Remarks on the Paper “Coupled Fixed Point Theorems for Single-Valued Operators in \(b\)-Metric Spaces”

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Abstract

In this paper, we improve some recent coupled fixed point results for single-valued operators in the framework of ordered \(b\)-metric spaces established by Bota et al. [M-F. Bota, A. Petrusel, G. Petrusel and B. Samet, Coupled fixed point theorems for single-valued operators in \(b\)-metric spaces, Fixed Point Theory Appl. (2015) 2015:231]. Also, we prove that Perov-type fixed point theorem in ordered generalized \(b\)-metric spaces is equivalent with Ran-Reurings-type theorem in ordered \(b\)-metric spaces.

Keywords: Vector-valued metric, ordered \(b\)-metric space, coupled fixed point, integral equation, well-posed fixed point problem.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 1966, Perov [11] formulated a fixed point theorem which extends the well-known contraction mapping principle to the case when the metric \(d\) takes values in \(R^n_+\), that is, to the case of a generalized (cone) metric space. In 1989, Bakhtin [2] introduced the concept of a \(b\)-metric space which is another generalization of the ordinary metric space. After that, several papers have appeared dealing with results in \(b\)-metric spaces (see, e.g., [5, 7, 9] as well as the references therein). For the concepts of \(b\)-convergence, \(b\)-Cauchy sequence, \(b\)-continuity and \(b\)-completeness in \(b\)-metric spaces, see for instance [5, 7]. Furthermore, several new kinds of spaces have appeared, as generalized \(b\)-metric spaces, ordered generalized \(b\)-metric spaces, etc.
In this paper, we will first show that most of the results of paper [5] on coupled fixed points in ordered (generalized) $b$-metric spaces can be obtained in a much easier way. Further, in Section 3, we will improve these results and show that Perov-type fixed point theorem in ordered generalized $b$-metric spaces is equivalent to Ran-Reurings-type theorem in ordered $b$-metric spaces. We finish by proving a result on well-posedness of the given fixed point problem.


Very recently, in [5], M-F. Bota et al. proved some coupled fixed point results for mixed monotone mappings in ordered generalized $b$-metric spaces. In this section, we will show that most of these results are basically not new.

Remark 2.1. (concerning [5, Theorem 2.2]). It is well known that from the condition (ii) of [5, Theorem 2.2], the mixed monotone property of $T$ and by induction it easily follows that the sequence $x_{n+1} = T^n(x_0, y_0) = T(x_n, y_n)$ is nondecreasing, while the sequence $y_{n+1} = T^n(y_0, x_0) = T(y_n, x_n)$ is nonincreasing. The rest of the proof of this theorem in [5] is also not new. That is, all is the same as in [4, Theorem 2.1] for ordinary metric spaces. Moreover, the proof that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, is again well known (see, e.g., [9, Lemma 3.1]).

Further, it is not hard to see that the estimates of $d(T^n(x_0, y_0), x)$ and $d(T^n(y_0, x_0), y)$ presented in the mentioned theorem hold without assumption that the $b$-metric $d$ is continuous. Indeed, our claim follows immediately from the following two inequalities:

$$\frac{1}{s} d(T^n(x_0, y_0), x^*) \leq d(T^n(x_0, y_0), T^{n+p}(x_0, y_0)) + d(T^{n+p}(x_0, y_0), x^*),$$

$$\frac{1}{s} d(T^n(y_0, x_0), y^*) \leq d(T^n(y_0, x_0), T^{n+p}(y_0, x_0)) + d(T^{n+p}(y_0, x_0), y^*),$$

as well as from the proof of [9, Lemma 3.1].

Finally, it is worth noticing that [5, Theorem 2.2] holds if the condition $k \in [0, \frac{1}{2})$ is relaxed to $k \in [0, 1)$ (see [6, Theorem 1], [7, Theorem 1.8], and Theorem 3.1 below).

In the sequel of [5], the well known Perov’s Theorem is proved for the case of so-called generalized $b$-metric spaces (see also [10]).

Remark 2.2. (concerning [5, Theorem 3.2]). Instead of the condition that $f$ has a closed graph (condition (3) in [5, Theorem 3.2]), one can suppose that $f$ is continuous or that $(X, d, \preceq)$ is regular (recall that an ordered (generalized) metric space is said to be regular if for each nondecreasing sequence $\{x_n\}$ in $X$, $x_n \to x$ as $n \to \infty$ implies that $x_n \preceq x$ for $n \in \mathbb{N}$). Otherwise, [5, Theorem 3.2] is simply classical Ran-Reuring’s result in the framework of ordered generalized $b$-metric spaces.
Remark 2.3. (concerning [5, Theorem 3.7]). It is not hard to see that the condition
\[ d(T(x,y), T(u,v)) \leq k_1 d(x,u) + k_2 d(y,v), \]
which is assumed in this theorem, implies the following:
\[ d_+(F_T(Y), F_T(V)) \leq k d_+(Y,V), \] for all \( Y \subseteq V \) or \( Y \supseteq V \),
where \( k = k_1 + k_2 \), \( Y = (x,y), V = (u,v) \), \( d_+(Y,V) = d(x,u) + d(y,v) \), \( F_T(Y) = (T(x,y), T(y,x)) \) and \( Y \subseteq V \iff x \leq u \) and \( y \geq v \). Further, (1) implies that
\[ D_+(F_T(Y), F_T(V)) \leq kD_+(Y,V), \] for all \( Y \subseteq V \) or \( Y \supseteq V \),
where \( D_+(Y,V) = \|d_+(Y,V)\| \). Since \((X \times X, D_+, \subseteq)\) is an ordered generalized \( b \)-metric space, then the proof of [5, Theorem 3.7] follows according to [6, Theorem 1]. Hence, in fact, [5, Theorem 3.7] is not new, that is, all ideas and methods in it are well known (for more details of respective results in the framework of metric spaces see [1, 3, 14, 12, 15]).

The authors of [5] discussed also the following system of integral equations:
\[
\begin{cases}
  x(t) = g(t) + \int_0^T G(s,t) f(s, x(s), y(s)) \, ds,
  \\
y(t) = g(t) + \int_0^T G(s,t) f(s, y(s), x(s)) \, ds,
\end{cases}
\]
where \( t \in [0,T] \). Using certain \( b \)-metric, they proved an existence result for solutions of the system (2).

Remark 2.4. (concerning [5, Theorem 4.1]). First of all, it follows from the condition (iii) of [5, Theorem 4.1], that
\[
|f(s, u_1(s), v_2(s)) - f(s, v_1(s), v_2(s))| \leq (\alpha(s) + \beta(s)) \max_{s \in [0,T]} \{|u_1(s) - v_1(s)|, |u_2(s) - v_2(s)|\},
\]
while the condition (iv) implies
\[
k := \max_{t \in [0,T]} \int_0^T G(s,t) (\alpha(s) + \beta(s)) \, ds < 1.
\]
Then, if \( S : X \times X \to X \) is defined as in [5], for all \((x \geq u \) and \( y \leq v \) or \((u \geq x \) and \( v \leq y \)), we have
\[
|S(x,y)(t) - S(u,v)(t)| = \left| \int_0^T G(s,t) [f(s, x(s), y(s)) - f(s, u(s), v(s))] \, ds \right|
\leq \int_0^T G(s,t) |f(s, x(s), y(s)) - f(s, u(s), v(s))| \, ds
\leq \int_0^T G(s,t) (\alpha(s) + \beta(s)) \max_{s \in [0,T]} \{|x(s) - u(s)|, |y(s) - v(s)|\} \, ds
= k\delta(Y,V),
\]
where \( \delta (Y, V) = \delta (x, y) = \max \{ D(x, u), D(y, v) \} \) is a metric on \( X^2 \) if \( D \) is a metric on \( X \). In this case \( D(x, y) = \max_{s \in [0, T]} |x(s) - y(s)| \) is a known metric on the space \( C[0, T] \).

Further, in the same manner, we have

\[
|S(x, y)(t) - S(v, u)(t)| = \left| \int_0^T G(s, t) [f(s, y(s), x(s)) - f(s, v(s), u(s))] \, ds \right|
\leq \int_0^T G(s, t) |f(s, y(s), x(s)) - f(s, v(s), u(s))| \, ds
\leq \int_0^T G(s, t) (\alpha(s) + \beta(s)) \max_{s \in [0, T]} \{|y(s) - v(s)|, |x(s) - u(s)|\} \, ds
= k\delta(Y, V).
\]

Hence, we obtain:

\[
\max_{t \in [0, T]} \{|S(x, y)(t) - S(u, v)(t)|, |S(y, x)(t) - S(v, u)(t)|\} \leq k\delta(Y, V),
\]

that is,

\[
\delta(F_S(Y), F_S(V)) \leq k\delta(Y, V),
\]

where \( F_S(Y) = F_S((x, y)) = (S(x, y), S(y, x)) \). Instead of the method used in the framework of \( b \)-metric spaces as in [5], we can use now simply Banach Contraction Principle for the proof that the system of integral equations (2) has a unique solution in the complete metric space \( C[0, T] \). It is clear that our approach is brief and natural. Hence, we may conclude that [5, Theorem 4.1] may be proved without using any technique involving \( b \)-metric spaces.

### 3. Improvements

Now, we announce our first result which generalizes [5, Theorem 2.2].

**Theorem 3.1.** Let \((X, d, \preceq)\) be a \( b \)-complete, partially ordered \( b \)-metric space with parameter \( s \geq 1 \). Let \( f : X \times X \rightarrow X \) be a mixed monotone mapping for which there exists a constant \( k \in [0, 1) \) such that for all \((x, y) \in X \times X\) such that for all \((x \preceq u \preceq y)\) or \((x \succeq u \preceq y)\),

\[
d(f(x, y), f(u, v)) + d(f(y, x), f(v, u)) \leq k[d(x, u) + d(y, v)].
\]

Suppose that

(a) \( f \) is continuous, or
(b) \((X, d, \preceq)\) is regular.

If there exist \( x_0, y_0 \in X \) such that \((x_0 \preceq f(x_0, y_0))\) and \((y_0 \preceq f(y_0, x_0))\) or \((x_0 \succeq f(x_0, y_0))\) and \((y_0 \succeq f(y_0, x_0))\), then there exist \( x^*, y^* \in X \) such that \( x^* = f(x^*, y^*)\) and \( y^* = f(y^*, x^*)\).
Proof. Consider the mapping $d_+: X^2 \times X^2 \to \mathbb{R}_+$ defined by $d_+(Y, V) = d(x, u) + d(y, v)$, for all $Y = (x, y)$, $V = (u, v) \in X^2$ and the relation $\sqsubseteq$ on $X^2$ defined by $Y \sqsubseteq V \iff x \preceq u$ and $y \succeq v$. It is a simple task to check that $(X^2, d_+, \sqsubseteq)$ is an ordered $b$-metric space. Also, $(X^2, d_+, \sqsubseteq)$ is $b$-complete and regular if $(X, d, \preceq)$ is such. Further, consider the mapping $F: X^2 \to X^2$ defined by $F(Y) = (f(x, y), f(y, x))$ for all $Y = (x, y) \in X^2$. It is clear that for $Y = (x, y)$, $V = (u, v) \in X^2$, in view of the definition of $d_+$, we have

$$d_+(F(Y), F(V)) = d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))$$

and

$$d_+(Y, V) = d(x, u) + d(y, v).$$

Hence, by the condition (3) we obtain a Banach type contraction (in a $b$-metric space):

$$d_+(F(Y), F(V)) \leq kd_+(Y, V),$$

for all $Y, V \in X^2$ with $Y \sqsubseteq V$ or $Y \sqsupseteq V$. The rest of proof follows by [6, 7, 9] or [13].

Remark 3.2. Theorem 3.1 is a proper generalization of [5, Theorem 2.2] in two ways. First of all, the condition $k \in [0, \frac{1}{s})$ is relaxed to $k \in [0, 1)$. Secondly, the contractive condition used in [5, Theorem 2.2] is strictly stronger than the condition (3). Appropriate examples can be easily constructed similarly as in [3, 12] and several other papers.

Also, Theorem 3.1 generalizes [3, Theorem 3].

Now, we shall prove the main result of this section.

Theorem 3.3. Theorem 3.2 from [5] is equivalent with the following result:

Let $(X, d, \preceq)$ be a $b$-complete ordered $b$-metric space with parameter $s \geq 1$ and let $f: X \to X$ be an operator. Suppose that:

1. For each $(x, y) \notin X_\preceq$ there exists $z \in X$ such that $(x, z), (y, z) \in X_\preceq$;
2. $X_\preceq \in I(f \times f)$;
3. $f: X \to X$ has a closed graph;
4. There exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_\preceq$;
5. There exists a constant $k \in [0, 1)$, such that

$$d(f(x), f(y)) \leq kd(x, y) \text{ for each } (x, y) \in X_\preceq.$$

Then $f$ is a Picard operator, i.e., $\text{Fix}(f) = \{x^\ast\}$ and $f^n(x) \to x^\ast$, as $n \to \infty$, for every $x \in X$. 

Proof. Putting \( m = 1 \), we obviously have that \([5, \text{Theorem } 3.2]\) implies the formulated result. Conversely, let the given result hold true. We shall show that in this case \([5, \text{Theorem } 3.2]\) also holds. It is known that each generalized \( b \)-metric space is also a cone \( b \)-metric space over normal solid cone with the normal constant \( K = 1 \) (for the details see \([8]\)). Therefore, the conditions (2), (5), as well as the normality of the cone imply that

\[
\|d(f^n(x), f^n(y))\|_R \leq \|A^n\| \|d(x, y)\|_R \quad \text{for each } (x, y) \in X. \tag{4}
\]

Further, from the condition (5) of \([5, \text{Theorem } 3.2]\) (that \( sA \) converges to zero), it follows that there exists \( n_0 \in \mathbb{N} \) such that \( \|A^{n_0}\| < 1 \). Hence, (4) becomes

\[
D(f^{n_0}(x), f^{n_0}(y)) \leq kD(x, y) \quad \text{for each } (x, y) \in X,
\]

where \( D(a, b) = \|d(a, b)\|, k = \|A^n\| < 1 \). Since \((X, D)\) is a \( b \)-metric space with the same parameter \( s \geq 1 \) and \( f^{n_0} : X \to X \), then by \([6, \text{Theorem } 1]\) \( f^{n_0} \) has a unique fixed point in \( X \). Hence, \( f \) has a unique fixed point. Moreover, \( f \) is a Picard operator in generalized metric space \((X, d)\). Indeed, by the assumption, \((f^{n_0})^n(x) \to x^* \) in the \( b \)-metric space \((X, D)\) from which we obtain that \( f^n(x) \to x^* \), also in \((X, D)\). Since the spaces \((X, d)\) and \((X, D)\) have the same convergent sequences, the result follows.

Remark 3.4. Theorem 3.3 and \([5, \text{Theorem } 3.2]\) show that the celebrated theorem of Ran-Reurings holds in both frameworks: ordered generalized \( b \)-metric spaces and ordered \( b \)-metric spaces.

We finish considering well-posedness of the problem treated in \([5, \text{Theorem } 3.2]\), i.e. of a Perov type operator in the framework of an ordered generalized (in the sense of Perov) \( b \)-metric space. Recall that the problem for an operator \( f \) with a unique fixed point \( x^* \in X \) is said to be well-posed if for each sequence \( \{y_n\} \) in \( X \), \( d(y_n, f y_n) \to \theta \) as \( n \to \infty \) implies that \( y_n \to x^* \) as \( n \to \infty \).

Theorem 3.5. Under the assumptions of \([5, \text{Theorem } 3.2]\), the fixed point problem for \( f \) is well-posed.

Proof. According to Theorem \([5, \text{Theorem } 3.2]\), the operator \( f \) has a unique fixed point \( x^* \). Suppose that \( \{y_n\} \) is a sequence in \( X \) such that \( d(y_n, f y_n) \to \theta \) as \( n \to \infty \) in the given generalized ordered \( b \)-complete \( b \)-metric space \((X, d, \preceq)\). Then we have that

\[
\frac{1}{s}d(y_n, x^*) \leq d(y_n, f y_n) + d(f y_n, x^*) = d(y_n, f y_n) + d(f y_n, f x^*) \\
\leq d(y_n, f x_n) + Ad(y_n, x^*),
\]

wherefrom \((I - sA)d(y_n, x^*) \leq sd(y_n f y_n)\) and

\[
d(y_n, x^*) \leq (I - sA)^{-1}sd(y_n, f y_n) \to \theta
\]

in \( \mathbb{R}^n \) since \((I - sA)^{-1}s \in M_{m \times m}(\mathbb{R}_+). \) Hence, \( d(y_n, x^*) \to \theta \) in the Banach space \( \mathbb{R}^m \), i.e., the given fixed point problem for \( f \) is well-posed.
Remark 3.6. Note that the condition (5) of [5, Theorem 3.2] (that $sA$ converges to zero, in other words, that $\rho(A) < \frac{1}{s}$ for the spectral radius of the matrix $A$) is crucial in the previous proof. In fact, the similar is true for a Banach-type contraction $f$ (satisfying $d(fx, fy) \leq kd(x, y)$) in an arbitrary $b$-metric space $(X, d)$ with parameter $s > 1$—it has a unique fixed point whenever $k \in [0, 1)$, however, it is well-posed only if $k \in [0, \frac{1}{s})$.

Acknowledgement. The first author is thankful to the Ministry of Education, Science and Technological Development of Serbia, Grant No. 174002.

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