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The Principle of Relativity: From Ungar's Gyrolanguage for Physics to Weaving Computation in Mathematics

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Abstract

This paper extends the scope of algebraic computation based on a non standard $\times$ to the more basic case of a non standard $+$, where standard means associative and commutative. Two physically meaningful examples of a non standard $+$ are provided by the observation of motion in Special Relativity, from either outside (3D) or inside (2D or more), We revisit the "gyro"-theory of Ungar to present the multifaceted information processing which is created by a metric cloth W, a relating computational construct framed in a normed vector space V, and based on a non standard addition denoted $\oplus$ whose commutativity and associativity are ruled (woven) by a relator, that is a map which assigns to each pair of admissible vectors in V an automorphism in Aut W. Special attention is given to the case where the relator is directional.

Keywords: Relator, noncommutativity, nonassociativity, induced addition, organ, metric cloth, weaving information processing, cloth geometry, hyperbolic geometry, special relativity, liaison, geodesic, organic line, action at a distance.

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1. About Relating Computation

1.1 Introduction

Hypercomputation, that is nonlinear computation in real multiplicative Dickson algebras $A_k \cong \mathbb{R}^{2^k}$, is developed in (Chatelin 2012 a). For $k \geq 2$ (resp. $k \geq$

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3) multiplication is not commutative (resp. not associative). However addition remains both associative and commutative.

The situation changes in an essential way when computation is merely additive and there exists a *relator* which rules the way any two multidimensional numbers in $\mathbb{R}^n$ (i.e vectors) are to be *added*. This kind of relating computation will be defined in precise terms in Section 2. It includes the special case of an *explicit metric reference* consisting of a positive *finite* number λ, $0 < \lambda < \infty$. The classical structure of an abelian additive group is weakened by considering an addition whose commutativity and associativity are controlled by the relator. A physically meaningful example was provided a century ago by 3D-Special Relativity (Einstein) where the role of λ as a metric reference is played by c, the speed of light in vacuum, and the relator is a plane rotation.

1.2 Special Relativity in the Early Days

It was soon recognised that hyperbolic geometry underlies Einstein's law of addition for admissible velocities (Varičak 1910, Borel 1914) creating the relativistic effect known today as *Thomas precession* (Silberstein 1914, Thomas 1926). But, despite Maxwell's valiant efforts (Maxwell 1871), Hamilton's noncommutative $\times$ of 4-vectors was still unacceptable for most scientists at the dawn of the 20th century. Therefore Einstein's noncommutative $+$ of 3-vectors (representing relativistically admissible velocities) was fully inconceivable: Einstein's geometric vision was far too much ahead of its time! An analytic version of Special Relativity with more appeal to physicists was conceived by Minkowski in 1907, by dressing up as physical concepts the Lorentz transformations which had been introduced by (Poincaré 1905) as the correction of Lorentz preliminary version (1904), see (Walter 1999, Auffray 2005, Damour 2008). This version was quickly grasped by leading physicits (Von Laue, Sommerfeld); it is the version adopted until today in most physics textbooks for students, which carefully avoids any reference to the underlying non commutative quaternionic field $\mathbb{H}$ invented by Hamilton (1843).

1.3 A Mathematical Revival in 1988

Einstein's intuition was left dormant for some 80 years until it was brought back to a new mathematical life in the seminal paper (Ungar 1988). During almost 30 years, Ungar has crafted an algebraic language for hyperbolic geometry lucidly presented in (Ungar, 2008). The book sheds a *natural light* on the physical theories of Special Relativity and Quantum Computation. It dissipates some of the mystery that has shrouded earlier expositions. At the same time, it provides new insight on hyperbolic geometry. Ungar's geometry, which is expressed in "gyrolanguage", is based on the key concepts of gyrator and gyrovector space. They are mathematical concepts abstracted from Thomas precession, a kinematic effect in 3D-special relativity. The *physical* effect was anticipated in (Borel 1913, 1914).

As we shall see, these concepts find an equally natural use beyond physics, in the realm of computation ruled by a relator.

1.4 Geometric Information Processing in Relating Computation

The gyrolanguage is geared towards Hyperbolic Geometry and Physics. In this paper, we export some of Ungar's tools developed for mathematical physics into mathematical computation in a *relating* context (Definition 2.1 below). The reward of the shift of focus from physics to computation is to gain insight about the *geometric* ways by which information can be organically processed in the mind of a computing agent when *relation* prevails. This processing exemplifies the computational thesis posited in (Chatelin 2012 a,b) by revealing geometric aspects of organic intelligence.

The change of focus entails some necessary changes in the vocabulary which are signalled by a reference to the original gyroterm defined in (Ungar 2008). The reader can find all the necessary theoretical background for the presentation to follow in Ungar's work, conveniently put together in his 2008 book which is an algebraic goldmine. Unless otherwise stated, all cited gyroresults are taken from this book.

1.5 Organisation of the Paper

Sections 2 to 6 export parts of Ungar's gyrotheory for physics into relating computation: an organ is a gyrocommutative gyrogroup (Section 2), a metric cloth is a gyrovector space (Section 3). The associated cloth geometry is studied by means of three basic organic lines, the first two corresponding to gyrolines and cogyrolines (Sections 4 to 6)). The rest of the paper (Sections 7 to 9) is original. In Section 7 we restrict our attention to those relators which are directional because they do not depend on the norm of the vectors. This restriction enables us to show that the third organic line enjoys a twofold interpretation in terms of each of two geodesics (Section 7). Section 8 develops the consequences for Weaving Information Processing based on cloth geometry. Finally, epistemological considerations are presented in Section 9.

2. Additive Relating Computation

2.1 Preliminaries

A *groupoid* $(S, \oplus)$ is a set S of elements on which a binary operation called *addition* and denoted $\oplus$ is defined : $(a, b) \in S \times S \mapsto a \oplus b \in S$. An element 0 such that $0 \oplus a = a$ (resp. $a \oplus 0 = a$) is called a left (resp. right) *neutral*. An *automorphism* for $(S, \oplus)$ is a bijective endomorphism φ which preserves $\oplus$: $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$

for all $a, b \in S$. The set of automorphisms forms a group (relative to $\oplus$) denoted Aut $(S, \oplus)$ with the identity map I as unit element. The subtraction is denoted $\ominus$: $a \ominus b = a \oplus (\ominus b)$. In particular $\ominus a$, the left opposite of a, satisfies $\ominus a \oplus a = 0$.

2.2 Relators

We suppose that we are given a map:

$$rel \quad : \quad S \times S \to \text{Aut } (S, \oplus)$$
$$(a, b) \mapsto rel(a, b)$$

such that
$$rel(a \oplus b, b) = rel(a, b). \tag{A1}$$

A map *rel* satisfying the *reduction* axiom (A1) is called a *relator*. We set $\boldsymbol{R} = rel(S, S)$ for the range of the relator in Aut $(S, \oplus)$.

2.3 Organs underlie Additive Relating Computation

We suppose that $\oplus$ satisfies the additional axioms:

$$a \oplus b = rel(a, b)(b \oplus a), \tag{A2}$$
$$a \oplus (b \oplus c) = (a \oplus b) \oplus rel(a, b)c, \tag{A3}$$

which express by means of $rel(a, b)$ a weak form of commutativity (A2) and associativity (A3). Then $(a \oplus b) \oplus c = a \oplus (b \oplus rel(b, a)c)$ by Theorem 2.35.
The algebraic structure (G, rel) consisting of the additive groupoid $G = (S, \oplus)$ and the relator *rel* is called an *organ*.

Definition 2.1 An additive relating computation refers to any algebraic computation taking place in an organ defined by the data $\{\oplus, rel\}$ satisfying the three axioms (A1), (A2), (A3).

Remark 1. In (Definition 2.7, Ungar 2008), the relator is called gyrator with (A1)$\Longleftrightarrow$ ($G5$) (=left loop property in multiplicative algebra vocabulary). Next (A3)$\Longleftrightarrow$ ($G3$) is gyroassociativity and (A2)$\Longleftrightarrow$ ($G6$) is gyrocommutativity which is optional in a gyrogroup. An organ is a gyrocommutative gyrogroup (Definition 2.8). And $\oplus$ is denoted either $+$ or $\oplus$ therein.

2.4 Some Properties of the Relator

The neutral 0 and the opposite $\ominus a$ are unique: (left=right), and $a \ominus a = \oplus a \ominus a = 0$.
The relator satisfies:

- $\ominus (a \oplus b) \;=\; rel(a, b)(\ominus b \ominus a)$, (Theorem 2.11)
  $\;=\; \ominus a \ominus b$ (Theorem 3.2)
- $rel^{-1}(a, b) \;=\; rel(\ominus b, \ominus a)$ (Theorem 2.32)
- $rel(b, a) \;=\; rel^{-1}(a, b)$ (Theorem 2.34)
  $\;=\; rel(a, \ominus rel(a, b)b)$ (Lemma 2.33)

More identities are found in Table 2.2 (Ungar 2008, p.50). In particular:

$$rel(\ominus a, a) = rel(a, \ominus a) = rel(0, a) = rel(a, 0) = rel(0, 0) = I \tag{2.1}$$

The identities in (2.1) follow from the reduction axiom (A1). Because $\ominus a \oplus a = 0 \oplus 0 = 0$, $rel(\ominus a, a)$ and $rel(0, 0)$ could be arbitrarily chosen in Aut $(S, \oplus)$. In full generality, 0 is a singularity with an indeterminate character for the relator. The indeterminacy disappears under the reduction axiom (A1).
The following additional hypotheses are useful:

- $g \oplus g = 0 \Longrightarrow g = 0$ holds for any $g \in G$ (H_1)
- for any $0 \neq g \in G$, there exists at least one half-vector h such that $h \oplus h = g$. (H_2)

(H_1) is satisfied in the Examples 2.1 to 2.3 that will be given in Section 2.6. It is the additive analogue of the multiplicative notion of 2-torsion free algebra, see Definition 3.32 on p. 72.
Under the 2 assumptions (H_1) and (H_2) for $\oplus$, the following statements hold:

- the half-vector h is unique (Theorem 3.34),
- $rel(a, b) \neq \ominus I$ (Theorem 3.36), that is anticommutativity is ruled out: $a \oplus b \neq \ominus (b \oplus a)$,
- $rel(a, b)b = \oplus b \Longrightarrow b = 0$ (Theorem 3.37).

2.5 The Two Basic Equations Associated with $\oplus$ and rel

Because $\oplus$ is not commutative we are led to consider $\mathcal{L} = \{L_a = a \oplus \cdot; a \in G\}$ and $\mathcal{R} = \{R_a = \cdot \oplus a; a \in G\}$ Left– (resp. right–) addition $\oplus$ is abbreviated $L\oplus$ (resp. $R\oplus$). We consider the left and right linear equations associated with a, b in G.

$$L_a x = a \oplus x = b, \tag{2.2}$$
$$R_a y = y \oplus a = b, \tag{2.3}$$

Each of them has the unique solution

$$x = \ominus a \oplus b, \text{ by Eq.(2.30)}, \tag{2.4}$$
$$y = b \ominus rel(b, a)a. \text{ by Eq.(2.32)}, \tag{2.5}$$

The equality (2.5) suggests to consider the composite map $\oplus rel(\cdot, \ominus \cdot)$ as an *induced* addition $\hat{+}$ defined by

$$(a, b) \in G \times G \mapsto a \,\hat{+}\, b = a \oplus rel(a, \ominus b)b \text{ (Theorem 2.14)}. \tag{2.6}$$

The corresponding subtraction, denoted $\hat{-}$, is such that (2.5) can be rewritten as $y = b \,\hat{-}\, a$ (Theorem 2.22). Definition (2.6) is equivalent to $a \oplus b = a \hat{+} rel(a, b)b$.

Three properties about $\oplus$ and $\hat{+}$, are noteworthy:

- Aut $(S, \oplus) = $ Aut $(S, \hat{+})$ (Theorem 2.28),
- $\hat{+}$ is classically *commutative* (Theorems 2.38 and 3.4).
- $\ominus a = \hat{-} a$ (Theorem 2.21).

The concept of an *organ* is determined by two data: the addition $\oplus$ and the associated relator (as a map into the automorphisms for $\oplus$). In the relating perspective, the source notion is the pair $(\oplus, rel)$ where the *relator* rules its associated addition $\oplus$. This addition precedes the secondary addition $\hat{+}$, which is induced by $R\oplus$ and the relator combined together. This novel concept reduces to the classical concept of an abelian additive group when the primitive operation is associative and commutative (hence $\oplus = \hat{+}$), that is when the range $\boldsymbol{R}$ reduces to $\{I\}$. By expanding its range to the larger subset $\boldsymbol{R} \subset$ Aut $(S, \oplus)$, the relator controls the weak (or relative) commutativity and associativity of $\oplus$, thus introducing anisotropy in the organic structure. This has the additional benefit to induce the existence of $\hat{+}$, another addition which is classically commutative.

In other words, the expansion $\{I\} \rightarrow \boldsymbol{R}$ loosens the rigid structure of an abelian group and provides the more flexible, relating, structure of an organ which lies at the foundation of relating computation.

When the range $\boldsymbol{R}$ is a proper subset of Aut $(S, \oplus)$, its role is to *reduce* the variety of possible automorphisms. The standard group structure appears as a limit case corresponding to the ultimate reduction $\boldsymbol{R} = \{I\}$.

Remark 2. The *group* structure which underlies classical computation guarantees the invariance of its logic. From their logical vantage point, many logicians view the whole mathematical enterprise as a mere giant tautology. It is clear that the reduction of mathematics to a formal axiomatic system does not do justice to the creative power of non linear computation which may lead to a non standard addition ruled by a relator (see Example 2.2 below). We believe that the concept of an *organ* is better suited than that of a group to describe some of the organic logics which are at work in life's computation and are *evolutive* by essence

Organic Information Processing (IP) is a *dynamical* process which reflects the variability of the *relator*. Its operations in G consist of $\oplus$, $\hat{+}$ and their automorphisms. One can view an organ as a new algebraic species, some kind of a "fieldoid", based on the groupoid, in which $\hat{+}$ plays the role attributed to $\times$ in an ordinary field (group-based) structure. The main difference with a field is that the neutral 0 (identical for $\oplus$ and $\hat{+}$) replaces the unit $1 \neq 0$. The analogy is commented next.

Remark 3. The induction $\{R\oplus, rel\} \rightarrow \hat{+}$ is analogous to the creation of the product $n \times a$ by n repeated additions of the real number a. In this most familiar case, the multiplication stems from an iterated addition.

2.6 Three Basic Examples

The following Examples are found in Sections 3.4, 3.8 and 3.10 respectively of (Ungar 2008) The explicit formula for $x \oplus y$ entails the determination of $rel(x, y)$.

Example 2.1 The subgroup of all Möbius transformations of the complex open unit disk $D = \{z \in \mathbb{C}; |z| < 1\}$ into itself is defined by : $(a, z) \mapsto e^{i\theta} \frac{a+z}{1+\bar{a}z}$ for $a, z \in D$ and $\theta \in \mathbb{R}$. If we set $a \oplus z = \frac{a+z}{1+\bar{a}z}$, then $z \oplus a = \dfrac{a+z}{1+\bar{z}a}$. The relator is defined by $rel(a, z) = \frac{1+a\bar{z}}{1+\bar{a}z} \in \text{Aut}\,(D, \oplus)$. Hence clearly $a \oplus z = rel(a, z)(z \oplus a)$. Endowed with $\oplus$ the unit disk becomes an organ. Observe that $\oplus$ is expressed by means of the 3 operations $+, \times$, conjugacy defined on $\mathbb{C}$. It is known in mathematics as a *hyperbolic translation* in the plane $\mathbb{R}^2$. The relator is a *rotation* since its modulus is 1. The pseudo-hyperbolic distance in $\bar{D}$ from a to b is $d(a, b) = \left|\dfrac{a-b}{1-\bar{a}b}\right| = |b \ominus a|$. The metric used by Poincaré(1882) in his disc-model for hyperbolic geometry in $\mathbb{R}^2$ is

$$\tanh^{-1} d(a, b) = \frac{1}{2} \ln \frac{1 + d(a, b)}{1 - d(a, b)},$$

cf. Ungar, Section 6.17, p. 216-217. There exists a *real* version of this addition defined in the open unit disk $B_1 = \{x \in \mathbb{R}^2,\ \|x\| < 1\}$ which reads:

$$x \oplus y = \frac{(1 + 2 < x, y > + \|y\|^2)x + (1 - \|x\|^2)y}{1 + 2 < x, y > + \|x\|^2\|y\|^2},$$

cf.(3.127) in Ungar.

Setting $X = \|x\|\|y\| \geq 0$, $\theta = \measuredangle(x, y)$ the denominator $X^2 + 2X\cos\theta + 1$ has no real roots unless $\cos\theta = -1$, then $X = \|x\|\|y\| = 1$ is a double root. The condition that $x, y \in \bar{B}_1$ entails $\|x\| = \|y\| = 1$, $x = -y$. We observe then that $x \oplus (-x) = \dfrac{x-x}{1-1} = \dfrac{0}{0}$ is an indeterminate form for $x \in \partial B_1$.

When x and y are linearly dependent, $y = rx$, $r \in \mathbb{R}$ (say) then the addition becomes associative and commutative for x, y inside B_1 (so that $1 + r\|x\|^2 = 1+ < x, y > \neq 0$)

$$x \oplus y = \frac{1}{1+ < x, y >}(x + y)$$

Example 2.2 Let c be the vacuum speed of light. We set $B_c = \{x \in \mathbb{R}^3; \|x\| < c\}$ to represent the ball of relativistically admissible velocities.

Einstein's law of addition of velocities $x, y \in B_c$ is

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}} \left[x + y + \frac{1}{c^2} \frac{\gamma_x}{1 + \gamma_x} x \wedge (x \wedge y)\right]$$

where $\gamma_x = \left(1 - \frac{1}{c^2}\|x\|^2\right)^{-1/2}$ is the inverse of Lorentz contraction $\sqrt{1 - \left(\frac{\|x\|}{c}\right)^2}$.

Using Grassmann identity in $\mathbb{R}^3$:

$$x \wedge (y \wedge z) =< x, z > y- < x, y > z,$$

(Lamotke 1998, Chapter 7, p. 207), one can also write

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}} \left[x + \frac{1}{\gamma_x} y + \frac{1}{c^2} \frac{\gamma_x}{1 + \gamma_x} < x, y > x\right]$$

a formula well defined, unless $1 + \dfrac{\|x\|\|y\|}{c^2} \cos\theta = 0$, where $\theta = \measuredangle(x, y)$. The relation $\cos\theta = -\dfrac{c^2}{\|x\|\|y\|}$ for $x, y \in \bar{B}_c$ entails that $x, y \in \partial B_c$ and $\cos\theta = -1$. Therefore $x = -y$, and $x \oplus (-x) = \dfrac{0}{0}$ is an indeterminate form.

The two velocity components, parallel and orthogonal to the relative velocity between inertial systems, were given by Einstein in his 1905-epoch-making paper. The above formula is valid for $n \geq 2$.

Einstein's addition is ruled by a relator which is the *rotation*: $y \oplus x \mapsto x \oplus y$ in the plane spanned by x and y (when independent) with axis parallel to $x \wedge y$ through the angle ε, $0 \leq |\varepsilon| < \pi$ (Borel 1913, Silberstein 1914). The angle ε is related non linearly to $\theta = \measuredangle(x, y)$ and to $\frac{1}{c}\|x\|, \frac{1}{c}\|y\|$ in the following way (Ungar 1988,1991): $\varepsilon = 0$ for $|\theta| \in \{0, \pi\}$ and for $|\theta| \in]0, \pi[$ x and y are independent, yielding:

$$\cos\varepsilon = \frac{(\rho + \cos\theta)^2 - \sin^2\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

$$\sin\varepsilon = -2\frac{(\rho + \cos\theta)\sin\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

with $\rho^2 = \frac{\gamma_x + 1}{\gamma_x - 1}\frac{\gamma_y + 1}{\gamma_y - 1}$, $\rho > 1$, and $|\varepsilon| < |\theta|$.

When $\|x\|$ and $\|y\|$ tend to c^-, γ_x and γ_y tend to ∞ and $\rho \to 1^+$. Then $\cos\varepsilon \to \cos\theta$ and $\sin\varepsilon \to -\sin\theta$, that is $\varepsilon \to -\theta$.

We recall that for x, y in $\Im\mathbb{H} \cong \mathbb{R}^3$, $x \times y = -< x, y > + x \wedge y \in \mathbb{H} \cong \mathbb{R}^4$, where $x \wedge y = \frac{1}{2}[x, y] = \frac{1}{2}[x \times y - y \times x] \in \Im\mathbb{H}$. Therefore Einstein's addition wraps up the two distinct operations $+$ and $\times$ in $\Im\mathbb{H}$ into a single *synthetic addition* denoted $\oplus$. The synthesis is realised on independent vectors at the expense of classical commutativity and associativity.

Example 2.3 $V = \mathbb{R}^n$, $n \geq 2$ is the euclidean linear vector space with scalar product $< \cdot, \cdot >$. Let be given λ, $0 < \lambda < \infty$, and define $v_\lambda = \frac{1}{\lambda}v$ for $v \in V$, $\beta_v = \left(1 + \|v_\lambda\|^2\right)^{-1/2}$, $0 < \beta_v \leq 1$. We consider

$$u \oplus v = \left(\frac{1}{\beta_v} + \frac{\beta_u}{1 + \beta_u} < u_\lambda, v_\lambda >\right) u + v$$

defined for $u, v \in V$. For $n = 3$ and $\lambda = c$, this additive law governs the relativistic addition of *proper* velocities expressed in traveller's time. The relator is again a *rotation*. If u and v are dependent, $u \oplus v = \frac{1}{\beta_v}u + \frac{1}{\beta_u}v$ (Eq. (3.214) on p. 96).

The reader can check that in each example above $x \oplus y$ is symmetric in x and y iff x and y are *dependent*.

2.7 Liaison Λ between rel, $\oplus$ and $\hat{+}$

To the linear equations (2.2), (2.3) for $\oplus$, we add the third equation for $\hat{+}$

$$a \hat{+} \hat{x} = \hat{x} \hat{+} a = b \tag{2.7}$$

which admits the unique solution

$$\hat{x} = \ominus(\ominus b \oplus a) = b \ominus a. \tag{2.8}$$

Observe that $x = rel(\ominus a, b)\hat{x}$ by (A2)$\Longleftrightarrow$ $\hat{x} = rel(b, \ominus a)x$.

Each of the solutions x, y and $\hat{x}$ is obtained by a respective call to the three following cancellation laws for $\oplus$ and $\hat{+}$:

- left cancellation for $\oplus$: $a \oplus (\ominus a \oplus b) = b$ (2.9)

• right cancellation for $\oplus$: $(b \,\hat{-}\, a) \oplus a = b$ (2.10)

• cancellation for $\hat{+}$: $(b \ominus a) \,\hat{+}\, a = a \,\hat{+}\, (b \ominus a) = b$ (2.11)

Identities (2.10) and (2.11) express a link by means of the relator between $R\oplus$ and $\hat{+}$ which is not present in (2.9) concerning $L\oplus$. If one uses $x = \ominus a \oplus b$,$y = b \ominus rel(a,b)a = b \hat{-} a$ and $\hat{x} = b \ominus a$, the three identities become respectively

$$a \oplus x = b \tag{2.12}$$
$$y \oplus a = b \tag{2.13}$$
$$\hat{x} \,\hat{+}\, a = a \,\hat{+}\, \hat{x} = b \tag{2.14}$$

This rewriting separates $R\oplus$ and $\hat{+}$ in the identities (2.10), (2.11) which appear now as (2.13) = right cancellation for $\oplus$, (2.14)=cancellation for $\hat{+}$.

None of the two writings is a faithful description of the complete computational reality which is, by essence, *connected*. Whichever writing is chosen, the reader should keep in mind that a liaison based on $rel(a, \cdot)$ exists between $\cdot \oplus a$ and $\cdot \hat{+} a = a \hat{+} \cdot$ for $rel(a, \cdot) \neq I$ when the additive cancellation laws are at work. This liaison reflects the existence of the relator which regulates any relating computation performed in its organ. The liaison concerns $L\oplus$ as well. Indeed, the equality (2.8) $\hat{x} = b \ominus a$ suggests to consider the equation involving L_b:

$$L_b \tilde{x} = b \oplus \tilde{x} = a$$

whose solution is $\tilde{x} = \ominus b \oplus a = \ominus (b \ominus a) = \ominus \hat{x}$.

Definition 2.2 We call liaison $\Lambda(rel, \oplus, \hat{+})$ the computational consequences of the three fundamental cancellation laws (2.9), (2.10) and (2.11).

The computational dynamics of organic IP results from the shifts $L\oplus$, $R\oplus$ and the automorphisms of G. Given a and b, we shall be concerned in Sections 4 and 7 with the evolution of $\hat{x} = b \ominus a$ (resp. $y = b \,\hat{-}\, a$) when a left (resp. right) shift by an arbitrary $g \in G$ is realised simultaneously on a and b.

Regarding left shift $g\oplus$ and $\ominus$, we have:

$(g \oplus b) \ominus (g \oplus a) = rel(g, b)(b \ominus a)$ (Theorem 3.13). (2.15)

For future reference we mention the following result with right shift:

$a \,\hat{-}\, b = (a \oplus k) \,\hat{-}\, (b \oplus g)$ with $k = rel(a, b)g$ (Theorem 2.23). (2.16)

3. Metric Cloths

3.1 The Normed Vector Space Frame

Let V be a linear vector space over $\mathbb{R}$ with finite dimension $n \geq 2$, endowed with a scalar product $< a, b >$ for $a, b \in V$ and derived norm $\|a\| = \sqrt{< a, a >}$.

The addition $+$ and scalar multiplication are standard operations in $V \cong \mathbb{R}^n$. Let λ be given, $0 < \lambda < \infty$ and set $B_\lambda = \{x \in V; \|x\| < \lambda\}$. We suppose that the ball B_λ, or V itself, are endowed with the *organic* structure $G = (S, \oplus)$ with relator rel, where S represents B_λ or V as the case may be. The neutral 0 for G is identified with $0 \in V$.

The linear vector space V is the *frame* of the organ G iff the relator preserves the scalar product: $< rel(u, v)x, rel(u, v)y >=< x, y >$ for any quadruple $(u, v, x, y) \in G^4$. It follows that $\|rel(x, y)\| = 1$ for $x, y \in G$, and G inherits from V its scalar product $< \cdot, \cdot >$ and norm $\| \cdot \|$ which are invariant under $\boldsymbol{R} \subset \text{Aut } G$.

We assume moreover that, if x and y are linearly dependent in G, then for $x = ry$, $r \in \mathbb{R}$ (say), $(ry) \oplus y = y \oplus (ry)$. Hence $rel(ry, y) = rel(x, y) = I$ ($\Longrightarrow x \oplus y = x \hat{+} y$). The formula for $\oplus$ becomes symmetric in x and y when x and y are colinear. The property is satisfied for the 3 Examples given in Section 2.6.

3.2 The Scalar Multiplication $\boxtimes$

We suppose that G admits a scalar multiplication $\boxtimes : (\mathbb{R} \times G$ or $G \times \mathbb{R} \to G)$ such that

- $r \boxtimes a = a \boxtimes r$, $r \in \mathbb{R}$, $a \in G$,
- $1 \boxtimes a = a$,
- $(r_1 + r_2) \boxtimes a = (r_1 \boxtimes a) \oplus (r_2 \boxtimes a)$,
- $(r_1 r_2) \boxtimes a = r_1 \boxtimes (r_2 \boxtimes a)$, $a \in G$, $r_1, r_2 \in \mathbb{R}$,
- for r and $a \neq 0$ $\frac{|r| \boxtimes a}{\|r \boxtimes a\|} = \frac{a}{\|a\|}$,
- $rel(u, v)(r \boxtimes a) = r \boxtimes (rel(u, v)a)$ for $u, v, a \in G$, $r \in \mathbb{R}$,
- $rel(r_1 \boxtimes u, r_2 \boxtimes u) = I$, $u \in G$, $r_1, r_2 \in \mathbb{R}$.
- $\|r \boxtimes a\| = |r| \boxtimes \|a\|$, $r \in \mathbb{R}$, $a \in G$.

Even though $\boxtimes$ does not distribute with $\oplus$ in general, the following special identity holds:

$$2 \boxtimes (a \oplus b) = a \oplus (2 \boxtimes b \oplus a) = a \hat{+} (a \oplus 2 \boxtimes b)$$

for any $a, b \in W$ (Theorem 6.7).

3.3 $n = 1$: the Measuring Rod $M = \{\pm\|a\|, a \in G\}$

All elements in M are colinear, hence the relator image reduces to $\{I_1 = 1\}$, and $\oplus = \hat{+}$ on M. M is a 1D-linear vector line equipped with $\oplus$, $\otimes$ and $\|\cdot\|$ deriving from G and V. These 3 operations usually *differ* from the standard operations $+, \cdot, |\cdot|$ defined on $\mathbb{R}$.

3.4 $n \geq 2$: the V-Framed Metric Cloth W

We suppose that $\|a \oplus b\| \leq \|a\| \oplus \|b\|$, $a, b \in G$. This ends the list of axioms satisfied by Ungar's gyrovector space in the carrier V (Definition 6.2, Ungar 2008).

To this list, we add that, when a and b are *dependent and nonzero*: $b = ra$, $r \neq 0$, there exists $l \in \mathbb{R}\backslash\{0\}$ such that $b = l \otimes a$ with $1 \cdot a = 1 \otimes a = a$ for $r = 1$. If $r = 0$, $0 \cdot a = 0 = 0 \otimes a = 0$. In other words, $r \cdot (\cdot) = r(\cdot) = l \otimes (\cdot)$: the map $l \mapsto r = \mu(l)$, $\mu(0) = 0$, $\mu(1) = 1$, is a change of scale on the axis spanned by $a \neq 0$, induced by the change of context from the linear vector space $V(+, \cdot)$ to the additive cloth $W(\oplus, \otimes)$. As a consequence $rel(a, b) = I$ and the vectors $a \oplus b = a \hat{+} b = (1 + l) \otimes a$ are colinear with $a + b = (1 + r)a$.

The structure $W = (S, \oplus, \otimes) = (G, \otimes)$ obeying the assumptions above is a *metric cloth* in the normed vector frame V. The cloth W is organically and metrically woven by $\{\oplus$, relator, $\otimes, \|\cdot\|\}$. It satisfies the

Proposition 3.1. *The addition $\oplus$ in the metric cloth W satisfies (H_1) and (H_2).*

Proof. 1) (H_1): observe that $g \oplus g = 2 \otimes g$. Hence if $2 \otimes g = 0$, $\|2 \otimes g\| = 2 \otimes \|g\| = 0$ and $\|g\| = 0$ in $\Longleftrightarrow g = 0$.

2) (H_2): $g = \frac{1}{2} \otimes (2 \otimes g) = \frac{1}{2}(g \oplus g)$. Now Ungar's Theorem 6.4 tells us that $\otimes$ distributes *axially* along the axis spanned by any $g \neq 0$ in G:

$$r \otimes (r_1 \otimes g \oplus r_2 \otimes g) = r \otimes (r_1 \otimes g) \oplus r \otimes (r_2 \otimes g) =$$

$$(rr_1) \otimes g \oplus (rr_2) \otimes g = (r(r_1 + r_2)) \otimes g.$$

Setting $r = 1/2$ and $r_1 = r_2 = 1$ we get $\frac{1}{2} \otimes (g \oplus g) = \frac{1}{2} \otimes g \oplus \frac{1}{2} \otimes g$. Therefore the half-vector h for g exists and is uniquely defined by $\frac{1}{2} \otimes g$. □

Because $\otimes$ distributes axially in W, it follows readily that anticommutativty is ruled out for $\oplus$

Example 3.1 The scalar multiplication for the organ B_c in Example 2.2 is such that $r \otimes 0 = 0$, $r \otimes x = \mu(r)x$ for $0 \neq x \in B_c$. We set $x_c = \frac{1}{c}x$, then Definition 6.86 on p. 218 gives

$$\mu(r) = \frac{1}{\|x_c\|} \tanh(r \tanh^{-1} \|x_c\|), \quad r \in \mathbb{R}, \tag{3.1}$$

with $\mu(0) = 0$, $\mu(1) = 1$. Then B_c becomes the $\mathbb{R}^3$-framed cloth W_E (based on Einstein's addition) which is an alternative framework for Special Relativity in Physics, classically presented by means of Lorentz transformations, hence implicitly on the field of quaternions $\mathbb{H}$.

Let $q = (c\alpha, X)$ be given in $\mathbb{H}$, with real part $c\alpha$, $\alpha \in \mathbb{R}$ and imaginary part X in $\mathbb{R}^3$. Then $q^2 = c^2\alpha^2 - \|X\|^2 + 2c\alpha X$. A Lorentz transformation in $\mathbb{H}$ leaves invariant the quantity

$$\Re q^2 = c^2\alpha^2 - \|X\|^2 = f \quad \text{constant for all} \quad q \in \mathbb{H}$$

(Poincaré 1905). Observe that $\|X\|^2 = c^2\alpha^2 - f$ and $\|\Im q^2\|^2 = 4c^2\alpha^2(c^2\alpha^2 - f)$ are nonnegative iff $c^2\alpha^2 \geq f$ which is always satisfied when $f \leq 0$.

By (11.2) in (Ungar 2008), the Lorentz transformation *without* rotation is a boost $L(u)$ for $u \in B_c$ such that, for $u_c = \frac{1}{c}u$, $q_c = \frac{1}{c}q = (\alpha, X_c)$

$$L(u)q_c = (\gamma_u[\alpha + < u_c, X_c >], \gamma_u u[\alpha + \frac{\gamma_u}{1+\gamma_u} < u_c, X_c >]).$$

Then by (11.10) for $u, v \in B_c$ we get the composition law:

$$L(u)L(v) = L(u \oplus v)rel(u,v) = rel(u,v)L(v \oplus u).$$

The general case (transformations with rotations in $SO(3)$ is given in (11.15), (11.20).

These formulae shed an interesting light about the connection between hypercomputation in $\mathbb{H}$ based on $\times$ and computation in the cloth W_E based on Einstein addition $\oplus_E$. The connection is developed in the references (Chatelin 2011, 2012b).

Example 3.2 For a given λ, $0 < \lambda < \infty$, we set $x_\lambda = \frac{1}{\lambda}x$ for $x \in \mathbb{R}^n$, and consider the organ $B_\lambda = \{x \in \mathbb{R}^n; \|x\| < \lambda\}$ where the addition is the Poincaré addition

$$x \oplus_P y = \frac{\left(1 + 2 < x_\lambda, y_\lambda > + \|y_\lambda\|^2\right) x + \left(1 - \|x_\lambda\|^2\right) y}{1 + 2 < x_\lambda, y_\lambda > + \|x_\lambda\|^2 \|y_\lambda\|^2}$$

which is not well-defined when $y = -x$ on the sphere ∂B_λ. The scalar multiplication $r \in \mathbb{R} \mapsto r \otimes x = \{0 \ for \ x = 0, \ \mu(r)x \ for \ 0 \neq x \in B_\lambda\}$ is defined by (3.1) where c is replaced by λ (Definition 6.83 where Möbius stands for Poincaré).

Using a common reference λ, $0 < \lambda < \infty$, we obtain two metric cloths W_E and W_P framed in $\mathbb{R}^n$. Remarkably, these two cloths are isomorphic in the following sense. The bijective map $\psi : W_E \to W_P$ defined by $x \mapsto x' = \psi(x) = \frac{1}{2} \otimes x$ preserves $\oplus$ and $\otimes$. See Table 6.1 on p. 226 for more, see also (Ungar 2012). The commutator $[x, y] = (x \oplus y) - (y \oplus x)$ is studied for $\oplus_E$ and $\oplus_P$ in (Chatelin, 2012c).

Example 3.3 Let $\oplus_{PV}$ represent the relativistic addition of velocities in the traveller's time defined in Example 2.3. The underscript PV for *proper velocity* used by Ungar indicates that time refers to the traveller, i.e. the moving observer (standing inside the phenomenon) rather than to an outside observer. Definitions 6.87 on p.223 gives

$$\mu_{PV}(r) = \frac{1}{\|x_\lambda\|} \sinh(r \sinh^{-1} \|x_\lambda\|), \ r \in \mathbb{R},$$

which is a modification of (3.1): λ replaces c and sinh replaces tanh. Then $(V \oplus_{PV}, \otimes)$ is the metric cloth W_{PV}.

Because $a - a = 0$ in V, $-a = (-1) \times a = \ominus a$ in W. In general $r \otimes (a \oplus b) \neq (r \otimes a) \oplus (r \otimes b)$, unless a and b are dependent. Scalar multiplication distributes axially (Theorem 6.4). The automorphisms of W form the group Aut (W): they consist of automorphisms of G which preserve also the scalar multiplication $\otimes$ and the scalar product $< \cdot, \cdot >$ (Definition 6.5).

The identification $-a = \ominus a = \hat{-} a$ which holds in W provides more insight on the induced addition $\hat{+}$ by considering the mirror equation for (2.2) where a and b are exchanged:

$$b \oplus \tilde{x} = a. \tag{3.2}$$

Lemma 3.2.

$$\tilde{x} = -\hat{x} \tag{3.3}$$

Proof. (3.1) yields $\tilde{x} = -b \oplus a$ by (2.4) and $\hat{x} = b \ominus a$ by (2.8). Now $\hat{x} = -(-b \oplus a) = -\tilde{x}$. □

In the larger context of a cloth, the liaison Λ includes $+$, as illustrated by the identification $\hat{x} = -\tilde{x}$.

Definition 3.1 An additive weaving computation refers to any algebraic computation taking place in a metric cloth $W = \{S, \oplus, \otimes\}$.

The set of operations that we shall consider in Weaving Information Processing (WIP) is restricted to $\mathrm{Op}(W) = \mathcal{L} \cup \mathcal{R} \cup \mathrm{Aut}\,(W)$.

Definition 3.2 The Weaving Information Processing (WIP) in a metric cloth W is realised in W by means of $\mathrm{Op}(W)$.

We shall study by geometric means the results of WIP. The metric cloth W inherits from its euclidean frame not only a scalar product/norm, but also its *affine* essence with respect to a *real* parameter. Therefore the geometry derived from a cloth is based on *lines* (as affine functions of a real parameter) and in particular on *geodesics* (for which the triangle inequality becomes an equality). In what follows, we build on Ungar's mathematical vision based on physical insight. We develop some aspects of the role of geometry in WIP. The existence of the three additions $\oplus, \hat{+}, +$ endows cloth geometry with several ways to carry information, shedding a new light on the role of *non euclidean* geometry in Information Processing (IP).

4. The Metrics Associated with $\oplus$ and $\hat{+}$

4.1 Definition

We revisit the three linear equations (2.2), (2.3) (2.7) and their three solutions x (2.4), y (2.5), $\hat{x}$ (2.7). A simplification occurs because $\|rel(a,b)\| = 1$ for $x = rel(-a,b)\hat{x}$, hence $\|x\| = \|\hat{x}\| =\neq \|y\|$. Thus one can associate *two* metrics d in W with the three cancellation laws. They are given by

$$\begin{aligned} \mathring{d}(a,b) &= \| -a \oplus b\| = \|b \ominus a\|, & (4.1)\\ \hat{d}(a,b) &= \|b \,\hat{-}\, a\| = \|b \ominus rel(b,a)a\|. & (4.2) \end{aligned}$$

where the upperscripts $^\circ$ and $\hat{}$ for d refer to the respective additions $\oplus$ and $\hat{+}$. The values are identical when a and b are dependent.

Ungar's inequality (6.14) (resp. (6.18)) expresses the following triangle (resp relating-triangle) inequality (4.3) (resp. (4.4)):

$$\mathring{d}(a,c) \leq \mathring{d}(a,b) \oplus \mathring{d}(b,c), \tag{4.3}$$

$$\hat{d}(a, rel(a \hat{-} b, b \hat{-} c)c) \leq \hat{d}(a,b) \hat{+} \hat{d}(b,c). \tag{4.4}$$

It is clear that $\mathring{d}$ defines a distance, whereas $\hat{d}$ does not (pp. 158 and 159). Curves for which (4.3) is an equality are the *geodesics* of $\oplus$ associated with the distance $\mathring{d}$.

4.2 Invariance Properties

The two metrics are invariant under Aut (W). Metric invariance under left shift in $\mathcal{L}$ holds for $\mathring{d}$ by (2.15): $\| -a \oplus b\| = \|b \ominus a\| = \|(g \oplus b) \ominus (g \oplus a)\|$ for any $g \in G$ (Theorem 6.12).

Regarding $\mathcal{R}$-invariance for $\hat{+}$ (based on $\oplus g$), if $rel(a,b) = I$, then: $a \hat{-} b = (a \oplus g) \hat{-} (b \oplus g)$ implies $\mathcal{R}$-invariance for $\hat{+}$. This is always true when a and b are dependent. In general (2.16) above holds with $k = rel(a,b)g$, $\|k\| = \|g\|$. The topic will be developed further in Section 8.1.

Remark 4. *On the notational dilemma*

It is important to keep in mind that, in the connecting context of weaving computation, the notation itself is, by force, ambiguous. For example the notation $\mathring{d}$ and $\hat{d}$ was suggested by the definitions (4.1), (4.2). But, of course, the notation $\mathring{d}$ measures in an equal fashion both $x = \ominus a \oplus b$ associated with $L\oplus$ and $\hat{x} = b \ominus a = rel(a,b)x$ associated with $\hat{+}$. And $\hat{d}$ reflects the *unique* aspect $R\oplus$ converted into $\hat{+}$. In the difficult task to capture as best as possible the subtle relational interplay between $\oplus$ and $\hat{+}$, cloth geometry will prove to be a precious ally.

5. About the Organic Lines Passing through 2 Distinct Points in Cloth Geometry

5.1 Introduction

Let be given $a \neq b$ in $\mathbb{R}^n$. In euclidean geometry there exists a unique straight line passing through a and b, which can be represented by the affine function: $t \in \mathbb{R} \mapsto a + (b-a)t \in \mathbb{R}^n$: the point a (resp. b) corresponds to $t = 0$ (resp. 1). The straight line is the geodesic of the euclidean metric. The segment $[a,b]$ is defined by $0 \leq t \leq 1$. It has a unique midpoint $m_{ab} = a + \frac{1}{2}(b-a) = \frac{1}{2}(b+a) = m_{ba}$. This simple euclidean picture will be modified in cloth geometry since there exist *more than one* affine curve passing through two points due to the existence of more than one cancellation law.

In what follows we restrict our attention to the three fundamental (cancellation) laws (2.9), (2.10), (2.11) that we put at the foundations of our geometric study. The three laws are ordered respectively as first, second and third. They define *three* types of affine functions defining *organic lines* L_i numbered by $i \in \{1, 2, 3\}$. It is important to distinguish whether a and b are dependent or not.

5.2 Three Fundamental Organic Lines through a and b Independent

To each fundamental law we can associate a unique fundamental (organic) line passing through a for $t = 0$ and b for $t = 1$. The non commutative addition $\oplus$ provides the left-(resp. right-) line $L - L_{ab}$ (resp. $R - L_{ab}$). The commutative addition $\hat{+}$ provides the unique line-$\hat{L}_{ab}$. These lines are given by the table below

symbol	definition	representation, $t \in \mathbb{R}$
$L_1 = L\text{-}L_{ab}$	left-line for $L\oplus$	$a \oplus ((-a \oplus b) \boxtimes t)$ (5.1)
$L_2 = R\text{-}L_{ab}$	right-line for $R\oplus$	$((b \hat{-} a) \boxtimes t) \oplus a$ (5.2)
$L_3 = \hat{L}_{ab}$	line for $\hat{+}$	$((b \ominus a) \boxtimes t) \hat{+} a = a \hat{+} ((b \ominus a) \boxtimes t)$ (5.3)

We call a the *origin* of the 3 lines ($t = 0$). The three solutions $x, y, \hat{x}$ are the respective coefficients of t for the lines; they are distinct iff a and b are independent. The 3 representations can be rewritten respectively under the form: $a \oplus x \boxtimes t$, $y \boxtimes t \oplus a$, $\hat{x} \boxtimes t \hat{+} a = a \hat{+} \hat{x} \boxtimes t$.

Lemma 5.1. *If a and b are dependent, non zero and distinct, $y = b \hat{-} a = b \ominus a = \hat{x} = -a \oplus b = x$ is a real multiple of a.*

Proof. Use $rel(a, b) = I$ to show that $x = y = \hat{x}$. If $b = l \boxtimes a$, $l \neq 1$, $x = -b \oplus a = (1 - l) \boxtimes a = \mu(1 - l)a$, where $\mu(1 - l) \neq 0$ for $l \neq 1$. □

5.3 L_1 and L_2 are Geodesics for $\mathring{d}$ and $\hat{d}$

The lines L_1 and L_2 define 2 notions of collinearity between a, b and a third point c which are distinct when a and b are independent.

By Definition 6.22 (resp. 6.55) the 3 points a, b, c are L_1- (resp. L_2-) collinear iff there exists $t \in \mathbb{R}$ such that c satisfies (5.1) (resp. (5.2)). The points c defined for $0 < t < 1$ are between a and b on L_1 (resp. L_2). They define the open organic arc $L_1 - (a, b)$ (resp. $L_2 - (a, b)$).

In view of (4.3), it is not surprising that L_1 is a geodesic for $\mathring{d}$ (Theorem 6.48, Remark 6.49). The less obvious Lemma 6.61 tells us that $rel(a \hat{-} c, c \hat{-} b) = I$ when

a, b, c are L_2-collinear. It follows that L_2 is a geodesic for $\hat{d}$ (Theorem 6.77, Remark 6.79).

We observe that the noncommutativity of $\oplus$ $(L\oplus \neq R\oplus)$ which is controlled by the relator entails the existence of two distinct geodesics related to the metrics (4.1) and (4.2) when a and b are independent, $rel(a, b) \neq I$.

Corollary 5.2. *When $a \neq 0$ and $b = l \otimes a$, $l \neq 1$, the 3 lines L_i, $i = 1, 2, 3$, coalesce into the geodesic for $\mathring{d} = \hat{d}$ which is the euclidean straight line spanned by a.*

Proof. Apply Lemma 5.1. The common geometric image is a euclidean straight line, more precisely the linear axis spanned by $a \neq 0$. □

If $l = 1$, the lines degenerate into the point $a \neq 0$. Observe that it is the *linear independence* of a and b which forces the organic lines to bend, indicating a non linearity in disguise.

6. About Midpoints on Organic Arcs

There are 3 types of fundamental organic arc (a, b) to consider which are denoted $L_i - (a, b)$. We first assume that a and b are independent.

6.1 Midpoints on L_1 and L_2 for $\oplus$

In Chapter 6, Ungar shows that a unique midpoint on $L_1 - (a, b)$ exists for (5.1) by Theorems 6.53, 6.34 and on $L_2 - (a, b)$ for (5.2) by Theorem 6.74:

- $m^L_{ab} = a \oplus (x \otimes \frac{1}{2}) = \frac{1}{2} \otimes (a \hat{+} b) = b \ominus \left(x \otimes \frac{1}{2}\right) = m^L_{ba},$ (6.1)
  $\|a \ominus m^L_{ba}\| = \|b \ominus m^L_{ba}\| = \|x\| \otimes \frac{1}{2},$
- $m^R_{ab} = \left(y \otimes \frac{1}{2}\right) \oplus a = b \ominus \left(y \otimes \frac{1}{2}\right) = m^R_{ba}$, with $\|y\| \neq \|x\|$, (6.2)
  $\|a \hat{-} m^R_{ab}\| = \|b \hat{-} m^R_{ab}\| = \|y\| \otimes \frac{1}{2}.$

The equality $m^L_{ab} = m^L_{ba} = \frac{1}{2} \otimes (a \hat{+} b)$, suggests that a and b could play a more symmetric role in the definition of the left line L_1 for $\oplus$ under an appropriate change of parameter.

Lemma 6.1. *The line $L_1 = L\text{-}L_{ab}$ can be represented in the four equivalent forms:*
$a \oplus x \otimes t = a \otimes (1 - t) \oplus b \otimes t$, $x = -a \oplus b$, *and* $b \oplus \tilde{x} \otimes t' = b \otimes (1 - t') \oplus \otimes t'$, $\tilde{x} = -b \oplus a$, *with* $t + t' = 1$.

Proof. $a \oplus (-a \otimes t \oplus b \otimes t) = a \otimes (1 - t) \oplus b \otimes t$ since $rel(a, a) = I$.
When t' replaces t, a and b are exchanged. □

Letting $t = t' = \frac{1}{2}$ yields m^L_{ab} which admits the fully symmetric representation $\frac{1}{2} \otimes (a \hat{+} b)$. This reflects an essential property of the scalar multiplication $\otimes$ by 2 (Theorem 6.7, Ungar 2008).

$$2 \otimes (a \oplus b) = a \oplus (2 \otimes b \oplus a) = a \hat{+} (a \oplus (2 \otimes b)) \tag{6.3}$$

for any $a, b \in W$. In (6.3), $2 \otimes a$ is split so that a occurs in two places in the rhs of $2 \otimes (a \oplus b)$, yielding three terms.

This yields the remarkable

Theorem 6.2. *For any two independent points $a \neq b$ the three additions $L\oplus$, $R\oplus$ and $\hat{+}$ provide the same arithmetic mean on the geodesic $L_1 = L\text{-}L_{ab}$:*

$$m^L_{ab} = \frac{1}{2} \otimes (a \oplus b) = \frac{1}{2} \otimes (b \oplus a) = \frac{1}{2} \otimes (a \hat{+} b).$$

Proof. This is Theorems 6.33 and 6.34. Observe that, in addition to the above coincidences, and to (6.1), we also have $m^L_{ab} = b \oplus \left(\tilde{x} \otimes \frac{1}{2}\right) = a \ominus \left(\tilde{x} \otimes \frac{1}{2}\right)$. □

No such remarkable property holds for m^R_{ab} on $L_2 = R\text{-}L_{ab}$. The identities about m^R_{ab} and m^R_{ba} given in (6.2) cannot be further rewritten in general.

6.2 On the Line $\hat{L}$ for $\hat{+}$

The third type of organic arc (a, b) on $\hat{L}$ defined by (5.3) above has *two* pseudo-means: $\hat{m}_{ab} = \left(\hat{x} \otimes \frac{1}{2}\right) \hat{+} a$ differs from $\hat{m}_{ba} = b \,\hat{-}\, \hat{x} \otimes \frac{1}{2}$ (Section 6.13 in Ungar). However, $\|x\| = \|\hat{x}\|$ guarantees the equality of the respective distances $\|a \,\hat{-}\, \hat{m}_{ab}\| = \|b \,\hat{-}\, \hat{m}_{ba}\| = \|\hat{x}\| \otimes \frac{1}{2}$ and of their counterparts on $L\text{-}L_{ab}$.

Lemma 6.3. *The two pseudo-means $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$ are such that*

$$\|x\| \otimes \frac{1}{2} = \mathring{d}(a, m^L_{ab}) = \mathring{d}(a, \hat{m}_{ab}) = \mathring{d}(b, \hat{m}_{ba}).$$

Proof. Clear by (6.1). □

When a and b are independent, the two midpoints: m^L on $L\text{-}L_{ab}$, m^R on $R\text{-}L_{ab}$, enable dichotomy inside the two arcs $L_1 - (a, b)$, $L_2 - (a, b)$. The existence of *two* pseudo-means $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$ forbids any appeal to a dichotomy argument on an L_3-arc. For this reason, the general study of $L_3 = \hat{L}_{ab}$ is stopped at this point by Ungar, see. p. 205.

Lemma 6.4. *If $a \neq 0$, $b = ra$, for $r \in \mathbb{R}$, the four means (or midpoints) coalesce into the single point $m = \frac{1}{2} \boxtimes (a \hat{+} b)$ on the unique line L_{ab}.*

Proof. When a and b are dependent, $\oplus = \hat{+}$, hence $x = y = \hat{x}$. Then, by Corollary 5.2, the three organic lines L_1, L_2 and L_3 coalesce into a unique one which is the geodesic for $\mathring{d} = \hat{d}$ through a and b. Clearly $m^L = m^R$, and $\hat{m}_{ba} = (b \ominus a) \oplus a \ominus (\hat{x} \boxtimes \frac{1}{2}) = (\hat{x} \boxtimes \frac{1}{2}) \oplus a = \hat{m}_{ab} = m_R = m^L$. □

7. Directional Relators

7.1 Definition

In this Section, the admissible relators belong to the subset $\boldsymbol{Q}$ of automorphisms in Aut W which satisfy (H$_3$):

$$rel(a, b) = rel(\frac{a}{\|a\|}, \frac{b}{\|b\|}) \qquad \text{(H}_3\text{)}$$

for any pair (a, b) of nonzero vectors in V.

In other words, the map *rel* is specified by the unit vectors $1_a = \dfrac{a}{\|a\|}$ and $1_b = \dfrac{b}{\|b\|}$ defining linear directions spanned by a and b. Any relator satisfying (H$_3$) is called *directional*. It follows that

$$rel(a, b) = rel(a, b \boxtimes t) = rel(a \boxtimes t, b) \text{ for any } 0 \neq t \in \mathbb{R}.$$

Example 7.1 Let $x, y \in V = \mathbb{R}^n$ be independent, then $\theta = (x, y) \notin \{0, \pi\}$. Let $R(\theta)$ denote the plane rotation $x \mapsto y$. We define $x \oplus y = x + R(\theta)y$, hence $y \oplus x = y + R(-\theta)x = R(-\theta)(x \oplus y)$ for independent x and y. Otherwise $\oplus = +$. The range $\boldsymbol{Q}$ of the relator is the set of plane rotations $SO(2)$ except the symmetry $-I_2$.

7.2 A Twofold Interpretation of L_3 under (H$_3$)

We use the generic notation $L_{ab} = L(a, x)$ where x is the coefficient of the parameter t in the equation for the associated line passing through the origin $a(t = 0)$ and $b(t = 1)$. For example, $\hat{L}_{ab} = L_3 = L(a, \hat{x})$, $\hat{x} = b \ominus a$. The line $\hat{L}_{ab}$ can be interpreted equally as a version of $(i), L_1$ or $(ii), L_2$.

Lemma 7.1. *(i) $a \hat{+} \hat{x} \boxtimes t = a \oplus x \boxtimes t$ with $rel(a, -b)\hat{x} = x$.*

(ii) $\hat{x} \boxtimes t \hat{+} a = \hat{x} \boxtimes t \oplus a_2$ with $a_2 = rel(b, -a)a$.

(iii) Moreover $\|\hat{x}\| = \|x\|$, $\|a\| = \|a_2\|$: $\hat{x}$ and a_2 are rotated about O from x and a through the same angle.

Proof. (i) $a \hat{+} \hat{x} \boxtimes t = a \oplus (rel(a, -\hat{x})\hat{x}) \boxtimes t$ by (2.6) and (H$_3$) with $rel(a, -b \oplus a) = rel^{-1}(-b \oplus a, a) = rel^{-1}(-b, a) = rel(a, -b)$ by (A1). And $rel(-a, b)\hat{x} = x$.

(ii) $\hat{x} \boxtimes t \hat{+} a = (b \ominus a) \boxtimes t \oplus rel(\hat{x}, -a)a$ by (2.6) and (H$_3$) again, and $rel(b \ominus a, -a) = rel(b \oplus (-a), -a) = rel(b, -a)$.

(iii) Clear when we observe that $rel^{-1}(a, -b) = rel(b, -a)$. □

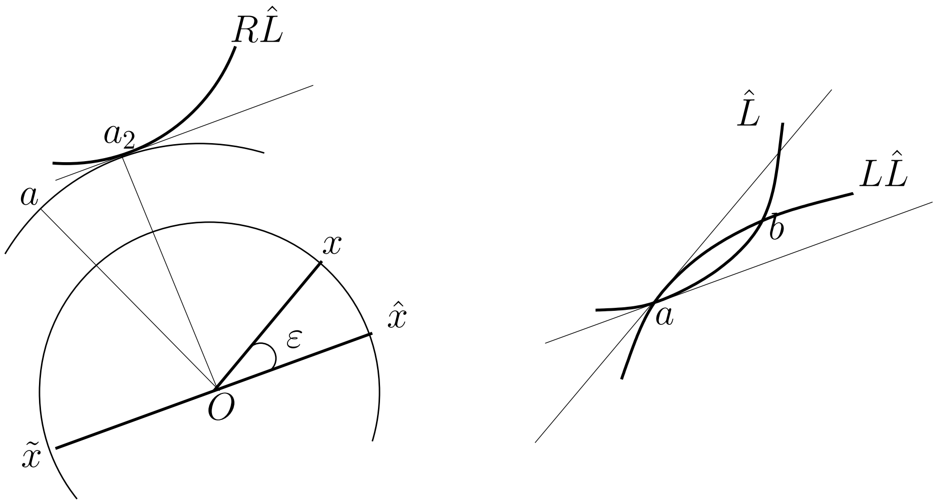

Figure 1: $\hat{L} = \hat{L}_{ab}$ *and its left and right interpretations/images:* $L\hat{L} = L\text{-}L_{ab}$ *and* $R\hat{L} = R\text{-}L(a_2, \hat{x})$.

Proposition 7.2. *When the relator is directional, the following two interpretations hold for* $\hat{L}_{ab}$*:*
(i) $\hat{L}_{ab} = L\text{-}L(a, x) = L\text{-}L_{ab}$ *with* $x = rel(a, -b)\hat{x}$, $b = a \oplus x = a \hat{+} \hat{x}$.
(ii) $\hat{L}_{ab} = R\text{-}L(a_2, \hat{x}) = R\text{-}L_{a_2 b_2}$ *with* $a_2 = rel(b, -a)a$, $b_2 = \hat{x} \oplus a_2$.

Proof. Apply Lemma 7.1. For $t = 1$, (i) $a \oplus x = b = a \hat{+} \hat{x}$, (ii) $\hat{x} \oplus a_2 = b_2 \iff \hat{x} = b_2 \hat{-} a_2 = b \ominus a$. ☐

The line $\hat{L}_{ab} = L(a, \hat{x})$ can be interpreted at the same time as the left or right image of two distinct sources. In the left (resp. right) image, the origin a is preserved (resp. moved to a_2) and the coefficient $\hat{x}$ is moved back to x (resp. preserved). Therefore the line $\hat{L} = \hat{L}_{ab}$ is a *composite* construction resulting from $\oplus$ and $rel(b, -a)$ with a *dual* character: it can be interpreted either leftwise or rightwise. Quite remarkably, the left interpretation $L\hat{L}$ is $L\text{-}L_{ab} = L_1$ itself. The right interpretation $R\hat{L} = R - L_{a_2 b_2}$ can be characterised by the rotation $x \mapsto \hat{x} = rel(b, -a)x$ about O through the angle ε. Then a is rotated into a_2 through ε. See Figure 1.

There are altogether *four* lines of interest associated with a pair (a, b): the three fundamental lines L_i, $i = 1, 2, 3$ through a, b plus the right interpretation or image $R\hat{L}$ through a_2, b_2.

7.3 a and b are Dependent

When a and b are dependent and distinct, nonzero, the 3 points O, a, b are collinear. As we know an *essential simplification* takes place: the *three organic* lines above coalesce geometrically into *one*. When the relator is *directional*, more can be said about the right image $R - L_{a_2 b_2}$ for $\hat{L}$.

Lemma 7.3. *If $a \neq b$ are dependent, then $rel(a,b) = I$, $a_2 = a$, $b_2 = b$ and $R\hat{L} \equiv L_i$, $i = 1,2,3$. If $a = b$, $x = 0$ and the line reduces to the point a.*

Proof. By assumption $rel(a,b) = I$ then $a \oplus b = a \hat{+} b$, hence $x = y = \hat{x} = b \ominus a$. The 3 lines L_1, L_2, L_3 coalesce into a unique line $a \oplus x \boxtimes t = x \boxtimes t \oplus a = x \boxtimes t \hat{+} a$, if $x \neq 0 \iff b \neq a$ (Corollary5.2). If $b = a$, $x = 0$, the lines reduce to the unique point $a = b \neq O$.

For the right image $R\hat{L}$ $a_2 = a$ and $b_2 = b$, yielding the identification $R\hat{L} \equiv L_i$ □

When a and b are *dependent* and distinct, a unification takes place. Not only the organic lines coalesce into the geodesic L_1, but also does, when the relator is directional, the right image $R\hat{L}$.

8. Weaving Information Processing (WIP)

8.1 Organic Lines

Among the three fundamental lines passing through a and b independent, the *first two* are geodesics (expressing two different views on the non commutativity of $\oplus$ (Section 5.3)). Remarkably, the third organic line L_3 offers, under (H$_3$) two geodesic images of itself, either $L\hat{L} = L_1$ as its left image, or $R\hat{L} = R - L_{a_2 b_2}$ as its right image.

This Section develops some consequences of these geometric properties on information processing in weaving computation.

Proposition 8.1. *The two additions $\oplus$ and $\hat{+}$ coalesce on the geodesic $L_1 = L\text{-}L_{ab}$.*

Proof. By successive dichotomy arguments based on Theorem 6.2 above: $x \oplus y = x \hat{+} y$ for any x, y between a and b on $L\text{-}L_{ab}$. □

Proposition 8.1 indicates that a sort of "differential" commutativity holds for $x \oplus y$ when x and y vary on L_1. Given a and b linearly independent the geodesic for $\mathring{d}$ through a, b describes the unique locus of points for which $\oplus$ is *commutative*, hence $\oplus = \hat{+}$ locally (on L_1). This mechanism underlies the emergence of the axiomatic role of commutativity for addition in classical mathematics.

Let us turn to L_2 which is a geodesic for $\hat{d}$; it plays a very different connecting role in IP that is discovered by revisiting (2.16) above:

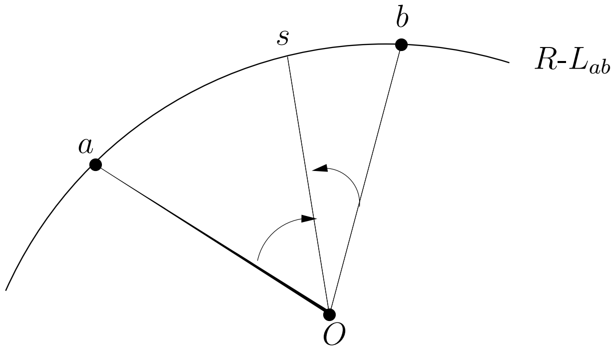

Figure 2: $a, b,\ s \in R\text{-}L_{ab} = L_2$.

Proposition 8.2. *The line $L_2 = R\text{-}L_{ab}$ is such that for any $w \in W$ and $s \in R\text{-}L_{ab}$, then*

$$b \,\hat{-}\, a = (b \oplus rel(b,s)w) \,\hat{-}\, (a \oplus rel(a,s)w) \tag{8.1}$$

Proof. See Theorem 6.76 in (Ungar 2008) and Figure 2. □

The identity (8.1) is one possible form of the kind of right shift-invariance enjoyed by $\hat{+}$ when a and b are independent, which generalises (2.16). Indeed, if $s = a$, (8.1) yields $b \hat{-} a = (b \oplus rel(a,b)w) \hat{-} (a \oplus w)$ which becomes (2.16) after exchanging a and b, and setting $w = g$. The coefficient $y = b \,\hat{-}\, a$ is *invariant* when the same right shift, chosen in $\{\cdot \oplus rel(\cdot, s)w,\ w \in W,\ s \in R\text{-}L_{ab}\}$, is equally applied to a and b, see Figure 2 . This *exact*, albeit special, kind of $\mathcal{R}$-invariance for $\hat{+}$ under right shift should be contrasted with the metric $\mathcal{L}$-invariance for $\oplus$ (which hides the rotation $rel(g,b)$ present in (2.15)).

Definition 8.1 Given $a \neq b$, the property (8.1) for $w \in W$, $s \in L_2 = R\text{-}L_{ab}$ defines the L_2- link between a and b assumed to be independent.

Any s on $R\text{-}L_{ab}$ is uniquely defined by $t \in \mathbb{R}$ through (5.2) which defines the map:

$$t \in \mathbb{R} \mapsto y(t) = \Big(y \boxtimes t\Big) \oplus a,\ t \in \mathbb{R},\ y = b \,\hat{-}\, a.$$

At any $(t, w) \in \mathbb{R} \times W$ we consider in W

$$z_a(t) = rel(a, y(t))w, \quad z_b(t) = rel(b, y(t))w,$$

with $z_a(0) = z_b(1) = w$. By (8.1), $b \,\hat{-}\, a = (b \oplus z_b(t)) \,\hat{-}\, (a \oplus z_a(t))$ for all $t \in \mathbb{R}$, with $\|z_a(t)\| = \|z_b(t)\| = \|w\|$ for any $w \in W$.

The L_2- link between a and b is ruled by the two values $rel(a, y)$ and $rel(b, y)$ for the relator. Indeed, $rel(a, (b \hat{-} a) \otimes t \oplus a) rel(b \hat{-} a, a) = I$ by (2.16) in Ungar (2008), and $rel^{-1}(b \hat{-} a, a) = rel(-a, a \hat{-} b)$ (Section 2.4 in Ungar).

Proposition 8.3. *When w varies on the sphere $S_r = \{w, \|w\| = r\}$ for $0 < r < \lambda$, the L_2- link between a and b maintains $z_a(t)$ and $z_b(t)$ on S_r for all $t \in \mathbb{R}$. In particular $z_a(0) = z_b(1) = w$.*

Proof.
Clear from the above discussion. □

When w is arbitrary in W, the double equality $\|w\| = \|z_a(t)\| = \|z_b(t)\|$ holds for any t, and hides the actual source of the L_2- link (8.1) between a and b which resides in the relator at the pairs $(a, b \hat{-} a)$ and $(b, b \hat{-} a)$.

As for the third line, Section 7 has told us that, when the relator is **directional**, the line $\hat{L} = L_3$ is a *shape-shifter*: it can be interpreted as $L\hat{L} \equiv L_1$ and equally as $R\hat{L} = R - L_{a_2 b_2}$ which differ when $rel(a, b) \neq I$.

8.2 Weaving Computation and Broadcasting Information

The broadcasting of information from a to b uses the *real* parameter t in $\mathbb{R}$ to channel through the three lines L_i with distinct features.

1) For the geodesic L_1, $\|b \ominus a\|$ is invariant under left shift. We say that L_1 *radiates metric* information. In other words, L_1 is a channel which is blind to rotations performed on the results produced by WIP: it is a *normative* channel. Because the two additions $\oplus$ and $\hat{+}$ yield identical results for any pair of points picked on itself, L_1 draws the *commutative* path from a to b: addition $\oplus$ is *locally* commutative on L_1.

2) By comparison, the geodesic L_2 through a and b (when independent) is a channel which *selects*, from the whole of WIP results, only the ones which enjoy the L_2-link (according to Definition 8.1). We say that L_2 *emanates* selected *exact* information. It is a *discriminative* or *filtering* channel.

The lines L_1 and L_2 are the *two* channels associated with $L\oplus$ and $R\oplus$ respectively: they differ geometrically when a and b are independent.

3) As we have already noticed, the third line $L_3 = \hat{L}$ is a computational construct which can be *interpreted* by means of any of the channels L_1 and L_2 when the relator is directional. The two interpretations differ markedly from $\hat{L}$ and between themselves.

The left image L_1 generally differs from $\hat{L}_{ab}$ for $t \neq 0$ and 1. The right image is $R\hat{L} = R\text{-}L_{a_2 b_2}$ which passes through $a_2 \neq a$ $(t = 0)$ and $b_2 \neq b$ $(t = 1)$ in general. This computational property lends weight to the notion of "action at a distance" for information, a possibility which is most often ruled out a priori in empirical science.

By contrast, if a and b are *dependent*, $a \neq b$, there exists a *unique* channel because all L_i coalesce into the axis spanned by $a \neq 0$. When the relator is directional, the right geometric image for $\hat{L}$ is $\hat{L}$ itself.

It appears that there are several *distinct* ways by which information can be broadcast from a to b:

(i) If a and b are independent, there exist two distinct channels of information based on $L\oplus$ and $R\oplus$: the left one is a geodesic for $\mathring{d}$ which is normative and the right one is a filtering geodesic for $\hat{d}$. Provided that the relator be *directional*, these channels enable the computing agent to get a left and right interpretations for the construct $L_3 = \hat{L}$. It is remarkable that the right interpretation sustains the ill-received concept of "action at a distance" for information.

(ii) If a and b are dependent, and if the relator is directional, the two channels L_1 and L_2 coalesce with the organic line L_3 and with its two images.

9. An Epistemological Appraisal

The fact that hyperbolic geometry underlies Special Relativity was quickly realised by a handful of physicists and geometers (Ungar, 2008, Section 3.8); Ungar (2012). But the scope of hyperbolic geometry reaches much further.

9.1 Hyperbolic Geometry in Nature

A number of natural shapes exhibit, at least locally, a hyperbolic character in their geometry. The most famous example is a horse saddle or a mountain pass. Among other natural hyperbolic surfaces, one can cite lettuce leaves, coral reef or some species of marine flatworms with hyperbolic ruffles. According to W. Thurston, if one moves away from a point in hyperbolic plane, the space around the point expands exponentially. The idea was implemented in crochet in 1997 by D. Taimina by ceaselessly increasing the number of stitches in each row of her crochet model (Henderson and Taimina 2001). Experiments have shown that the visual information seen through the eyes and processed by our brain is better explained by hyperbolic geometry (Luneburg 1950). This explains the popularity of hyperbolic browsers among information professionals (Lamping et al. 1995, Allen 2002). Einstein gyrovector spaces are used in (Urribarri et al. 2013) to program an efficient tree layout, with varying levels of detail for data enclosed in a 3D-volume.

9.2 Axiomatic vs. Cloth Geometries

The classical concept of a *group* underlies the three geometries which can be axiomatically derived from three versions of the parallel postulate: by a point not on a given line in a plane, one can draw a number p of parallels to the line with $p \in \{0, 1, \infty\}$. The best-known case $p = 1$ corresponds to a linear vector space endowed with a scalar product and derived norm. The cases $p = 0$ (elliptic) and $p = \infty$ (hyperbolic) are modifications of the euclidean case, each with many equivalent models.

By comparison, cloth geometry is derived from a metric cloth framed in a linear normed space with dimension $n \geq 2$, and based on an organ $G(\oplus$,relator). It is *not* axiomatically defined, but is a *computational construct* based on $\oplus$ and on the corresponding choice of automorphisms in the relator's range $\boldsymbol{R}$. The computation results in a **trimorphic** geometry in which the relator for $\oplus$, by inducing a secondary addition $\hat{+}$, blurs the clear-cut distinctions created by axiomatisation based on an abelian group. For example, it can be proved that $p = \infty$ and $p = 1$ are co-existing properties (Fig. 8.50 on p. 370). Depending on the choice $\boldsymbol{R}$ of isometries, the computed geometry will exhibit, in addition to the euclidean structure of the frame V, *new non-euclidean* features, among which some are considered as characteristic of either hyperbolic or elliptic geometries defined axiomatically. To witness, Chapter 7 in (Ungar 2008) ends on p. 259 with the following statement:

"In modern physics, hyperbolic geometry is the study of manifolds with Riemannian metrics with contant negative curvature. However, we can see from Table 7.1 that in classical hyperbolic geometry, that is, the hyperbolic geometry of Bolyai and Lobacheivsky, constant negative curvatures and variable positive ones are inseparable."

The clear-cut distinction between the three aspects of geometry is relative rather than absolute: it can be by-passed by *weaving computation*.

9.3 Cloth Geometry in the Mind

In (Calude and Chatelin 2010,Chatelin 2012 a,b,d,2015) we have argued that hypercomputation in multiplicative Dickson algebras is part of the algorithmic toolkit for the human mind. Experimental evidence provided by Special Relativity indicates that the mental reconstruction of the observed outside 3D-reality is controlled by cloth geometry based on Einstein addition of 3D-velocities. This may offer a possible clue to what is perceived by some physicists as a pre-established harmony between mathematics and physics (Minkowski 1908,Wigner 1960, Pyenson 1982, Ungar 2003). The paper (Ungar 2003) analyses the twofold harmony which takes place in Special Relativity. Two complementary aspects of *equal importance* are useful to understand SR: either physics and geometry in 3D (Einstein 1905) or analysis in 4D (Poincaré 1905, Minkowski 1908). These complementary aspects are but the two sides of the same coin: mathematical *computations* in the mind.Both aspects have not been equally understandable in the beginnings. Therefore Minkowskian relativity prevailed for a long time, leaving certain theoretical gaps which can be filled elegantly with an appeal to the original idea of Einstein in its geometrically more mature form developed later by Ungar, see (Ungar 2003).

Going back to the human intellectual reconstruction of relativity, we **posit** that, more generally, there exists a commonly shared set of relators for mind computation. This would explain why most people agree on the general appearance of the external landscape, if not on all the details. Two eye-witnesses never agree on the minute details about the scene they both observed at the same place and time. The existence of a common cloth geometry in 3D could be the reason why we, human beings, have the feeling that we share more or less the same external reality, our habitat called Nature.

As for the inner world inside each of us, it differs widely from one individual to the next. Why? Because the number n of dimensions for the frame is not bound to be 3 anymore, but may vary arbitrarily at will, $n \geq 2$.

Cloth geometry provides a plausible mechanism for outer action and inner understanding after observation. In WIP perspective, both processes result from a drive in the mind toward explanation. The observer is *free* to choose to relate a and b by outer or inner observation. However the reader should remember that the physical reference $\lambda = c$ for the speed of light is imposed by physical reality and defines the limit of observable velocities. No such constraint exists for inner observation; in other words the inner reference λ is self-imposed (or chosen).

9.4 On the Poincaré vs. Einstein debate about Relativity and Geometry

During the first two decades of the 20th century the intellectual debate about the "true" nature of physical space was structured around Poincaré (and his legacy after 1912) and Einstein, see (Paty 1992). These giants stood at the two endpoints of a continuum of ideas running from Mathematics to Physics. The issues at stakes have been heatedly debated, including a priority dispute which appears rather futile in view of Ungar's isomorphism between W_E and W_P.

On the one hand Poincaré had an axiomatic vision of Geometry based on *groups* which led him to anticipate the "law of relativity" (Poincaré 1902). In special relativity he proved the dynamical invariance of physical laws for Mechanics and Electromagnetism (slightly ahead of Einstein). The relativistic dynamics presented in (Poincaré 1905) bears on group theory and (implicitly) on the field $\mathbb{H}$ of quaternions, two advanced mathematical notions which are now common in theoretical physics. His work wraps up more than 250 years of discoveries about the baffling behaviour of light (Auffray 2005). Poincaré is often criticised because – as Lorentz, Maxwell and Fresnel did before him – he occasionally mentions *ether*, a notion which is considered unnecessary in current physics. We remark that in the cognitive perspective of information processing in the mind, a background reference is required for weaving computation, whatever name is given to it, ether, or cloth geometry, or even riemannian geometry for General Relativity.

On the other hand, it is clear that Einstein did not at first feel the need for a non-euclidean geometry, because he only slowly became aware of the physical consequences of his non symmetric composition law. Together with Ehrenfest, Max Born and others, he realised that an accelerated motion would not permit exact rigidity for the moving body, but would imply elastic deformations and possible explosion. In order to save the relativity principle (by showing that it can apply to all kinds of motions including accelerations) Einstein had to *modify the geometry*, thus uncovering the full breadth of the 1905 paper.

Following (Paty 1992), we may say that: “Poincaré thought Physics with his geometric mind, as much as Einstein viewed Geometry through his physicist’s eyes”.

The principle of relativity has been observed in light phenomena since the 17th Century. In this intellectual odyssey, history has chosen to emphasise the year 1905 and the sole contribution of the physicist Einstein, This is an ironical twist of fate since the version of Special Relativity which survives today in textbooks rests upon the group structure of Lorentz transformations due to the mathematician Poincaré, while it overlooks the information role played *implicitly* by Einstein’s non commutative addition of 3-vectors for the construction of the human image of the world.

In retrospect, one realises that special relativity in physics has two intricate aspects based on *two* algebraic structures: the metric cloth W_E (based on $\oplus_E$) envisioned by Einstein *and* the noncommutative field $\mathbb{H}$ (based on $\times$) implicit in Poincaré. A thorough comparison between the *distinct* computational roles played by these two structures is given in (Chatelin 2011).

9.5 Einstein’s Vision of Relativity

In 10 years (1905-1915) Einstein’s vision evolved from the commonly shared euclidean view to a highly personal one. By transmuting ideas borrowed from Riemann and Poincaré he was led to General Relativity in 1916. This larger vision he would maintain and refine for the rest of his life (Einstein 1921). Hence his work presents a remarkable continuity of thought since the day he planted the seed of Relativity by positing that admissible velocities do *not* add in a symmetric fashion. The simplicity of this idea – so daring at the time – should strike a chord in any mathematically inclined mind! Simplicity is not triviality ...; it means depth and beauty, conferring a flavour of eternity to Einstein’s revolutionary idea. The new idea ran against a couple of centuries of scientific development for physics, which had climaxed in the 19th century with a commutative addition for 2- or 3-vectors in classical Mechanics, symbolised by the parallelogram law. It is fair to say that there exists a world of difference between the two physics papers authored by Einstein and Ungar which are 83 years apart (1905-1988): the difference illustrates the progress of algebraic knowledge in the 20th century. More than a century had to elapse to allow the slow coming of age of the idea of relativity: from its birthplace in experimental physics to its original habitat in the human mind which can add vectors in a *non*commutative way. This evolution would not surprise the perceptive Mach who wrote in *Die Mechanik* (1883): "We should not consider as *foundations* for the real universe the auxiliary intellectual means that we use for the *representation* of the world on the *stage of thought." (italics in original).*

The relativistic formula is routinely put to good use by engineers in telecommunications, geolocalisation and space industries. But is it really understood? A look at textbooks for physics undergraduates casts some doubts. The pristine clarity of Einstein's addition is obscured behind the cloud of Lorentz transformation and its inherent technicalities. The essence is lost in the mist of Minkowski's 4D-spacetime as this is recalled in (Ungar, 2003). It is not uncommon to find only the symmetric formula (valid for parallel velocities) as any quick websurf will confirm. It is no surprise that history has chosen to tout the (physically more difficult) equation $E = mc^2$, which is but one of the many consequences of Einstein's seminal law of noncommutative addition. Why is the analytic Poincaré/Minkowski version still prefered? Because it was the *first* to be accepted in Physics and it offers a satisfactory answer to most questions which have been raised to-date (Ungar 2003). The gaps uncovered in Theoretical Physics have not yet reach the critical mass which would force the physics community to fully endorse the geometric version of Einstein on an equal footing. Hence Ungar is still a lone pioneer.

The result of this unsatisfactory -but all too human- state of affairs is that relativity is not yet fully embraced: it is, at best, interpreted as an exotic law of Nature, with no deeper consequences on everyday life than the use of cellular phones and GPS devices. Relativity is not perceived as giving us a clue about the ways by which the human mind builds its *"imago mundi"*, its image of the world (Chatelin 2012a,b,d). The role of relativity in western science is mostly confined to physics research (nanoscale or high energy) in order to develop ever more sophisticated technologies. More than one century after Einstein's ground breaking invention, relativity has not yet been taken seriously by social scientists. They do not venture beyond the overly simplified version that is called *relativism*, a mental construct which does not do justice to the philosophical depth of relativity.

Information Processing is of paramount importance for human affairs. Information-based activities such as education, medicine, economy and ecology, could benefit greatly from a new relativity-based scientific approach to cognition.

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