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Diameter Two Graphs of Minimum Order with Given Degree Set

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Abstract

The *degree set* of a graph is the set of its degrees. Kapoor et al. [Degree sets for graphs, Fund. Math. 95 (1977) 189-194] proved that for every set of positive integers, there exists a graph of diameter at most two and radius one with that degree set. Furthermore, the minimum order of such a graph is determined. A graph is *2-self-centered* if its radius and diameter are two. In this paper for a given set of natural numbers greater than one, we determine the minimum order of a 2-self-centered graph with that degree set.

Keywords: Degree set, self-centered graph, radius, diameter.

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, the *degree* of v in G, denoted by $deg_G(v)$. We denote the minimum and maximum degrees of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* between two vertices u and v of a connected graph G is denoted by $d_G(u, v)$ and it is the number of edges in a shortest path connecting u and v. The *eccentricity* $e_G(u)$ of a vertex u, of a connected graph G, is $max\{d_G(u, v)|v \in V(G)\}$. The radius of a connected graph G, *rad(G)*, is the minimum eccentricity among the vertices of G, while the diameter of G, *diam(G)*, is the maximum eccentricity. If *rad(G)=diam(G)=r*, then G is an *r-self-centered* graph. We use *r-sc* as a notation for r-self-centered graph. F. Buckley [2] worked on *r-sc* graphs, but the concept of r-sc graphs was developed independently by Akiyama, Ando, and Avis [1], who called them *r-equi* graphs. They proved that if G is an r-sc graph, then G is a block and $\Delta(G) \le |V(G)| - 2(r - 1)$.

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Hence, for $r = 2$ we have the following corollary.

Corollary 1.1. *If G is a 2-sc graph and v is a vertex of G, then* $2 \leq deg_G(v) \leq |V(G)| - 2$.

In this paper we study 2-sc graphs in terms of the degree sets, where for a given graph G the *degree set* of G, denoted by $D(G)$, is the set of degrees of the vertices of G. It is a simple observation that any set of positive integers forms the degree set of a graph. So it is natural to investigate the minimum order of such graphs. This question is completely answered by a result of Kapoor, Polimeni and Wall [4]. Their result can be stated as follows.

Theorem 1.2 (S. F. Kapoor et al. [4])**.** *For every set $S = \{a_1, ..., a_n\}$ of positive integers, with $a_1 < \cdots < a_n$, there exists a graph G such that $D(G) = S$ and furthermore,*

$$\mu(S) = a_n + 1,$$

where $\mu(S)$ represents the minimum order of such a graph G.

The graph G in Theorem 1.2 has order $a_n + 1$. Therefore G has diameter at most two and radius one. Hence G is not a 2-sc graph. Corollary 1.1 implies that every 2-sc graph has no vertex of degree less than or equal to 1. In this paper, we show that for a finite, nonempty set S of positive integers greater than 1, there exists a 2-sc graph G such that $D(G) = S$. Furthermore, the minimum order of such a graph G is determined.

2. Results

We write K_n and C_n for the *Complete* graph and the *Cycle* of order n, respectively. Also for a graph G, the graph $\bar{G}$ is the *Complement* of G. The *union* of graphs G and H is the graph $G \cup H$ which consists of copies of graphs G and H. Two graphs are *disjoint* if they have no vertex in common. If a graph G consists of $k(\geq 2)$ disjoint copies of a graph H, then we write $G = kH$.

Let S be a set of positive integers, where $S = \{a_1, ..., a_n\}$ and $1 < a_1 < \cdots < a_n$. We define $\mu_r(S)$ to be the minimum order of an r-sc graph G for which $D(G) = S$. In the case when $S = \{a_1\}$, the following theorem implies that there exists an a_1-regular 2-sc graph of minimum order.

Theorem 2.2. *Let a_1 be a positive integer greater than 1 and $S = \{a_1\}$. There exists a 2-sc graph G such that $D(G) = S$ and furthermore,*

$$\mu_2(S) = \begin{cases} a_1 + 2 & \text{if } a_1 \text{ is even} \\ a_1 + 3 & \text{if } a_1 \text{ is odd} \end{cases}.$$

Proof. Let a_1 be a positive integer greater than 1. If a_1 is even, then the graph

$$G = \overline{(\frac{a_1}{2} + 1)K_2},$$

is clearly an a_1-regular graph with a_1+2 vertices. The graph G is also a 2-sc graph [3]. Additionally, Corollary 1.1 implies that, every 2-sc graph of order a_1+1 has no vertex of degree a_1. Therefore, we need at least a_1+2 vertices to construct an a_1-regular 2-sc graph. Hence $\mu_2(S) = a_1 + 2$.

If a_1 is odd, then the graph

$$H = \overline{C}_{a_1+3},$$

is an a_1-regular graph of order a_1+3. The graph H is also a 2-sc graph. Since the graph H has order at least 6 and for each pair of nonadjacent vertices u and v of H there exists at least one common neighbour, it follows that $d_H(u, v) = 2$. Since, in any graph, the number of vertices of odd degree is even. Thus there is no a_1-regular graph of order a_1+2. Therefore, the graph H has the minimum order among all such 2-sc a_1-regular graphs. Hence $\mu_2(S) = a_1 + 3$. □

The following lemma which is obtained by Z. Stanic [5] has an interesting applications for constructing 2-sc graphs from other not necessarily 2-sc graphs and also it will be needed in the proof of our results for non-regular graphs. Recall that the *join* $G + H$ of two disjoint graphs G and H is the graph consisting of the union $G \cup H$, together with edges xy, where $x \in V(G)$ and $y \in V(H)$.

Lemma 2.3 (Z. Stanic [5]). *Let G and H be simple nontrivial graphs with $\Delta(G) \leq |V(G)| - 2$ and $\Delta(H) \leq |V(H)| - 2$, then $G + H$ is a 2-sc graph.*

Now, we extend Theorem 2.2 for non-regular graphs in following theorems.

Theorem 2.4. *Let a_1 be even and S be a set of positive integers, where $S = \{a_1, \ldots, a_n\}$, $2 \leq a_1 < \cdots < a_n$ and $n > 1$. Then there exists a 2-sc graph G such that $D(G) = S$ and furthermore,*

$$\mu_2(S) = a_n + 2.$$

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = S_1$. Consider the graph

$$G = (H \cup K_1) + F,$$

where $F = \overline{\frac{a_1}{2} K_2}$. The graph G has order a_n+2. We observe that for each vertices v of G, one of the following cases occurs:
1) If $v \in V(K_1)$, then $deg_G(v) = |V(F)| = a_1$.
2) If $v \in V(F)$, then $deg_G(v) = deg_F(v) + |V(K_1)| + |V(H)| = (a_1 - 2) + 1 + (a_n - a_1 + 1) = a_n$.
3) If $v \in V(H)$, then $deg_G(v) = deg_H(v) + |V(F)| = deg_H(v) + a_1$.

Thus $D(G) = S$. Moreover, by considering Lemma 2.3, G is a 2-sc graph and since there is no 2-sc graph of order $a_n + 1$, hence $\mu_2(S) = a_n + 2$. □

Theorem 2.5. *Let a_1 be odd and S be a set of positive integers, where $S = \{a_1, ..., a_n\}$, $3 \le a_1 < \cdots < a_n$ and $n > 1$. Then there exists a 2-sc graph G of order $a_n + 3$ such that $D(G) = S$.*

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$, where for $1 \le i \le n$, $a_i \in S$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = \{a_2 - a_1, ..., a_n - a_1\}$. Consider the graph

$$G = (H \cup 2K_1) + F_1,$$

where $F_1 = \overline{C}_{a_1}$. The graph G has order $a_n + 3$. We observe that for each vertices v of G, one of the following cases occurs.
1) If $v \in V(2K_1)$, then $deg_G(v) = |V(F_1)| = a_1$.
2) If $v \in V(F_1)$, then $deg_G(v) = deg_{F_1}(v) + |V(2K_1)| + |V(H)| = a_n$.
3) If $v \in V(H)$, then $deg_G(v) = deg_H(v) + |V(F_1)| = deg_H(v) + a_1$.
Thus $D(G) = S$. Moreover, by Lemma 2.3, G is a 2-sc graph. □

Note that we considered $S = \{a_1, ..., a_n\}$ and presented a construction method in Theorem 2.4 to ascertain the value of $\mu_2(S)$, where a_1 is even, whereas if a_1 is odd, the graph G described in the proof of Theorem 2.5 has not necessarily the minimum order. As an example, for $S = \{3, 4\}$, the graph G_1 of Figure 1 which is obtained by the method of Theorem 2.5 has order 7, whereas the 2-sc graph G_2 where $G_2 = \overline{P}_6$ with 6 vertices has also the same degree set (see Figure 1).

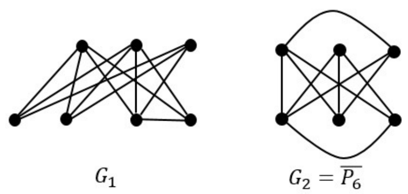

Figure 1: The 2-sc graphs G_1 and G_2 with different orders and the same degree sets.

In this section, we prove that if at least one element of S is even then $\mu_2(S) = a_n + 2$. We begin with a simple case.

Theorem 2.6. *Let S be a set of positive integers, where $= \{a_1, ..., a_n\}$, $n > 1$, $1 < a_1 < a_2 < \cdots < a_n$, a_1 is odd and $a_n = a_{n-1} + 1$ then $\mu_2(S) = a_n + 2$.*

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = S_1$. Consider the graph

$$F = (H \cup K_1) + \overline{C}_{a_1}.$$

Lemma 2.3 implies that the graph F is a 2-sc graph. The graph F has order $a_n + 2$, and $D(F) = S$. Since there is no 2-sc graph of order $a_n + 1$, therefore, $\mu_2(S) = a_n + 2$. □

Now we consider the set $S = \{a_1, ..., a_n\}$ of positive integers. We prove that if all the elements of S are odd, then $\mu_2(S) = a_n + 3$, otherwise $\mu_2(S) = a_n + 2$. Before proving the main result, we need to have the following theorem.

Theorem 2.7 (I. Zverovich [6])**.** *Let S be a set of positive integers, where $S = \{a_1, ..., a_n\}$ and $3 \leq a_1 < \cdots < a_n$. Then there exists a Hamiltonian graph G such that $D(G) = S$ and $|V(G)| = a_n + 1$.*

Lemma 2.8. *For a graph G, if $\Delta(G) = |V(G)| - 2$ and G contains at least two non−adjacent vertices of degree $\Delta(G)$, then G is a 2-sc graph.*

Proof. Let x and y be two non−adjacent vertices of G with $deg_G(x) = deg_G(y) = \Delta(G) = |V(G)| - 2$. Obviously, x and y are adjacent to all other vertices of G. Therefore, $e_G(x) = e_G(y) = d_G(x, y) = 2$. Moreover, since $\Delta(G) = |V(G)| - 2$, it follows that for all other vertices v of G there is at least one non−adjacent vertex. Hence $e_G(v) = 2$. Therefore G is a 2-sc graph. □

Lemma 2.9. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$ and $2 \leq a_1 < \cdots < a_n$. Then there exists a graph G of order $a_n + 1$ such that $D(G) = S$ and G has a Hamilton path.*

Proof. Let $S' = \{a_1 + 1, ..., a_n + 1\}$. Since $a_1 + 1 \geq 3$, Theorem 2.7 implies that there exists a Hamiltonian graph G' of order $a_n + 2$ such that $D(G') = S'$. Let C' be a Hamilton cycle in G' where $C' = (v_1, v_2, ..., v_{a_n+2}, v_1)$. Without loss of generality, let v_1 be a vertex of degree $a_n + 1$ which is connected to all other vertices of G'. Let $G = G' - v_1$. Thus $D(G) = S$, $|V(G)| = a_n + 1$. Furthermore, by removing the vertex v_1 of C' we obtain the Hamilton path P where $P = (v_2, v_3, ..., v_{a_n+1}, v_{a_n+2})$. □

Lemma 2.10. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$, a_n be odd and $3 \leq a_1 < \cdots < a_n$. Then there exists a graph G of order $a_n + 1$ such that $D(G) = S$ and G has at least two adjacent vertices x and y of degree a_n. Moreover, G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy.*

Proof. Let $S' = \{a_1 - 1, ..., a_n - 1\}$. By Lemma 2.9, there is a graph G' of order a_n with $D(G') = S'$ and a Hamilton path P such that $P = (v_1, v_2, ..., v_{a_n})$ where $v_i \in V(G')$ for $1 \leq i \leq a_n$. Let x be a vertex of degree $a_n - 1$ of G'. We construct G by adding a new vertex y to G' adjacent to all vertices of G'. For $1 \leq i \leq a_n$ we have

$$deg_G(v_i) = deg_{G'}(v_i) + 1.$$

Clearly, we have two adjacent vertices x and y of degree a_n and also G is a graph of order $a_n + 1$ such that $D(G) = S$. We claim that G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy. Obviously, P is a path in G. Let M' be a maximal matching of P such that the vertex x is unsaturated. The size of matching M' is at least $\frac{a_n-3}{2}$. Let $M = M' \cup \{xy\}$. Clearly M is a matching of G such that $|M| = \frac{a_n-1}{2}$, which completes the proof. □

Now we prove our main theorem.

Theorem 2.11. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$ and $1 < a_1 < \cdots < a_n$. If all elements of S are odd, then $\mu_2(S) = a_n + 3$, otherwise $\mu_2(S) = a_n + 2$.*

Proof. Consider the case when all elements of S are odd. Theorem 2.5 implies that there exists a 2-sc graph G of order $a_n + 3$ such that $D(G) = S$. Moreover, as noted earlier, in any graph, there is an even number of odd vertices. Hence there is no graph of order $a_n + 2$ with S as its degree set. Therefore G is a 2-sc graph of minimum order such that $D(G) = S$, Hence $\mu_2(S) = a_n + 3$.

Now assume that at least one even element a_i exists in S where $1 \leq i \leq n$. If a_1 is even, then by Theorem 2.4, $\mu_2(S) = a_n + 2$. Now let a_1 be odd. There exists at least one i where $2 \leq i \leq n$ such that a_i is even. Now we have two cases as follows:

First we consider the case in which a_n is even. Hence $|V(G)|$ is odd. Since $a_1 \geq 3$, Theorem 2.7 implies that there exists a Hamiltonian graph G of order $a_n + 1$ such that $D(G) = S$. Let C be a Hamilton cycle in G such that $C = (v_1, v_2, ..., v_{a_n+1}, v_1)$ where $v_i \in V(G)$ for $1 \leq i \leq a_n + 1$. Without loss of generality, let v_1 be a vertex of degree a_n.

Let M be a matching of G where $M = \{v_2v_3, v_4v_5, ..., v_{a_n}v_{a_n+1}\}$ and the edge v_iv_{i+1} for $2 \leq i \leq a_n$ is an edge of the Hamilton cycle C (Notice that exactly one vertex v_1 of G is not saturated by M, hence $|M| = \frac{|V(G)|-1}{2}$). Let $G^* = G - M$. Clearly, for $2 \leq i \leq a_n + 1$, we have $deg_{G^*}(v_i) = deg_G(v_i) - 1$ and also $deg_{G^*}(v_1) = deg_G(v_1)$. Now, we construct a new graph H by adding a new vertex v adjacent to each vertex of G^* except v_1. Since $deg_H(v) = deg_H(v_1) = deg_G(v_1) = a_n$ and for $2 \leq i \leq a_n$, we have $deg_H(v_i) = deg_G(v_i)$, it follows immediately that $D(H) = D(G) = S$. Furthermore, since $|V(H)| = a_n + 2$ and H has at least two non$-$adjacent vertices v and v_1 of degree a_n, by Lemma 2.8, H is a 2-sc graph. Therefore, $\mu_2(S) = a_n + 2$.

Now we consider the case in which a_n is odd. Lemma 2.10 implies that there exists a graph G of order $a_n + 1$ such that $D(G) = S$. Furthermore, the graph G has at least two adjacent vertices x and y of degree a_n and also G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy . Let v_i be a vertex of degree a_i where a_i is even and $2 \leq i \leq n - 1$. Consider the matching M of size $\frac{a_i}{2}$ of G which contains the edge xy. Let

$$G^* = G - M.$$

We construct H by adding a new vertex v to G^* such that

$$E(H) = E(G^*) \cup \{vu_i \mid u_i \textit{ is the vertex of } G \textit{ which is saturated by } M, \textit{ where } 1 \leq i \leq n\}.$$

Clearly, $D(H) = S$ and H has an order $a_n + 2$. Since H has at least two non$-$adjacent vertex x and y such that $deg_H(x) = deg_H(y) = a_n$, Lemma 2.8 implies that the graph H is a 2-sc graph and $\mu_2(S) = a_n + 2$. □

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