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Unconditionally Stable Difference Scheme for the Numerical Solution of Nonlinear Rosenau-KdV Equation

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Abstract

In this paper we investigate a nonlinear evolution model described by the Rosenau-KdV equation. We propose a three-level average implicit finite difference scheme for its numerical solutions and prove that this scheme is stable and convergent in the order of $O(\tau^2 + h^2)$. Furthermore we show the existence and uniqueness of numerical solutions. Comparing the numerical results with other methods in the literature show the efficiency and high accuracy of the proposed method.

Keywords: Finite difference scheme, solvability, unconditional stability, convergence.

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1. Introduction

Nonlinear partial differential equations are useful in describing various phenomena. These equations arise in various areas of physics, mathematics and engineering. Analytical solutions of these equations are usually not available. Since only limited classes of equations are solved by analytical means, numerical solution of these nonlinear partial differential equations is of practical importance. KdV equation is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model and is as follows

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known KdV equation

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[4], so Rosenau [6, 7] proposed the so-called Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{2}$$

The existence and the uniqueness of the solution for (2) were proved in [7], but it is difficult to find the analytical solution for (2). So, much works has been done on the numerical methods for (2) [1, 5]. On the other hand, for the further consideration of the nonlinear wave, the viscous term $+u_{xxx}$ needs to be included [4]

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0, \tag{3}$$

which is usually called the Rosenau-KdV equation. Some analytical methods for the solution of this equation are given in [2, 9]. The authors of [4] proposed a conservative three-level linear finite difference scheme for the numerical solution of Rosenau-KdV equation. They proved the stability and convergency of method and existence and uniqueness of numerical solutions. In this paper, we propose a linear three-level average implicit finite difference scheme for the Rosenau-KdV equation (3) with the following boundary conditions

$$\begin{aligned} &u\left(x_L, t\right) = u\left(x_R, t\right) = 0, \qquad && u_x\left(x_L, t\right) = u_x\left(x_R, t\right) = 0, \\ &u_{xx}\left(x_L, t\right) = u_{xx}\left(x_R, t\right) = 0, && t \in [0, T], \end{aligned} \tag{4}$$

and initial condition

$$u\left(x, 0\right) = u_0\left(x\right), \qquad x \in \left[x_L, x_R\right]. \tag{5}$$

The solitary wave solution for (3) is [3, 9]

$$\begin{aligned} u\left(x, t\right) = &\left(-\tfrac{35}{24} + \tfrac{35}{312}\sqrt{313}\right) \\ &\times \sec h^4 \left[\tfrac{1}{24}\sqrt{-26 + 2\sqrt{313}} \times \left(x - \left(\tfrac{1}{2} + \tfrac{1}{26}\sqrt{313}\right) t\right)\right]. \end{aligned} \tag{6}$$

The structure of this paper is as follows. In Section 2, we will describe a three level average implicit finite difference scheme for the Rosenau-KdV equation and discuss the estimate for the difference solution. In Section 3, we will show that the scheme is uniquely solvable. Then, in Section 4, we will prove the convergence and stability for the difference scheme. Finally numerical results are given in Section 5 to verify our theoretical analysis and efficiency of proposed method in comparison with other methods in the literature.

2. Proposed Finite Difference Scheme

Let $h = (x_R - x_L)/J$ and τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = x_L + jh$ $(j = -1, 0, 1, 2 \ldots, J, J+1)$, $t_n =$

$n\tau(n=0,1,2,\ldots,N, N=[T/\tau])$, $u_j^n \approx u(x_j,t_n)$ and $Z_h^0=\{u=(u_j)\,|\,u_{-1}=u_0=u_J=u_{J+1}=0,\quad j=-1,0,1,...,J,J+1\}$. Throughout this paper, we denote C as a generic positive constant independent of h and τ, which may have different values in different occurrences. We introduce the following notations [4]

$$\begin{aligned}
&\left(u_j^n\right)_x=\tfrac{1}{h}\left(u_{j+1}^n-u_j^n\right), \qquad \left(u_j^n\right)_{\bar{x}}=\tfrac{1}{h}\left(u_j^n-u_{j-1}^n\right),\\
&\left(u_j^n\right)_{\hat{x}}=\tfrac{1}{2h}\left(u_{j+1}^n-u_{j-1}^n\right), \qquad (u^n,v^n)=h\sum_j u_j^n v_j^n,\\
&\left(\bar{u}_j^n\right)=\tfrac{1}{2}\left(u_j^{n+1}+u_j^{n-1}\right), \qquad \|u^n\|^2=(u^n,u^n),\\
&\left(u_j^n\right)_{\hat{t}}=\tfrac{1}{2\tau}\left(u_j^{n+1}-u_j^{n-1}\right), \qquad \|u^n\|_\infty=\sup_j\left|u_j^n\right|.
\end{aligned} \tag{7}$$

We note that
$$\left(\frac{u^{p+1}}{p+1}\right)_x=\frac{1}{p+2}\left[u^p u_x+\left(u^{p+1}\right)_x\right], \tag{8}$$
and
$$\left(u_j^n\right)_{\bar{x}x}=\left(u_j^n\right)_{x\bar{x}}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}.$$

We propose the following implicit finite difference scheme for solving Eqs. (3)-(5)

$$\left(u_j^n\right)_{\hat{t}}+\left(u_j^n\right)_{xx\bar{x}\bar{x}\hat{t}}+\left(u_j^n\right)_{\hat{x}}+\left(u_j^n\right)_{\hat{x}x\bar{x}}+\tfrac{1}{3}\left[u_j^n\left(\overline{u}_j^n\right)_{\hat{x}}+\left(u_j^n\overline{u}_j^n\right)_{\hat{x}}\right]=0, \tag{9}$$

$$j=1,2,3,\ldots,J-1, \qquad n=1,2,3,\ldots,N-1, \tag{10}$$

$$u_j^0=u_0(x_j), \quad j=0,1,2,3,\ldots,J, \tag{11}$$

$$\begin{aligned}
&u^n\in Z_h^0, \quad (u_0^n)_{\hat{x}}=(u_J^n)_{\hat{x}}=0,\\
&(u_0^n)_{x\bar{x}}=(u_J^n)_{x\bar{x}}=0, \quad n=1,2,3,\ldots,N.
\end{aligned} \tag{12}$$

We now state some lemmas which are needed to prove stability and convergence of scheme.

Lemma 2.1. *[8]. For any two mesh functions $u,v\in Z_h^0$ we have the following relations*

1. $(u_x,v)=-(u,v_{\bar{x}})$,
2. $(u_{x\bar{x}},v)=-(u_x,v_x)$,

3. $(u_{x\overline{x}}, u) = -(u_x, u_x) = -\|u_x\|^2,$

4. If $(u_0)_{x\overline{x}} = (u_J)_{x\overline{x}} = 0$, *then* $(u_{xx\overline{x}\overline{x}}, u) = \|u_{xx}\|^2.$

Lemma 2.2. *[8]. There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \leq C_1 \|u^n\| + C_2 \|u_x^n\|. \tag{13}$$

Lemma 2.3. *[8]. Suppose that $\omega(k)$ and $\rho(k)$ is nondecreasing. If $C > 0$, and*

$$\omega(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} \omega(l), \qquad \forall k, \tag{14}$$

then

$$\omega(k) \leq \rho(k) e^{C\tau k}, \qquad \forall k. \tag{15}$$

Theorem 2.4. *If u^n be the solution of (9)-(12), $u_0 \in H_0^2[x_L, x_R]$ and $u(x,t) \in C_{x,t}^{5,3}$ then we have the following relations*

$$\|u^n\| \leq C, \quad \|u_x^n\| \leq C, \quad \|u^n\|_\infty \leq C, \quad n = 1, 2, \ldots N.$$

Proof. Taking an inner product of (9) with $2\overline{u}^n (i.e., u^{n+1} + u^{n-1})$, considering the boundary conditions (12) and Lemma 2.1, we obtain

$$\begin{aligned}\tfrac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2\right) + \tfrac{1}{2\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \left\|u_{xx}^{n-1}\right\|^2\right) + 2\left(u_{\hat{x}}^n, \overline{u}^n\right) + \\ 2(u_{\hat{x}x\overline{x}}^n, \overline{u}^n) + 2\left(P, \overline{u}^n\right) = 0,\end{aligned} \tag{16}$$

where $P_j = \frac{1}{3}\left[u_j^n \left(\overline{u}_j^n\right)_{\hat{x}} + \left(u_j^n \overline{u}_j^n\right)_{\hat{x}}\right]$. We can write

$$(P, \overline{u}^n) = 0,$$

so we get

$$\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \left\|u_{xx}^{n-1}\right\|^2\right) \quad = -2\,(u_{\hat{x}x\overline{x}}^n, \overline{u}^n) - 2\,({u^n}_{\hat{x}}, \overline{u}^n). \tag{17}$$

By Cauchy-Schwarz inequality and Lemma 2.1, we find

$$(u_{\hat{x}x\overline{x}}^n, 2\overline{u}^n) = -\left(u_{\hat{x}x}^n, 2\overline{u}_x^n\right),$$

$$\left|\left(u_{\hat{x}x}^n, u_x^{n+1} + u_x^{n-1}\right)\right| \leqslant \|u_{xx}^n\|^2 + \frac{1}{2}\left(\left\|u_x^{n+1}\right\|^2 + \left\|u_x^{n-1}\right\|^2\right), \tag{18}$$

$$\begin{aligned}\left\|u_x^{n+1}\right\|^2 &\leq \tfrac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u_{xx}^{n+1}\right\|^2\right), \\ \left\|u_x^{n-1}\right\|^2 &\leq \tfrac{1}{2}\left(\left\|u^{n-1}\right\|^2 + \left\|u_{xx}^{n-1}\right\|^2\right).\end{aligned} \tag{19}$$

Substituting (19) into (18), we get

$$|(u^n_{\hat{x}x\bar{x}}, 2\overline{u}^n)| \le \|u^n_{xx}\|^2 + \frac{1}{4}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 + \left\|u^{n-1}\right\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right), \quad (20)$$

and

$$\left((u^n)_{\hat{x}}, u^{n+1} + u^{n-1}\right) \le \|u^n_x\|^2 + \frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n-1}\right\|^2\right). \quad (21)$$

It follows from (17)-(21) that

$$\begin{aligned} &\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 - \left\|u^{n-1}_{xx}\right\|^2 \\ &\le 2\tau\left(\|u^n_{xx}\|^2 + \tfrac{1}{4}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 + \left\|u^{n-1}\right\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right)\right. \\ &\left.+ \|u^n_x\|^2 + \tfrac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n-1}\right\|^2\right)\right). \end{aligned} \quad (22)$$

Using Lemma 2.1 and Cauchy-Schwarz inequality, we obtain

$$\|u^n_x\|^2 \le \tfrac{1}{2}\left(\|u^n\|^2 + \|u^n_{xx}\|^2\right), \quad (23)$$

hence, we can write (22) as follows

$$\begin{aligned} &\left(\left\|u^{n+1}\right\|^2 + \|u^n\|^2\right) - \left(\|u^n\|^2 + \left\|u^{n-1}\right\|^2\right) + \left(\left\|u^{n+1}_{xx}\right\|^2 + \|u^n_{xx}\|^2\right) - \\ &\left(\|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right) \\ &\le C\tau\left(\left\|u^{n+1}_{xx}\right\|^2 + \|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2 + \left\|u^{n+1}\right\|^2 + \|u^n\|^2 + \left\|u^{n-1}\right\|^2\right). \end{aligned} \quad (24)$$

Let $B^n = \|u^n\|^2 + \left\|u^{n-1}\right\|^2 + \|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2$. It follows from (24) that

$$B^{n+1} - B^n \le C\tau\left(B^{n+1} + B^n\right),$$

so

$$(1 - C\tau)\left(B^{n+1} - B^n\right) \le 2C\tau B^n.$$

If τ is sufficiently small which satisfies $1 - C\tau = \delta > 0$, then

$$B^{n+1} - B^n \le C\tau B^n. \quad (25)$$

Summing up (25) from 0 to n-1, we have

$$B^n \le B^0 + C\tau\sum_{l=0}^{n-1} B^l.$$

It follows from Lemma 2.3 that

$$\|u^n\| \le C, \qquad \|u^n_{xx}\| \le C.$$

From (23), we have $\|u^n_x\| \le C$. Using Lemma 2.2, we get $\|u^n\|_\infty \le C$. $\square$

3. Solvability

Theorem 3.1. *The difference scheme (9)-(12) has a unique solution.*

Proof. We use from mathematical induction to prove. It is obvious that u^0 is uniquely determined by the initial condition (11). We also can get u^1 in order $O(h^2+\tau^2)$ by two-level C-N scheme (that is, u^0 and u^1 are uniquely determined). Now suppose $u^0, u^1, \dots, u^n$ be solved uniquely. Considering equation (9) for u^{n+1} we can get

$$\tfrac{1}{2\tau}u_j^{n+1} + \tfrac{1}{2\tau}\left(u_j^{n+1}\right)_{xx\bar{x}\bar{x}} + \tfrac{1}{6}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}} + \left(u_j^n u_j^{n+1}\right)_{\hat{x}}\right] = 0. \tag{26}$$

Taking an inner product of (26) with u^{n+1}, we obtain

$$\tfrac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \tfrac{1}{2\tau}\left\|u_{xx}^{n+1}\right\|^2 + \tfrac{h}{6}\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}} + u_j^{n-1}\left(u_j^n\right)_{\hat{x}}\right]u_j^{n+1} = 0. \tag{27}$$

We can write

$$\begin{aligned} &\frac{1}{6}h\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\widehat{x}} + \left(u_j^n u_j^{n+1}\right)_{\widehat{x}}\right]u_j^{n+1} \qquad (28)\\ =\;& \frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^n u_{j+1}^{n+1}u_j^{n+1} + u_{j+1}^n u_{j+1}^{n+1}u_j^{n+1}\right]\\ -\;& \frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^n u_{j-1}^{n+1}u_j^{n+1} + u_{j-1}^n u_{j-1}^{n+1}u_j^{n+1}\right] = 0, \end{aligned}$$

It follows from (27) and (28) that

$$\left\|u^{n+1}\right\|^2 + \left\|u_{xx}^{n+1}\right\|^2 = 0.$$

That is, (26) has only a trivial solution. Therefore, (9)-(12) determines u_j^{n+1} uniquely. □

4. Convergence and Stability

Let $v(x,t)$ be the solution of problem (3)-(5), $v_j^n = v(x_j, t_n)$, then the truncation error of the difference scheme (9)-(12) is as follows:

$$r_j^n = \left(v_j^n\right)_{\widehat{t}} + \left(v_j^n\right)_{xx\overline{x}\overline{x}\widehat{t}} + \left(v_j^n\right)_{\widehat{x}} + \left(v_j^n\right)_{\widehat{x}x\overline{x}} + \tfrac{1}{3}\left[v_j^n\left(\overline{v}_j^n\right)_{\widehat{x}} + \left(v_j^n\overline{v}_j^n\right)_{\widehat{x}}\right]. \tag{29}$$

Using Taylor expansion, we know that $r_j^n = O(h^2+\tau^2)$ holds if $h, \tau \to 0$.

Theorem 4.1. *Suppose that* $u_0 \in H_0^2[x_L, x_R]$*, then the solution* u^n *of (9)-(12) converges to the solution* $v(x,t)$ *of problem (3)-(5) in norm* $\|.\|_\infty$ *and the rate of convergence is* $O(\tau^2 + h^2)$.

Proof. Subtracting (9) from (29) and letting $e_j^n = v_j^n - u_j^n$, we have

$$r_j^n = \left(e_j^n\right)_{\hat{t}} + \left(\overline{e}_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(e_j^n\right)_{\hat{x}} + \left(e_j^n\right)_{\widehat{x}x\overline{x}} + R_{1,j} + R_{2,j}, \tag{30}$$

where

$$R_{1,j} = \tfrac{1}{3}\left[v_j^n\left(\overline{v}_j^n\right)_{\hat{x}} - u_j^n\left(\overline{u}_j^n\right)_{\hat{x}}\right],$$

$$R_{2,j} = \tfrac{1}{3}\left[\left(v_j^n\overline{v}_j^n\right)_{\hat{x}} - \left(u_j^n\overline{u}_j^n\right)_{\hat{x}}\right].$$

Computing the inner product of (30) with $2\overline{e}^n$, we obtain

$$(r^n, 2\overline{e}^n) = \frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) +$$

$$(e_{\hat{x}}^n, 2\overline{e}^n) + (e_{\hat{x}x\bar{x}}^n, 2\overline{e}^n) + (R_1, 2\overline{e}^n) + (R_2, 2\overline{e}^n)\,. \tag{31}$$

We can write (31) as follows

$$\begin{aligned}&\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) = \\ &2\tau\left[(r^n, 2\overline{e}^n) - (e_{\hat{x}x\bar{x}}^n, 2\overline{e}^n) - ((e^n)_{\hat{x}}, 2\overline{e}^n) - (R_1, 2\overline{e}^n) - (R_2, 2\overline{e}^n)\right].\end{aligned} \tag{32}$$

By Lemma 2.1, Theorem 2.1, and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&(R_1, 2\overline{e}^n)\\
&= \tfrac{2}{3} h \sum_j \left(v_j^n \left(\overline{v}_j^n\right)_{\hat{x}} - u_j^n \left(\overline{u}_j^n\right)_{\hat{x}} \right) \left(\overline{e}_j^n\right)\\
&= \tfrac{1}{3} h \sum_j \left[v_j^n \left(v_j^{n+1} + v_j^{n-1}\right)_{\widehat{x}} - v_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\widehat{x}} \right.\\
&\left. + v_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\widehat{x}} - u_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\widehat{x}} \right] \left(\overline{e}_j^n\right)\\
&= \tfrac{2}{3} h \sum_j \left(v_j^n \left(\overline{e}_j^n\right)_{\hat{x}} - e_j^n \left(\overline{u}_j^n\right)_{\hat{x}} \right) \left(\overline{e}_j^n\right)\\
&= \tfrac{2}{3} h \sum_j v_j^n \left(\overline{e}_j^n\right)_{\hat{x}} \left(\overline{e}_j^n\right) - \tfrac{2}{3} h \sum_j e_j^n \left(\overline{u}_j^n\right)_{\hat{x}} \left(\overline{e}_j^n\right)\\
&\leq \tfrac{2}{3} C h \sum_j \left(\left| \left(\overline{e}_j^n\right)_{\hat{x}} \right| + \left| e_j^n \right| \right) \left| \overline{e}_j^n \right|\\
&\leq C \left[\|\overline{e}_x^n\|^2 + \|e^n\|^2 + 2\|\overline{e}^n\|^2 \right]\\
&\leq C \left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + 2\left\|e^{n+1}\right\|^2 + \|e^n\|^2 + 2\left\|e^{n-1}\right\|^2 \right),
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
&(R_2, 2\overline{e}^n)\\
&= \tfrac{2}{3} h \sum_j \left(\left(v_j^n \overline{v}_j^n\right)_{\hat{x}} - \left(u_j^n \overline{u}_j^n\right)_{\hat{x}} \right) \left(\overline{e}_j^n\right)\\
&= \tfrac{2}{3} h \sum_j \left\{ \left(v_j^n \overline{v}_j^n\right)_{\widehat{x}} - \left(v_j^n \overline{u}_j^n\right)_{\widehat{x}} + \left(v_j^n \overline{u}_j^n\right)_{\widehat{x}} - \left(u_j^n \overline{u}_j^n\right)_{\widehat{x}} \right\} \left(\overline{e}_j^n\right)\\
&= -\tfrac{2}{3} h \sum_j \left\{ \left(v_j^n\right) \overline{e}_j^n \left(\overline{e}_j^n\right)_{\widehat{x}} + \left[v_j^n - u_j^n\right] \overline{u}_j^n \left(\overline{e}_j^n\right)_{\widehat{x}} \right\}\\
&\leq \tfrac{2}{3} C h \sum_j \left(\left|\left(\overline{e}_j^n\right)\right| + \left|e_j^n\right| \right) \left| \left(\overline{e}_j^n\right)_{\hat{x}} \right|\\
&\leq C \left[\|\overline{e}_x^n\|^2 + \|e^n\|^2 + \|\overline{e}^n\|^2 \right]\\
&\leq C \left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e^{n-1}\right\|^2 \right).
\end{aligned} \tag{34}
$$

Noting that similar to (18)-(21) we have

$$\left(r^n, 2\bar{e}^n\right) = \left(r^n, e^{n+1} + e^{n-1}\right) \leq \|r^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right), \tag{35}$$

$$\left((e^n)_{\hat{x}x\bar{x}}, 2\bar{e}^n\right) = -\left((e^n)_{\hat{x}x}, e_x^{n+1} + e_x^{n-1}\right) \leqslant \|e_{xx}^n\|^2 + \frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right), \tag{36}$$

$$\left((e^n)_{\hat{x}}, 2\bar{e}^n\right) = \left((e^n)_{\hat{x}}, e^{n+1} + e^{n-1}\right) \leq \|e_x^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right). \tag{37}$$

From (32)-(37), we have

$$\begin{aligned}
&\left(\left\|e^{n+1}\right\|^2 + \|e^n\|^2\right) - \left(\|e^n\|^2 + \left\|e^{n-1}\right\|^2\right) + \\
&\left(\left\|e_{xx}^{n+1}\right\|^2 + \|e_{xx}^n\|^2\right) - \left(\|e_{xx}^n\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right) \\
&\leq C\tau\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 + \|e^n\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e_x^n\|^2 + \left\|e_x^{n-1}\right\|^2 + \|e_{xx}^n\|^2\right) + \\
&2\tau\|r^n\|^2.
\end{aligned} \tag{38}$$

Similar to the proof of (23), we obtain

$$\begin{aligned}
&\left\|e_x^{n+1}\right\|^2 \leq \tfrac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e_{xx}^{n+1}\right\|^2\right), \\
&\|e_x^n\|^2 \leq \tfrac{1}{2}\left(\|e^n\|^2 + \|e_{xx}^n\|^2\right), \\
&\left\|e_x^{n-1}\right\|^2 \leq \tfrac{1}{2}\left(\left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right).
\end{aligned} \tag{39}$$

Let $D^n = \|e^n\|^2 + \|e_{xx}^n\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2$, then (38) can be rewritten as follows

$$D^{n+1} - D^n \leq 2\tau\|r^n\|^2 + C\tau\left(D^{n+1} + D^n\right), \tag{40}$$

which yields

$$(1 - C\tau)\left(D^{n+1} - D^n\right) \leq 2C\tau D^n + 2\tau\|r^n\|^2. \tag{41}$$

If τ is sufficiently small, which satisfies $1 - C\tau > 0$, then

$$D^{n+1} - D^n \leq C\tau D^n + C\tau\|r^n\|^2. \tag{42}$$

Summing up (42) from 1 to n, we have

$$D^n \leq D^0 + C\tau\sum_{l=0}^{n-1}\left\|r^l\right\|^2 + C\tau\sum_{l=0}^{n-1} D^l. \tag{43}$$

First, we can get u^1 in order $O(h^2+\tau^2)$ that satisfies $D^0 \leq C(h^2+\tau^2)^2$ by two-level C-N scheme. Since

$$\tau \sum_{l=0}^{n-1} \left\|r^l\right\|^2 \leq n\tau \max_{0\leq l\leq n-1} \left\|r^l\right\|^2 \leq T.O\big(\tau^2+h^2\big)^2, \tag{44}$$

we obtain

$$D^n \leq O\big(\tau^2+h^2\big)^2 + C\tau \sum_{l=0}^{n-1} D^l. \tag{45}$$

From Lemma 2.3 we get

$$D^n \leq O\big(\tau^2+h^2\big)^2, \tag{46}$$

that is

$$\|e^n\| \leq O\left(\tau^2+h^2\right), \qquad \|e^n_{xx}\| \leq O\left(\tau^2+h^2\right). \tag{47}$$

From (39) we have

$$\|e^n_x\| \leq O\left(\tau^2+h^2\right). \tag{48}$$

By Lemma 2.2, we obtain

$$\|e^n\|_\infty \leq O\left(\tau^2+h^2\right). \tag{49}$$

This completes the proof. □

Finally, we can state similarly the following theorem.

Theorem 4.2. *Under the conditions of Theorem 4.1, the solution u^n of (9)-(12) is stable in norm $\|.\|_\infty$.*

5. Numerical Results

In this section we present the numerical results of the proposed method on a test problem. We performed our computations using **Matlab** 7 software on a PC with Intel Core 2 Duo, 2.8 GHz CPU and 2 GB RAM. We tested the accuracy and stability of the method presented in this paper by performing the mentioned method for different values of Δt and h. Also we calculated the computational orders of the method presented in this article (denoted by C-Order) with the following formula

$$\frac{\log(\frac{E_1}{E_2})}{\log(\frac{h_1}{h_2})},$$

in which E_1 and E_2 are errors correspond to grids with mesh size h_1 and h_2 respectively. Also we put $x_L = -40$ and $x_R = 100$.

Table 1: Errors and computational orders obtained at different final times.

	$h=\tau=0.2$	$h=\tau=0.1$	$h=\tau=0.05$
$T=10$			
$\|e^n\|_\infty$	2.718820×10^{-4}	6.853283×10^{-5}	1.718933×10^{-5}
$C-Order$	$-$	1.988	1.995
$T=20$			
$\|e^n\|_\infty$	5.026183×10^{-4}	1.261183×10^{-4}	3.157146×10^{-5}
$C-Order$	$-$	1.995	1.998
$T=30$			
$\|e^n\|_\infty$	7.217771×10^{-4}	1.810695×10^{-4}	4.532327×10^{-5}
$C-Order$	$-$	1.995	1.998
$T=40$			
$\|e^n\|_\infty$	9.396398×10^{-4}	2.356919×10^{-4}	5.899417×10^{-5}
$C-Order$	$-$	1.995	1.998

5.1 Propagation of a Single Solitary Wave

We consider the equation (3) with the following exact solution

$$\begin{aligned} u(x,t) = & \left(-\tfrac{35}{24}+\tfrac{35}{312}\sqrt{313}\right) \\ & \times \sec h^4\left[\tfrac{1}{24}\sqrt{-26+2\sqrt{313}}\right. \\ & \left.\times\left(x-\left(\tfrac{1}{2}+\tfrac{1}{26}\sqrt{313}\right)t\right)\right]. \end{aligned}$$

The initial condition can be obtained from exact solution. Table 1 shows the computational orders and errors of proposed method with different values of $h=\tau$ at different final times. Numerical results of this table confirm the second order of accuracy of method. In Tables 2,3 we compare the errors of proposed method with the results of [4]. As we see the new method has better accuracy. Figure 1 shows the surface plot of approximate solution (left panel) and plot of absolute error (right panel) with $h=0.05$, $\tau=0.05$ at $T=20$.

5.2 Interaction of Two Solitary Waves

We investigate the interaction of two solitary waves for equation (3) using the following initial condition

$$u(x,0)=\sum_{j=1}^{2} 3d_j \sec h^2\left(k_j\left(x-x_j\right)\right),$$

Table 2: Comparison of $\|e^n\|_\infty$ error at various time steps.

$\|e^n\|_\infty$	$h=\tau=0.1$		$h=\tau=0.05$	
Method	Present	Scheme [4]	Present	Scheme [4]
$t=10$	2.718×10^{-4}	2.507×10^{-3}	1.719×10^{-5}	1.585×10^{-4}
$t=20$	5.026×10^{-4}	4.489×10^{-3}	3.157×10^{-5}	2.836×10^{-4}
$t=30$	7.218×10^{-4}	6.081×10^{-3}	4.532×10^{-5}	3.834×10^{-4}
$t=40$	9.396×10^{-4}	7.444×10^{-3}	5.899×10^{-5}	4.709×10^{-4}

Table 3: Comparison of $\|e^n\|$ error at various time steps.

$\|e^n\|$	$h=\tau=0.2$		$h=\tau=0.05$	
Method	Present	Scheme [4]	Present	Scheme [4]
$t=10$	7.389×10^{-4}	6.525×10^{-3}	4.663×10^{-5}	4.113×10^{-4}
$t=20$	1.443×10^{-3}	1.209×10^{-2}	9.070×10^{-5}	7.631×10^{-4}
$t=30$	2.132×10^{-3}	1.683×10^{-2}	1.339×10^{-5}	1.063×10^{-3}
$t=40$	2.818×10^{-3}	2.101×10^{-2}	1.769×10^{-5}	1.328×10^{-3}

in which $k_1 = 0.4$, $k_2 = 0.3$, $x_1 = 15$, $x_2 = 35$, and

$$d_j = \frac{4k_j^2}{1-4k_j^2}, \qquad j = 1,2.$$

From the above initial conditions, the solitary waves are propagated rightwards. Shapes of both waves at times $t = 10, 15, 20, 25$ and with $h = \tau = 0.1$ are shown in Figure 2. We see that as the time progresses the collision occurs and after collision two waves leave each other without changing their shape.

6. Conclusion

In this article, we constructed an implicit finite difference scheme for the solution of Rosenau-KdV equation. We proved that this scheme is stable and convergent in the order of $O(\tau^2 + h^2)$. Furthermore we showed the existence and uniqueness of numerical solutions. We compared the numerical results of this paper with other methods in the literature and concluded that the proposed method has better results.

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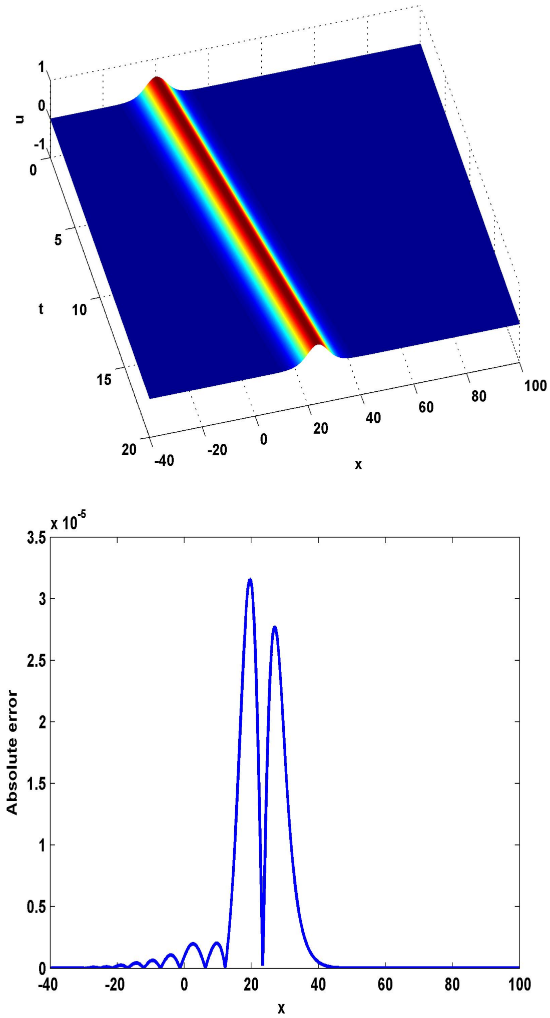

Figure 1: Surface plot of approximate solution (left panel) and plot of absolute error (right panel) with $h = 0.05$, $\tau = 0.05$ at $T = 20$.

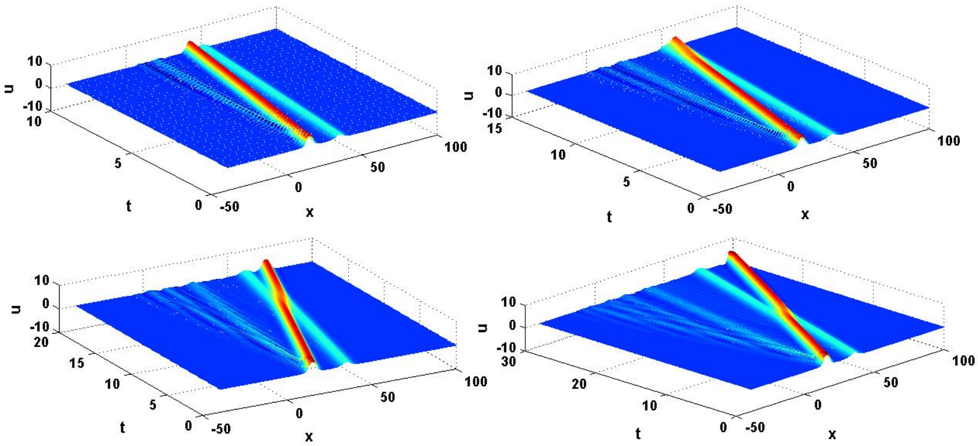

Figure 2: The numerical solutions of two solitary waves with $h = \tau = 0.1$ obtained at final times $t = 10$ (left-top), $t = 15$ (right-top), $t = 20$ (left-bottom) and $t = 30$ (right-bottom).