

## Remarks on the Paper “Coupled Fixed Point Theorems for Single-Valued Operators in $b$ -Metric Spaces”

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### Abstract

In this paper, we improve some recent coupled fixed point results for single-valued operators in the framework of ordered  $b$ -metric spaces established by Bota et al. [M-F. Bota, A. Petrusel, G. Petrusel and B. Samet, Coupled fixed point theorems for single-valued operators in  $b$ -metric spaces, Fixed Point Theory Appl. (2015) 2015:231]. Also, we prove that Perov-type fixed point theorem in ordered generalized  $b$ -metric spaces is equivalent with Ran-Reurings-type theorem in ordered  $b$ -metric spaces.

**Keywords:** Vector-valued metric, ordered  $b$ -metric space, coupled fixed point, integral equation, well-posed fixed point problem.

**2010 Mathematics Subject Classification:** 47H10, 54H25.

## 1. Introduction

In 1966, Perov [11] formulated a fixed point theorem which extends the well-known contraction mapping principle to the case when the metric  $d$  takes values in  $\mathbb{R}_+^m$ , that is, to the case of a generalized (cone) metric space. In 1989, Bakhtin [2] introduced the concept of a  $b$ -metric space which is another generalization of the ordinary metric space. After that, several papers have appeared dealing with results in  $b$ -metric spaces (see. e.g., [5, 7, 9] as well as the references therein). For the concepts of  $b$ -convergence,  $b$ -Cauchy sequence,  $b$ -continuity and  $b$ -completeness in  $b$ -metric spaces, see for instance [5, 7]. Furthermore, several new kinds of spaces have appeared, as generalized  $b$ -metric spaces, ordered generalized  $b$ -metric spaces, etc.

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Academic Editor: Ali Reza Ashrafi  
Received 27 June 2016, Accepted 10 October 2016  
DOI:

In this paper, we will first show that most of the results of paper [5] on coupled fixed points in ordered (generalized)  $b$ -metric spaces can be obtained in a much easier way. Further, in Section 3, we will improve these results and show that Perov-type fixed point theorem in ordered generalized  $b$ -metric spaces is equivalent to Ran-Reurings-type theorem in ordered  $b$ -metric spaces. We finish by proving a result on well-posedness of the given fixed point problem.

## 2. Remarks on the Paper [5]

Very recently, in [5], M-F. Bota et al. proved some coupled fixed point results for mixed monotone mappings in ordered generalized  $b$ -metric spaces. In this section, we will show that most of these results are basically not new.

*Remark 1.* (concerning [5, Theorem 2.2]). It is well known that from the condition (ii) of [5, Theorem 2.2], the mixed monotone property of  $T$  and by induction it easily follows that the sequence  $x_{n+1} = T^n(x_0, y_0) = T(x_n, y_n)$  is nondecreasing, while the sequence  $y_{n+1} = T^n(y_0, x_0) = T(y_n, x_n)$  is nonincreasing. The rest of the proof of this theorem in [5] is also not new. That is, all is the same as in [4, Theorem 2.1] for ordinary metric spaces. Moreover, the proof that the sequences  $\{x_n\}$  and  $\{y_n\}$  are Cauchy sequences, is again well known (see, e.g., [9, Lemma 3.1]).

Further, it is not hard to see that the estimates of  $d(T^n(x_0, y_0), x)$  and  $d(T^n(y_0, x_0), y)$  presented in the mentioned theorem hold without assumption that the  $b$ -metric  $d$  is continuous. Indeed, our claim follows immediately from the following two inequalities:

$$\begin{aligned} \frac{1}{s}d(T^n(x_0, y_0), x^*) &\leq d(T^n(x_0, y_0), T^{n+p}(x_0, y_0)) + d(T^{n+p}(x_0, y_0), x^*), \\ \frac{1}{s}d(T^n(y_0, x_0), y^*) &\leq d(T^n(y_0, x_0), T^{n+p}(y_0, x_0)) + d(T^{n+p}(y_0, x_0), y^*), \end{aligned}$$

as well as from the proof of [9, Lemma 3.1].

Finally, it is worth noticing that [5, Theorem 2.2] holds if the condition  $k \in [0, \frac{1}{s})$  is relaxed to  $k \in [0, 1)$  (see [6, Theorem 1], [7, Theorem 1.8], and Theorem 3.1 below).

In the sequel of [5], the well known Perov's Theorem is proved for the case of so-called generalized  $b$ -metric spaces (see also [10]).

*Remark 2.* (concerning [5, Theorem 3.2]). Instead of the condition that  $f$  has a closed graph (condition (3) in [5, Theorem 3.2]), one can suppose that  $f$  is continuous or that  $(X, d, \preceq)$  is regular (recall that an ordered (generalized) metric space is said to be regular if for each nondecreasing sequence  $\{x_n\}$  in  $X$ ,  $x_n \rightarrow x$  as  $n \rightarrow \infty$  implies that  $x_n \preceq x$  for  $n \in \mathbb{N}$ ). Otherwise, [5, Theorem 3.2] is simply classical Ran-Reuring's result in the framework of ordered generalized  $b$ -metric spaces.

*Remark 3.* (concerning [5, Theorem 3.7]). It is not hard to see that the condition

$$d(T(x, y), T(u, v)) \leq k_1 d(x, u) + k_2 d(y, v),$$

which is assumed in this theorem, implies the following:

$$d_+(F_T(Y), F_T(V)) \leq k d_+(Y, V), \text{ for all } Y \sqsubseteq V \text{ or } Y \supseteq V, \quad (1)$$

where  $k = k_1 + k_2$ ,  $Y = (x, y)$ ,  $V = (u, v)$ ,  $d_+(Y, V) = d(x, u) + d(y, v)$ ,  $F_T(Y) = (T(x, y), T(y, x))$  and  $Y \sqsubseteq V \iff x \preceq u$  and  $y \succeq v$ . Further, (1) implies that

$$D_+(F_T(Y), F_T(V)) \leq k D_+(Y, V), \text{ for all } Y \sqsubseteq V \text{ or } Y \supseteq V,$$

where  $D_+(Y, V) = \|d_+(Y, V)\|$ . Since  $(X \times X, D_+, \sqsubseteq)$  is an ordered generalized  $b$ -metric space, then the proof of [5, Theorem 3.7] follows according to [6, Theorem 1]. Hence, in fact, [5, Theorem 3.7] is not new, that is, all ideas and methods in it are well known (for more details of respective results in the framework of metric spaces see [1, 3, 12, 13, 15]).

The authors of [5] discussed also the following system of integral equations:

$$\begin{cases} x(t) = g(t) + \int_0^T G(s, t) f(s, x(s), y(s)) ds, \\ y(t) = g(t) + \int_0^T G(s, t) f(s, y(s), x(s)) ds, \end{cases} \quad (2)$$

where  $t \in [0, T]$ . Using certain  $b$ -metric, they proved an existence result for solutions of the system (2).

*Remark 4.* (concerning [5, Theorem 4.1]). First of all, it follows from the condition (iii) of [5, Theorem 4.1], that

$$\begin{aligned} & |f(s, u_1(s), u_2(s)) - f(s, v_1(s), v_2(s))| \\ & \leq (\alpha(s) + \beta(s)) \max_{s \in [0, T]} \{|u_1(s) - v_1(s)|, |u_2(s) - v_2(s)|\}, \end{aligned}$$

while the condition (iv) implies

$$k := \max_{t \in [0, T]} \int_0^T G(s, t) (\alpha(s) + \beta(s)) ds < 1.$$

Then, if  $S : X \times X \rightarrow X$  is defined as in [5], for all  $(x \succeq u$  and  $y \preceq v)$  or  $(u \succeq x$  and  $v \preceq y)$ , we have

$$\begin{aligned} |S(x, y)(t) - S(u, v)(t)| &= \left| \int_0^T G(s, t) [f(s, x(s), y(s)) - f(s, u(s), v(s))] ds \right| \\ &\leq \int_0^T G(s, t) |f(s, x(s), y(s)) - f(s, u(s), v(s))| ds \\ &\leq \int_0^T G(s, t) (\alpha(s) + \beta(s)) \max_{s \in [0, T]} \{|x(s) - u(s)|, |y(s) - v(s)|\} ds \\ &= k \delta(Y, V), \end{aligned}$$

where  $\delta(Y, V) = \delta((x, y), (u, v)) = \max\{D(x, u), D(y, v)\}$  is a metric on  $X^2$  if  $D$  is a metric on  $X$ . In this case  $D(x, y) = \max_{s \in [0, T]} |x(s) - y(s)|$  is a known metric on the space  $C[0, T]$ .

Further, in the same manner, we have

$$\begin{aligned} |S(y, x)(t) - S(v, u)(t)| &= \left| \int_0^T G(s, t) [f(s, y(s), x(s)) - f(s, v(s), u(s))] ds \right| \\ &\leq \int_0^T G(s, t) |f(s, y(s), x(s)) - f(s, v(s), u(s))| ds \\ &\leq \int_0^T G(s, t) (\alpha(s) + \beta(s)) \max_{s \in [0, T]} \{|y(s) - v(s)|, |x(s) - u(s)|\} ds \\ &= k\delta(Y, V). \end{aligned}$$

Hence, we obtain:

$$\max_{t \in [0, T]} \{|S(x, y)(t) - S(u, v)(t)|, |S(y, x)(t) - S(v, u)(t)|\} \leq k\delta(Y, V),$$

that is,

$$\delta(F_S(Y), F_S(V)) \leq k\delta(Y, V),$$

where  $F_S(Y) = F_S((x, y)) = (S(x, y), S(y, x))$ . Instead of the method used in the framework of  $b$ -metric spaces as in [5], we can use now simply Banach Contraction Principle for the proof that the system of integral equations (2) has a unique solution in the complete metric space  $C[0, T]$ . It is clear that our approach is brief and natural. Hence, we may conclude that [5, Theorem 4.1] may be proved without using any technique involving  $b$ -metric spaces.

### 3. Improvements

Now, we announce our first result which generalizes [5, Theorem 2.2].

**Theorem 3.1.** *Let  $(X, d, \preceq)$  be a  $b$ -complete, partially ordered  $b$ -metric space with parameter  $s \geq 1$ . Let  $f : X \times X \rightarrow X$  be a mixed monotone mapping for which there exists a constant  $k \in [0, 1)$  such that for all  $(x \preceq u$  and  $y \succeq v)$  or  $(x \succeq u$  and  $y \preceq v)$ ,*

$$d(f(x, y), f(u, v)) + d(f(y, x), f(v, u)) \leq k[d(x, u) + d(y, v)]. \quad (3)$$

Suppose that

- (a)  $f$  is continuous, or
- (b)  $(X, d, \preceq)$  is regular.

If there exist  $x_0, y_0 \in X$  such that  $(x_0 \preceq f(x_0, y_0)$  and  $y_0 \succeq f(y_0, x_0))$  or  $(x_0 \succeq f(x_0, y_0)$  and  $y_0 \preceq f(y_0, x_0))$ , then there exist  $x^*, y^* \in X$  such that  $x^* = f(x^*, y^*)$  and  $y^* = f(y^*, x^*)$ .

*Proof.* Consider the mapping  $d_+ : X^2 \times X^2 \rightarrow \mathbb{R}_+$  defined by  $d_+(Y, V) = d(x, u) + d(y, v)$ , for all  $Y = (x, y), V = (u, v) \in X^2$  and the relation  $\sqsubseteq$  on  $X^2$  defined by  $Y \sqsubseteq V \iff x \preceq u$  and  $y \succeq v$ . It is a simple task to check that  $(X^2, d_+, \sqsubseteq)$  is an ordered  $b$ -metric space. Also,  $(X^2, d_+, \sqsubseteq)$  is  $b$ -complete and regular if  $(X, d, \preceq)$  is such. Further, consider the mapping  $F : X^2 \rightarrow X^2$  defined by  $F(Y) = (f(x, y), f(y, x))$  for all  $Y = (x, y) \in X^2$ . It is clear that for  $Y = (x, y), V = (u, v) \in X^2$ , in view of the definition of  $d_+$ , we have

$$d_+(F(Y), F(V)) = d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ \text{and } d_+(Y, V) = d(x, u) + d(y, v).$$

Hence, by the condition (3) we obtain a Banach type contraction (in a  $b$ -metric space):

$$d_+(F(Y), F(V)) \leq kd_+(Y, V),$$

for all  $Y, V \in X^2$  with  $Y \sqsubseteq V$  or  $Y \supseteq V$ . The rest of proof follows by [6, 7, 9] or [14].  $\square$

*Remark 5.* Theorem 3.1 is a proper generalization of [5, Theorem 2.2] in two ways. First of all, the condition  $k \in [0, \frac{1}{s})$  is relaxed to  $k \in [0, 1)$ . Secondly, the contractive condition used in [5, Theorem 2.2] is strictly stronger than the condition (3). Appropriate examples can be easily constructed similarly as in [3, 13] and several other papers.

Also, Theorem 3.1 generalizes [3, Theorem 3].

Now, we shall prove the main result of this section.

**Theorem 3.2.** *Theorem 3.2 from [5] is equivalent with the following result:*

*Let  $(X, d, \preceq)$  be a  $b$ -complete ordered  $b$ -metric space with parameter  $s \geq 1$  and let  $f : X \rightarrow X$  be an operator. Suppose that:*

- (1) *for each  $(x, y) \notin X_{\preceq}$  there exists  $z \in X$  such that  $(x, z), (y, z) \in X_{\preceq}$ ;*
- (2)  *$X_{\preceq} \in I(f \times f)$ ;*
- (3)  *$f : X \rightarrow X$  has a closed graph;*
- (4) *there exists  $x_0 \in X$  such that  $(x_0, f(x_0)) \in X_{\preceq}$ ;*
- (5) *there exists a constant  $k \in [0, 1)$ , such that*

$$d(f(x), f(y)) \leq kd(x, y) \text{ for each } (x, y) \in X_{\preceq}.$$

*Then  $f$  is a Picard operator, i.e.,  $Fix(f) = \{x^*\}$  and  $f^n(x) \rightarrow x^*$ , as  $n \rightarrow \infty$ , for every  $x \in X$ .*

*Proof.* Putting  $m = 1$ , we obviously have that [5, Theorem 3.2] implies the formulated result. Conversely, let the given result hold true. We shall show that in this case [5, Theorem 3.2] also holds. It is known that each generalized  $b$ -metric space is also a cone  $b$ -metric space over normal solid cone with the normal constant  $K = 1$  (for the details see [8]). Therefore, the conditions (2), (5), as well as the normality of the cone imply that

$$\|d(f^n(x), f^n(y))\|_{\mathbb{R}^m} \leq \|A^n\| \|d(x, y)\|_{\mathbb{R}^m} \text{ for each } (x, y) \in X_{\preceq}. \quad (4)$$

Further, from the condition (5) of [5, Theorem 3.2] (that  $sA$  converges to zero), it follows that there exists  $n_0 \in \mathbb{N}$  such that  $\|A^{n_0}\| < 1$ . Hence, (4) becomes

$$D(f^{n_0}(x), f^{n_0}(y)) \leq kD(x, y) \text{ for each } (x, y) \in X_{\preceq},$$

where  $D(a, b) = \|d(a, b)\|$ ,  $k = \|A^{n_0}\| < 1$ . Since  $(X, D)$  is a  $b$ -metric space with the same parameter  $s \geq 1$  and  $f^{n_0} : X \rightarrow X$ , then by ([6, Theorem 1])  $f^{n_0}$  has a unique fixed point in  $X$ . Hence,  $f$  has a unique fixed point. Moreover,  $f$  is a Picard operator in generalized metric space  $(X, d)$ . Indeed, by the assumption,  $(f^{n_0})^n(x) \rightarrow x^*$  in the  $b$ -metric space  $(X, D)$  from which we obtain that  $f^n(x) \rightarrow x^*$ , also in  $(X, D)$ . Since the spaces  $(X, d)$  and  $(X, D)$  have the same convergent sequences, the result follows.  $\square$

*Remark 6.* Theorem 3.2 and [5, Theorem 3.2] show that the celebrated theorem of Ran-Reurings holds in both frameworks: ordered generalized  $b$ -metric spaces and ordered  $b$ -metric spaces.

We finish considering well-posedness of the problem treated in [5, Theorem 3.2], i.e. of a Perov type operator in the framework of an ordered generalized (in the sense of Perov)  $b$ -metric space. Recall that the problem for an operator  $f$  with a unique fixed point  $x^* \in X$  is said to be well-posed if for each sequence  $\{y_n\}$  in  $X$ ,  $d(y_n, fy_n) \rightarrow \theta$  as  $n \rightarrow \infty$  implies that  $y_n \rightarrow x^*$  as  $n \rightarrow \infty$ .

**Theorem 3.3.** *Under the assumptions of [5, Theorem 3.2], the fixed point problem for  $f$  is well-posed.*

*Proof.* According to Theorem [5, Theorem 3.2], the operator  $f$  has a unique fixed point  $x^*$ . Suppose that  $\{y_n\}$  is a sequence in  $X$  such that  $d(y_n, fy_n) \rightarrow \theta$  as  $n \rightarrow \infty$  in the given generalized ordered  $b$ -complete  $b$ -metric space  $(X, d, \preceq)$ . Then we have that

$$\begin{aligned} \frac{1}{s}d(y_n, x^*) &\leq d(y_n, fy_n) + d(fy_n, x^*) = d(y_n, fy_n) + d(fy_n, fx^*) \\ &\leq d(y_n, fx_n) + Ad(y_n, x^*), \end{aligned}$$

wherefrom  $(I - sA)d(y_n, x^*) \leq sd(y_n, fx_n)$  and

$$d(y_n, x^*) \leq (I - sA)^{-1}sd(y_n, fx_n) \rightarrow \theta$$

in  $\mathbb{R}^m$  since  $(I - sA)^{-1}s \in M_{m \times m}(\mathbb{R}_+)$ . Hence,  $d(y_n, x^*) \rightarrow \theta$  in the Banach space  $\mathbb{R}^m$ , i.e., the given fixed point problem for  $f$  is well-posed.  $\square$

*Remark 7.* Note that the condition (5) of [5, Theorem 3.2] (that  $sA$  converges to zero, in other words, that  $\rho(A) < \frac{1}{s}$  for the spectral radius of the matrix  $A$ ) is crucial in the previous proof. In fact, the similar is true for a Banach-type contraction  $f$  (satisfying  $d(fx, fy) \leq kd(x, y)$ ) in an arbitrary  $b$ -metric space  $(X, d)$  with parameter  $s > 1$ —it has a unique fixed point whenever  $k \in [0, 1)$ , however, it is well-posed only if  $k \in [0, \frac{1}{s})$ .

**Acknowledgement.** The first author is thankful to the Ministry of Education, Science and Technological Development of Serbia, Grant No. 174002.

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