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Remarks on the Paper "Coupled Fixed Point Theorems for Single-Valued Operators in b-Metric Spaces"

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Abstract

In this paper, we improve some recent coupled fixed point results for single-valued operators in the framework of ordered b-metric spaces established by Bota et al. [M-F. Bota, A. Petrusel, G. Petrusel and B. Samet, Coupled fixed point theorems for single-valued operators in b-metric spaces, Fixed Point Theory Appl. (2015) 2015:231]. Also, we prove that Perov-type fixed point theorem in ordered generalized b-metric spaces is equivalent with Ran-Reurings-type theorem in ordered b-metric spaces.

Keywords: Vector-valued metric, ordered b-metric space, coupled fixed point, integral equation, well-posed fixed point problem.

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1. Introduction

In 1966, Perov [11] formulated a fixed point theorem which extends the well-known contraction mapping principle to the case when the metric d takes values in $\mathbb{R}^m_+$, that is, to the case of a generalized (cone) metric space. In 1989, Bakhtin [2] introduced the concept of a b-metric space which is another generalization of the ordinary metric space. After that, several papers have appeared dealing with results in b-metric spaces (see. e.g., [5, 7, 9] as well as the references therein). For the concepts of b-convergence, b-Cauchy sequence, b-continuity and b-completeness in b-metric spaces, see for instance [5, 7]. Furthermore, several new kinds of spaces have appeared, as generalized b-metric spaces, ordered generalized b-metric spaces, etc.

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In this paper, we will first show that most of the results of paper [5] on coupled fixed points in ordered (generalized) b-metric spaces can be obtained in a much easier way. Further, in Section 3, we will improve these results and show that Perov-type fixed point theorem in ordered generalized b-metric spaces is equivalent to Ran-Reurings-type theorem in ordered b-metric spaces. We finish by proving a result on well-posedness of the given fixed point problem.

2. Remarks on the Paper [5]

Very recently, in [5], M-F. Bota et al. proved some coupled fixed point results for mixed monotone mappings in ordered generalized b-metric spaces. In this section, we will show that most of these results are basically not new.

Remark 2.1. (concerning [5, Theorem 2.2]). It is well known that from the condition (ii) of [5, Theorem 2.2], the mixed monotone property of T and by induction it easily follows that the sequence $x_{n+1} = T^n(x_0, y_0) = T(x_n, y_n)$ is nondecreasing, while the sequence $y_{n+1} = T^n(y_0, x_0) = T(y_n, x_n)$ is nonincreasing. The rest of the proof of this theorem in [5] is also not new. That is, all is the same as in [4, Theorem 2.1] for ordinary metric spaces. Moreover, the proof that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, is again well known (see, e.g., [9, Lemma 3.1]).

Further, it is not hard to see that the estimates of $d(T^n(x_0, y_0), x)$ and $d(T^n(y_0, x_0), y)$ presented in the mentioned theorem hold without assumption that the b-metric d is continuous. Indeed, our claim follows immediately from the following two inequalities:

$$\frac{1}{s} d\left(T^n(x_0, y_0), x^*\right) \le d\left(T^n(x_0, y_0), T^{n+p}(x_0, y_0)\right) + d\left(T^{n+p}(x_0, y_0), x^*\right),$$
$$\frac{1}{s} d\left(T^n(y_0, x_0), y^*\right) \le d\left(T^n(y_0, x_0), T^{n+p}(y_0, x_0)\right) + d\left(T^{n+p}(y_0, x_0), y^*\right),$$

as well as from the proof of [9, Lemma 3.1].

Finally, it is worth noticing that [5, Theorem 2.2] holds if the condition $k \in [0, \frac{1}{s})$ is relaxed to $k \in [0, 1)$ (see [6, Theorem 1], [7, Theorem 1.8], and Theorem 3.1 below).

In the sequel of [5], the well known Perov's Theorem is proved for the case of so-called generalized b-metric spaces (see also [10]).

Remark 2.2. (concerning [5, Theorem 3.2]). Instead of the condition that f has a closed graph (condition (3) in [5, Theorem 3.2]), one can suppose that f is continuous or that $(X, d, \preceq)$ is regular (recall that an ordered (generalized) metric space is said to be regular if for each nondecreasing sequence $\{x_n\}$ in X, $x_n \to x$ as $n \to \infty$ implies that $x_n \preceq x$ for $n \in \mathbb{N}$). Otherwise, [5, Theorem 3.2] is simply classical Ran-Reuring's result in the framework of ordered generalized b-metric spaces.

Remark 2.3. (concerning [5, Theorem 3.7]). It is not hard to see that the condition

$$d\left(T\left(x,y\right),T\left(u,v\right)\right)\leq k_1 d\left(x,u\right)+k_2 d\left(y,v\right),$$

which is assumed in this theorem, implies the following:

$$d_+\left(F_T\left(Y\right),F_T\left(V\right)\right)\leq k d_+\left(Y,V\right),\ \text{for all } Y\sqsubseteq V \text{ or } Y\sqsupseteq V, \tag{1}$$

where $k=k_1+k_2$, $Y=(x,y)$, $V=(u,v)$, $d_+\left(Y,V\right)=d\left(x,u\right)+d\left(y,v\right)$, $F_T\left(Y\right)=\left(T\left(x,y\right),T\left(y,x\right)\right)$ and $Y\sqsubseteq V\iff x\preceq u$ and $y\succeq v$. Further, (1) implies that

$$D_+\left(F_T\left(Y\right),F_T\left(V\right)\right)\leq kD_+\left(Y,V\right),\ \text{for all } Y\sqsubseteq V \text{ or } Y\sqsupseteq V,$$

where $D_+\left(Y,V\right)=\left\|d_+\left(Y,V\right)\right\|$. Since $\left(X\times X,D_+,\sqsubseteq\right)$ is an ordered generalized b-metric space, then the proof of [5, Theorem 3.7] follows according to [6, Theorem 1]. Hence, in fact, [5, Theorem 3.7] is not new, that is, all ideas and methods in it are well known (for more details of respective results in the framework of metric spaces see [1, 3, 14, 12, 15]).

The authors of [5] discussed also the following system of integral equations:

$$\begin{cases} x\left(t\right)=g\left(t\right)+\int_0^T G\left(s,t\right)f\left(s,x\left(s\right),y\left(s\right)\right)ds, \\ y\left(t\right)=g\left(t\right)+\int_0^T G\left(s,t\right)f\left(s,y\left(s\right),x\left(s\right)\right)ds, \end{cases} \tag{2}$$

where $t\in[0,T]$. Using certain b-metric, they proved an existence result for solutions of the system (2).

Remark 2.4. (concerning [5, Theorem 4.1]). First of all, it follows from the condition (iii) of [5, Theorem 4.1], that

$$\begin{aligned}&\left|f\left(s,u_1\left(s\right),u_2\left(s\right)\right)-f\left(s,v_1\left(s\right),v_2\left(s\right)\right)\right|\\ &\qquad\leq\left(\alpha\left(s\right)+\beta\left(s\right)\right)\max_{s\in[0,T]}\left\{\left|u_1\left(s\right)-v_1\left(s\right)\right|,\left|u_2\left(s\right)-v_2\left(s\right)\right|\right\},\end{aligned}$$

while the condition (iv) implies

$$k:=\max_{t\in[0,T]}\int_0^T G\left(s,t\right)\left(\alpha\left(s\right)+\beta\left(s\right)\right)ds<1.$$

Then, if $S:X\times X\to X$ is defined as in [5], for all ($x\succeq u$ and $y\preceq v$) or ($u\succeq x$ and $v\preceq y$), we have

$$\begin{aligned}\left|S\left(x,y\right)\left(t\right)-S\left(u,v\right)\left(t\right)\right|&=\left|\int_0^T G\left(s,t\right)\left[f\left(s,x\left(s\right),y\left(s\right)\right)-f\left(s,u\left(s\right),v\left(s\right)\right)\right]ds\right|\\ &\leq\int_0^T G\left(s,t\right)\left|f\left(s,x\left(s\right),y\left(s\right)\right)-f\left(s,u\left(s\right),v\left(s\right)\right)\right|ds\\ &\leq\int_0^T G\left(s,t\right)\left(\alpha\left(s\right)+\beta\left(s\right)\right)\max_{s\in[0,T]}\left\{\left|x\left(s\right)-u\left(s\right)\right|,\left|y\left(s\right)-v\left(s\right)\right|\right\}ds\\ &=k\delta\left(Y,V\right),\end{aligned}$$

where $\delta(Y,V)=\delta((x,y),(u,v))=\max\{D(x,u),D(y,v)\}$ is a metric on X^2 if D is a metric on X. In this case $D(x,y)=\max_{s\in[0,T]}|x(s)-y(s)|$ is a known metric on the space $C[0,T]$.

Further, in the same manner, we have

$$\begin{aligned}|S(y,x)(t)-S(v,u)(t)| &= \left|\int_0^T G(s,t)\left[f(s,y(s),x(s))-f(s,v(s),u(s))\right]ds\right| \\ &\le \int_0^T G(s,t)\left|f(s,y(s),x(s))-f(s,v(s),u(s))\right|ds \\ &\le \int_0^T G(s,t)(\alpha(s)+\beta(s))\max_{s\in[0,T]}\{|y(s)-v(s)|,|x(s)-u(s)|\}\,ds \\ &= k\delta(Y,V).\end{aligned}$$

Hence, we obtain:

$$\max_{t\in[0T]}\{|S(x,y)(t)-S(u,v)(t)|,|S(y,x)(t)-S(v,u)(t)|\}\le k\delta(Y,V),$$

that is,

$$\delta(F_S(Y),F_S(V))\le k\delta(Y,V),$$

where $F_S(Y)=F_S((x,y))=(S(x,y),S(y,x))$. Instead of the method used in the framework of b-metric spaces as in [5], we can use now simply Banach Contraction Principle for the proof that the system of integral equations (2) has a unique solution in the complete metric space $C[0,T]$. It is clear that our approach is brief and natural. Hence, we may conclude that [5, Theorem 4.1] may be proved without using any technique involving b-metric spaces.

3. Improvements

Now, we announce our first result which generalizes [5, Theorem 2.2].

Theorem 3.1. *Let $(X,d,\preceq)$ be a b-complete, partially ordered b-metric space with parameter $s\ge 1$. Let $f:X\times X\to X$ be a mixed monotone mapping for which there exists a constant $k\in[0,1)$ such that for all ($x\preceq u$ and $y\succeq v$) or ($x\succeq u$ and $y\preceq v$),*

$$d(f(x,y),f(u,v))+d(f(y,x),f(v,u))\le k\left[d(x,u)+d(y,v)\right]. \tag{3}$$

Suppose that

(a) f is continuous, or

(b) $(X,d,\preceq)$ is regular.

If there exist $x_0,y_0\in X$ such that ($x_0\preceq f(x_0,y_0)$ and $y_0\succeq f(y_0,x_0)$) or ($x_0\preceq f(x_0,y_0)$ and $y_0\succeq f(y_0,x_0)$), then there exist $x^*,y^*\in X$ such that $x^*=f(x^*,y^*)$ and $y^*=f(y^*,x^*)$.

Proof. Consider the mapping $d_+ : X^2 \times X^2 \to \mathbb{R}_+$ defined by $d_+(Y,V) = d(x,u) + d(y,v)$, for all $Y = (x,y)$, $V = (u,v) \in X^2$ and the relation $\sqsubseteq$ on X^2 defined by $Y \sqsubseteq V \iff x \preceq u$ and $y \succeq v$. It is a simple task to check that $\left(X^2, d_+, \sqsubseteq\right)$ is an ordered b-metric space. Also, $\left(X^2, d_+, \sqsubseteq\right)$ is b-complete and regular if $(X, d, \preceq)$ is such. Further, consider the mapping $F : X^2 \to X^2$ defined by $F(Y) = (f(x,y), f(y,x))$ for all $Y = (x,y) \in X^2$. It is clear that for $Y = (x,y), V = (u,v) \in X^2$, in view of the definition of d_+, we have

$$\begin{aligned} d_+(F(Y), F(V)) &= d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ \text{and } d_+(Y,V) &= d(x,u) + d(y,v). \end{aligned}$$

Hence, by the condition (3) we obtain a Banach type contraction (in a b-metric space):

$$d_+(F(Y), F(V)) \leq k d_+(Y,V),$$

for all $Y, V \in X^2$ with $Y \sqsubseteq V$ or $Y \sqsupseteq V$. The rest of proof follows by [6, 7, 9] or [13]. □

Remark 3.2. Theorem 3.1 is a proper generalization of [5, Theorem 2.2] in two ways. First of all, the condition $k \in [0, \frac{1}{s})$ is relaxed to $k \in [0,1)$. Secondly, the contractive condition used in [5, Theorem 2.2] is strictly stronger than the condition (3). Appropriate examples can be easily constructed similarly as in [3, 12] and several other papers.

Also, Theorem 3.1 generalizes [3, Theorem 3].

Now, we shall prove the main result of this section.

Theorem 3.3. *Theorem 3.2 from [5] is equivalent with the following result:*

Let $(X, d, \preceq)$ be a b-complete ordered b-metric space with parameter $s \geq 1$ and let $f : X \to X$ be an operator. Suppose that:

(1) *for each $(x,y) \notin X_{\preceq}$ there exists $z \in X$ such that $(x,z), (y,z) \in X_{\preceq}$;*

(2) $X_{\preceq} \in I(f \times f)$;

(3) *$f : X \to X$ has a closed graph;*

(4) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\preceq}$;*

(5) *there exists a constant $k \in [0,1)$, such that*

$$d(f(x), f(y)) \leq k d(x,y) \text{ for each } (x,y) \in X_{\preceq}.$$

Then f is a Picard operator, i.e., $Fix(f) = \{x^*\}$ and $f^n(x) \to x^*$, as $n \to \infty$, for every $x \in X$.

Proof. Putting $m = 1$, we obviously have that [5, Theorem 3.2] implies the formulated result. Conversely, let the given result hold true. We shall show that in this case [5, Theorem 3.2] also holds. It is known that each generalized b-metric space is also a cone b-metric space over normal solid cone with the normal constant $K = 1$ (for the details see [8]). Therefore, the conditions (2), (5), as well as the normality of the cone imply that

$$\|d(f^n(x), f^n(y))\|_{\mathbb{R}^m} \leq \|A^n\| \, \|d(x,y)\|_{\mathbb{R}^m} \text{ for each } (x,y) \in X_{\preceq}. \qquad (4)$$

Further, from the condition (5) of [5, Theorem 3.2] (that sA converges to zero), it follows that there exists $n_0 \in \mathbb{N}$ such that $\|A^{n_0}\| < 1$. Hence, (4) becomes

$$D(f^{n_0}(x), f^{n_0}(y)) \leq kD(x,y) \text{ for each } (x,y) \in X_{\preceq},$$

where $D(a,b) = \|d(a,b)\|$, $k = \|A^{n_0}\| < 1$. Since (X, D) is a b-metric space with the same parameter $s \geq 1$ and $f^{n_0} : X \to X$, then by ([6, Theorem 1]) f^{n_0} has a unique fixed point in X. Hence, f has a unique fixed point. Moreover, f is a Picard operator in generalized metric space (X, d). Indeed, by the assumption, $(f^{n_0})^n(x) \to x^*$ in the b-metric space (X, D) from which we obtain that $f^n(x) \to x^*$, also in (X, D). Since the spaces (X, d) and (X, D) have the same convergent sequences, the result follows. □

Remark 3.4. Theorem 3.3 and [5, Theorem 3.2] show that the celebrated theorem of Ran-Reurings holds in both frameworks: ordered generalized b-metric spaces and ordered b-metric spaces.

We finish considering well-posedness of the problem treated in [5, Theorem 3.2], i.e. of a Perov type operator in the framework of an ordered generalized (in the sense of Perov) b-metric space. Recall that the problem for an operator f with a unique fixed point $x^* \in X$ is said to be well-posed if for each sequence $\{y_n\}$ in X, $d(y_n, fy_n) \to \theta$ as $n \to \infty$ implies that $y_n \to x^*$ as $n \to \infty$.

Theorem 3.5. *Under the assumptions of [5, Theorem 3.2], the fixed point problem for f is well-posed.*

Proof. According to Theorem [5, Theorem 3.2], the operator f has a unique fixed point x^*. Suppose that $\{y_n\}$ is a sequence in X such that $d(y_n, fy_n) \to \theta$ as $n \to \infty$ in the given generalized ordered b-complete b-metric space $(X, d, \preceq)$. Then we have that

$$\begin{aligned} \frac{1}{s} d(y_n, x^*) &\leq d(y_n, fy_n) + d(fy_n, x^*) = d(y_n, fy_n) + d(fy_n, fx^*) \\ &\leq d(y_n, fx_n) + Ad(y_n, x^*), \end{aligned}$$

wherefrom $(I - sA)d(y_n, x^*) \leq sd(y_n fy_n)$ and

$$d(y_n, x^*) \leq (I - sA)^{-1} sd(y_n, fy_n) \to \theta$$

in $\mathbb{R}^n$ since $(I - sA)^{-1} s \in M_{m \times m}(\mathbb{R}_+)$. Hence, $d(y_n, x^*) \to \theta$ in the Banach space $\mathbb{R}^m$, i.e., the given fixed point problem for f is well-posed. □

Remark 3.6. Note that the condition (5) of [5, Theorem 3.2] (that sA converges to zero, in other words, that $\rho(A) < \frac{1}{s}$ for the spectral radius of the matrix A) is crucial in the previous proof. In fact, the similar is true for a Banach-type contraction f (satisfying $d(fx, fy) \leq kd(x, y)$) in an arbitrary b-metric space (X, d) with parameter $s > 1$—it has a unique fixed point whenever $k \in [0, 1)$, however, it is well-posed only if $k \in [0, \frac{1}{s})$.

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