranslation of a gyroline is again a gyroline.

Since the map $t \in [0, 1] \mapsto A\#_t B$ for any $A, B \in \mathbb{P}$ is introduced by two-variable geometric mean, a variety of approaches to extend it to multivariable geometric means have been recently developed. Among them we introduce a least squares mean as a hot topic of matrix means.

Remark 4.2. It is known in [1, Chapter 6] that $(\mathbb{P}, \delta)$ is a Bruhat-Tits space (a Hadamard space or a non-positive curvature space), which is a complete metric space satisfying the semi-parallelogram law. For an n-dimensional positive probability vector $\omega = (w_1, \ldots, w_n)$ and positive definite matrices $A_1, \ldots, A_n$, there exists a unique minimizer of the weighted sum of squares of Riemannian distances to each point.

$$\arg\min_{Z \in \mathbb{P}} \sum_{i=1}^{n} w_i \delta^2(Z, A_i). \tag{8}$$

This is called the least squares mean (the Karcher mean or Riemannian barycenter), and denoted by $\Lambda(\omega; A_1, \ldots, A_n)$. Vanishing the gradient of the objective function $f(Z) = \sum_{i=1}^{n} w_i \delta^2(Z, A_i)$, we obtain that the least squares mean coincides with the unique positive solution of the Karcher equation

$$\sum_{i=1}^{n} w_i \log(Z^{1/2} A_i^{-1} Z^{1/2}) = O. \tag{9}$$

Many interesting properties for the least squares mean including the monotonicity have been studied; see [2, 8, 9, 10, 11].

A. Ungar has introduced in [16] a *gyrocentroid* as a barycenter of points on the gyrovector space. It would be interesting to find a connection between the least squares mean and the gyrocentroid.

We finally give a formula of the gyroline on the gyrovector space $(\mathbb{D}_n, \odot, \star)$.

Theorem 4.3. *For any $\rho, \sigma \in (\mathbb{D}_n, \odot, \star)$ and $t \in [0, 1]$*

$$L(t; \rho, \sigma) = \frac{1}{\mathrm{tr}(\rho \#_t \sigma)} \rho \#_t \sigma.$$

Proof. Let $\rho, \sigma \in (\mathbb{D}_n, \odot, \star)$ and $t \in [0, 1]$. From Example 3.3 we have

$$(-1) \star \rho \odot \sigma = \frac{(-1) \circ \rho \oplus \sigma}{\mathrm{tr}((-1) \circ \rho \oplus \sigma)},$$

and

$$t \star [(-1) \star \rho \odot \sigma] = \frac{t \circ [(-1) \circ \rho \oplus \sigma]}{\mathrm{tr}(t \circ [(-1) \circ \rho \oplus \sigma])}.$$

Thus, we obtain

$$L(t; \rho, \sigma) = \rho \odot t \star [(-1) \star \rho \odot \sigma] = \frac{\rho \oplus t \circ [(-1) \circ \rho \oplus \sigma]}{\mathrm{tr}(\rho \oplus t \circ [(-1) \circ \rho \oplus \sigma])} = \frac{\rho \#_t \sigma}{\mathrm{tr}(\rho \#_t \sigma)}.$$

□

Remark 4.4. It has been shown in Proposition 3.8 of [5] that the map $L(t; \rho, \sigma)$ is a minimal geodesic on $\mathbb{D}_n$ with respect to the Hilbert projective metric. In Theorem 4.2 and Remark 4.3 of [5], moreover,

$$L\left(\frac{1}{2}; \rho_{\mathbf{u}}, \rho_{\mathbf{v}}\right) = \frac{\rho_{\mathbf{u}} \# \rho_{\mathbf{v}}}{\mathrm{tr}(\rho_{\mathbf{u}} \# \rho_{\mathbf{v}})} = \frac{\gamma_{\mathbf{u}} \mathbf{u} + \gamma_{\mathbf{v}} \mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{B}_{s=1}$. This is known as the *Einstein gyromidpoint* in Theorem 6.92, [16].

5. Applications and Remarks

We have seen in Example 3.1 that $(\mathbb{P}, \oplus, \circ)$ is a gyrovector space. Let us denote $\mathbb{H}$ as the real vector space of all Hermitian matrices. The exponential map from $\mathbb{H}$ to $\mathbb{P}$ given by

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is a diffeomorphism, and its inverse is the logarithm map denoted by log.

We define a map $f : \mathbb{H} \to \mathbf{B}(\mathbb{H}) := \{A \in \mathbb{H} : \|A\| < 1\}$ by

$$f(X) := \tanh \|X\| \frac{X}{\|X\|}, \ X \neq O,$$

and $f(O) := O$, where $\| \cdot \|$ denotes the Frobenius or Hilbert-Schmidt norm. Since the function $g(x) = \dfrac{\tanh x}{x}$ for $x > 0$ is bijective, so is f. Then the composition defined as

$$g = f \circ \log : \mathbb{P} \to \mathbf{B}(\mathbb{H}), \ g(A) := \tanh \| \log A\| \frac{\log A}{\| \log A\|}, \ A \neq I \tag{10}$$

and $g(I) = O$, is also a bijection. It means that every element in $\mathbf{B}(\mathbb{H})$ can be uniquely written as $g(A)$ for some $A \in \mathbb{P}$.

Furthermore, defining a binary operation $\diamond$ on $\mathbf{B}(\mathbb{H})$ by

$$g(A) \diamond g(B) := g(A \oplus B)$$

gives us an isomorphism g from $(\mathbb{P}, \oplus)$ onto $(\mathbf{B}(\mathbb{H}), \diamond)$. So the binary system $(\mathbf{B}(\mathbb{H}), \diamond)$ is a gyrocommutative gyrogroup. Also, setting

$$t * g(A) := g(t \circ A) = g(A^t)$$

for all $t \in \mathbb{R}$ and $A \in \mathbb{P}$ gives us a gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$. Indeed, the following are satisfied for any $s, t \in \mathbb{R}$ and $A, B, X \in \mathbb{P}$.

(V1) $1 * g(A) = g(A)$ and $0 * g(A) = g(I) = O$.

(V2) Using an isomorphism g we have

$$\begin{aligned}(s+t) * g(A) &= g(A^{s+t}) = g(A^{s/2} A^t A^{s/2}) \\ &= g(A^s \oplus A^t) = g(A^s) \diamond g(A^t) = s * g(A) \diamond t * g(A).\end{aligned}$$

(V3) $(st) * g(A) = g(A^{st}) = g((A^s)^t) = t * (s * g(A))$.

(V4) We note by the gyroassociativity and an isomorphism g that

$$\mathrm{gyr}[g(A), g(B)]g(X) = g(\mathrm{gyr}[A, B]X). \tag{11}$$

Then

$$\begin{aligned}\mathrm{gyr}[g(A), g(B)](t * g(X)) &= \mathrm{gyr}[g(A), g(B)]g(t \star X)\\ &= g(\mathrm{gyr}[A, B](t \star X)) = g(t \star \mathrm{gyr}[A, B]X)\\ &= t * g(\mathrm{gyr}[A, B]X) = t * \mathrm{gyr}[g(A), g(B)]g(X).\end{aligned}$$

Remark 5.1. Since $\mathrm{gyr}[s \circ A, t \circ A] = \mathrm{id}_{\mathbb{P}}$ on the gyrovector space $(\mathbb{P}, \oplus, \circ)$ for any $s, t \in \mathbb{R}$, we also have

$$\begin{aligned}\mathrm{gyr}[s * g(A), t * g(A)]g(X) &= \mathrm{gyr}[g(s \circ A), g(t \circ A)]g(X)\\ &= g(\mathrm{gyr}[s \circ A, t \circ A]X) = g(X).\end{aligned}$$

The second equality follows from the equation (11). This means that

$$\mathrm{gyr}[s * g(A), t * g(A)] = \mathrm{id}_{\mathbf{B}(\mathbb{H})}\,.$$

The following gives us a formula of the gyroline on the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$.

Proposition 5.2. *For given $A, B \in \mathbb{P}$, the gyroline connecting from $g(A)$ to $g(B)$ on the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$ is $g(A \#_t B)$.*

Proof. By the general equation (6) of gyroline on the gyrovector space,

$$\begin{aligned}L(t; g(A), g(B)) &= g(A) \diamond t * ((-1) * g(A) \diamond g(B))\\ &= g(A) \diamond t * (g(A^{-1}) \diamond g(B))\\ &= g(A) \diamond t * g(A^{-1/2}BA^{-1/2})\\ &= g(A) \diamond g((A^{-1/2}BA^{-1/2})^t)\\ &= g(A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}).\end{aligned}$$

□

Remark 5.3. On the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$, it would be interesting to investigate any geometric aspect such as metric relations and gyrocentroids.

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