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Hartley Series Direct Method for Variational Problems

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Abstract

The computational method based on using the operational matrix of an orthogonal function for solving variational problems is computer-oriented. In this approach, a truncated Hartley series together with the operational matrix of integration and integration of the cross product of two case vectors are used for finding the solution of variational problems. Two illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Hartley series, operational matrix, numerical methods, calculus of variations.

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1. Introduction

In the large number of scientific and engineering problems, it is necessary to determine the maximal or minimal of a certain functional. Such problems are called variational problems. Most variational problems do not have closed form solutions, so approximation and numerical techniques must be used. In the literature, the direct method of Ritz and Galerkin has been investigated for solving variational problems by Schecher [11]. Some orthogonal polynomials are applied on variational problems to find continuous solutions for these problems. The authors of [6, 7, 8, 9, 12, 3] introduced the rationalized Haar method, Legendre wavelets method, differential transformation method, Chebyshev finite difference method, variational iteration method and the hybrid of block-pulse functions and Lagrange interpolating polynomials for solving variational problems, respectively.

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In this work, we introduce a new direct computational method to solve variational problems. The method consists of reducing the variational problem into a set of algebraic equations by first expanding the candidate function as Hartley series with unknown coefficients. The operational matrix of integration and integration of the cross product of two *cas* function vectors are then used to evaluate the coefficients of the Hartley series in such a way that the necessary conditions for extremization is imposed.

The organization of this paper is as follows. In the next section, we describe the basic formulation of Hartley series required for our subsequent development. Section 3 summarizes the application of Hartley series method to the solution of variational problems. As a result a set of algebraic equations is formed and a solution of the considered problem is introduced. In Section 4, some numerical results are given to clarify the method. Section 5 ends this paper with a conclusion.

2. Properties of Hartley Series

In Fourier series the basis functions are the complex exponential, $e^{i\omega t}$. The Hartley series utilizes a similar frequency based function, the function $\cos(\omega t) + \sin(\omega t)$, also known as the $cosine - and - sine$ function or $cas(\omega t)$. The most important properties of the *cas* function are given in [1, 5, 10]. A periodic function $f(t)$ with period ℓ can be approximated by Hartley series as follows:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k T_k(t), \tag{1}$$

where

$$T_k(t) = cas\left(\frac{2k\pi t}{\ell}\right) = \cos\left(\frac{2k\pi t}{\ell}\right) + \sin\left(\frac{2k\pi t}{\ell}\right), \quad k \in Z,$$

and

$$c_k = 1/\ell \int_0^\ell f(t) T_k(t), \quad k \in Z.$$

If $f(t)$ is truncated up to $(2n+1)$ terms, then (1) can be written as

$$f(t) \simeq \sum_{k=-n}^{n} c_k T_k(t) = C^T T(t), \tag{2}$$

where the *cas* coefficient vector C and *cas* vector $T(t)$ are given by

$$C = [c_{-n}, \ldots, c_{-1}, c_0, c_1, \ldots, c_n]^T,$$
$$T(t) = [T_{-n}(t), \ldots, T_{-1}(t), T_0(t), T_1(t), \ldots, T_n(t)]^T.$$

The elements of the vector $T(t)$ are orthogonal in the interval $(0,\ell)$ and the integration of the vector $T(t)$ can be approximated by

$$\int_0^t T(\tau)d\tau \simeq \mathbf{P}T(t), \tag{3}$$

where $\mathbf{P}$ is $(2n+1)\times(2n+1)$ Hartley series operational matrix for integration and can be found in [4]

$$\mathbf{P} = \frac{\ell}{2\pi}\begin{pmatrix} & & & & \frac{-1}{n} & & & & \frac{1}{n} \\ & & & & \vdots & & & & \\ & & & & \frac{-1}{2} & & \frac{1}{2} & & \\ & & & & -1 & 1 & & & \\ \frac{1}{n} & \cdots & \frac{1}{2} & 1 & \pi & -1 & \frac{-1}{2} & \cdots & \frac{-1}{n} \\ & & & -1 & 1 & & & & \\ & & \frac{-1}{2} & & \frac{1}{2} & & & & \\ & & & & \vdots & & & & \\ \frac{-1}{n} & & & & \frac{1}{n} & & & & \end{pmatrix}. \tag{4}$$

The cross-product of two *cas* vectors is

$$T(t)T^T(t) = \begin{pmatrix} T_{-n}^2 & \cdots & T_{-n}T_0 & \cdots & T_{-n}T_n \\ \vdots & & \vdots & & \vdots \\ T_0T_{-n} & \cdots & T_0^2 & \cdots & T_0T_n \\ \vdots & & \vdots & & \vdots \\ T_nT_{-n} & \cdots & T_nT_0 & \cdots & T_n^2 \end{pmatrix}.$$

By using $\int_0^\ell T_n(t)T_m(t)dt = \ell\delta_{n,m}$ we obtain

$$\mathbf{H} = \int_0^\ell T(t)T^T(t)dt = \ell\mathbf{I}_{2n+1,2n+1}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix.

3. Hartley Series Direct Method

Consider the problem of finding the extremum of the functional

$$J(x) = \int_0^1 F[t, x(t), \dot{x}(t)]dt. \tag{6}$$

Here we consider a Ritz direct method for solving (6) using Hartley series. Let $\ell = 1$, suppose first that the rate variable $\dot{x}(t)$ can be expanded approximately as

$$\dot{x}(t) \simeq \sum_{k=-n}^{n} c_k T_k(t) = C^T T(t). \tag{7}$$

Using operational matrix of integration in (3) we have

$$x(t) \simeq \int_0^t \dot{x}(\tau)d\tau + x(0) \simeq (C^T\mathbf{P} + [0,\ldots,0,x(0),0,\ldots,0])T(t). \tag{8}$$

Notice that $T_0(t) = 1$, using (1) for $f(t) = t$ we can express t in $(0,1)$ in terms of Hartley series as

$$t = \sum_{k=-\infty}^{\infty} d_k T_k(t), \qquad d_k = \begin{cases} \frac{1}{2}, & k = 0, \\ \frac{-1}{2k\pi}, & k \in Z - \{0\}. \end{cases} \tag{9}$$

If (9) is truncated up to $2n+1$ terms, then we have

$$t \simeq \frac{1}{2\pi}\left[\frac{1}{n},\ldots,\frac{1}{2},1,\pi,-1,\frac{-1}{2},\ldots,\frac{-1}{n}\right]T(t) = d^T T(t). \tag{10}$$

Substituting (7), (8) and (10) in (6) the functional $J(x)$ becomes a function of $c_{-n},...,c_n$. Hence, to find the extremum of $J(x)$ we solve $\frac{\partial J}{\partial c_k} = 0, \quad k = -n,...,n$. The above procedure is now used to solve the following variational problems.

4. Illustrative Examples

To show the efficiency of the method described above, we present some examples. These examples are chosen such that there exist exact solutions for them.

Example 1: Consider the problem of finding the minimum of [2, 6, 7]

$$J(x) = \int_0^1 \left[\dot{x}^2(t) + t\dot{x}(t)\right] dt, \tag{11}$$

with the boundary conditions

$$x(0) = 0, \quad x(1) = \frac{1}{4}, \tag{12}$$

using (5), (7), (10) and (11) we obtain

$$\begin{aligned} J(x) &\simeq \int_0^1 \left[C^T T(t) T^T(t) C + d^T T(t) T^T(t) C\right] dt = C^T\mathbf{H}C + d^T\mathbf{H}C \\ &= \frac{1}{2}\sum_{k=-n,k\neq 0}^{n}\left(c_k^2 - \frac{1}{2k\pi}c_k\right) + c_0^2 + \frac{1}{2}c_0, \end{aligned} \tag{13}$$

now using (7), we have

$$x(t) \simeq \sum_{k=-n,k\neq 0}^{n} \frac{c_k}{2k\pi}(\sin 2k\pi t + 1 - \cos 2k\pi t) + x(0) + c_0 t. \tag{14}$$

Imposing the boundary conditions (12) we obtain $c_0 = \frac{1}{4}$. Thus (13) becomes

$$J(x) \simeq \frac{3}{16} + \frac{1}{2} \sum_{k=-n, k\neq 0}^{n} \left(c_k^2 - \frac{1}{2k\pi} c_k \right). \tag{15}$$

Therefore

$$\frac{\partial J}{\partial c_k} = c_k - \frac{1}{4k\pi} \quad k = -n, \ldots, -1, 1, \ldots, n,$$

setting the derivative of J with respect to c_k, equal to zero we obtain $c_k = \frac{1}{4k\pi}$. Hence, the extremal function is

$$\dot{x}(t) \simeq \frac{1}{4} + \sum_{k=-n, k\neq 0}^{n} \frac{1}{4k\pi} \left(\cos 2k\pi t + \sin 2k\pi t \right),$$

and

$$x(t) \simeq \frac{1}{4} t + \sum_{k=-n, k\neq 0}^{n} \frac{1}{8k^2\pi^2} \left(1 + \sin 2k\pi t - \cos 2k\pi t \right)$$

$$= \frac{1}{4} t + \frac{1}{8\pi^2} \sum_{k=-n, k\neq 0}^{n} \frac{1}{k^2} + \frac{1}{8\pi^2} \sum_{k=-n, k\neq 0}^{n} \frac{\sin 2k\pi t}{k^2} - \frac{1}{8\pi^2} \sum_{k=-n, k\neq 0}^{n} \frac{\cos 2k\pi t}{k^2}.$$

But $\sum_{k=-n, k\neq 0}^{n} \frac{\sin 2k\pi t}{k^2} = 0$, thus

$$x(t) \simeq \frac{1}{4} t + \frac{1}{4\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{4\pi^2} \sum_{k=1}^{n} \frac{\cos 2k\pi t}{k^2}.$$

Now noting that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $\sum_{k=1}^{\infty} \frac{\cos 2k\pi t}{k^2} = \frac{\pi^2}{6} - \pi^2 t(1-t)$, therefore

$$\lim_{n\to\infty} x(t) = \frac{1}{4} t + \frac{1}{4} t(1-t) = \frac{1}{2} t - \frac{1}{4} t^2,$$

which is the exact solution of the problem.

Example 2: In this example we use the Hartley series to a heat conduction problem. Consider the extermization of [6, 7]

$$J = \int_0^1 \left[\frac{1}{2} \dot{y}^2 - y g(x) \right] dx = \int_0^1 F[x, y, \dot{y}] dx \tag{16}$$

where $g(x)$ is a known function satisfying $\int_0^1 g(x)dx = 0$, with the boundary conditions

$$y(0) = 0, \quad \dot{y}(0) = \dot{y}(1) = 0. \tag{17}$$

Consider the case where $g(x)$ is given by

$$g(x) = \begin{cases} -1, & 0 \leq x < \frac{1}{4}, \quad \frac{1}{2} \leq x < 1, \\ 3, & \frac{1}{4} \leq x < \frac{1}{2}. \end{cases} \tag{18}$$

Note that the exact solution is

$$y_{exact}(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x \le \frac{1}{4}, \\ -\frac{3}{2}x^2 + x - \frac{1}{8}, & \frac{1}{4} \le x \le \frac{1}{2}, \\ \frac{1}{2}x^2 - x + \frac{3}{8}, & \frac{1}{2} \le x \le 1. \end{cases} \tag{19}$$

To solve this problem using Hartley series let

$$u(x) = y(x) - xy(1), \tag{20}$$

thus using (17) we have $u(0) = u(1) = 0$. Now suppose

$$\dot{u}(x) \simeq C^T T(x), \quad \text{where} \quad C^T = [c_{-n}, \dots, c_{-1}, c_0, c_1, \dots, c_n], \tag{21}$$

integration both sides of (21) and using (8), we obtain

$$u(x) \simeq C^T \mathbf{P} T(x) + u(0), \tag{22}$$

we can expanded $g(x)$ in Hartley series as

$$g(x) \simeq \beta^T T(x), \quad \beta_i = \int_0^1 g(x) T_i(x) dx \qquad i = -n, \dots, n. \tag{23}$$

Using (20) and suppose $y(1) = y_1$, we can write (16) as

$$J = \frac{1}{2}\int_0^1 \dot{u}^2 dx + y_1 \int_0^1 \dot{u} dx + \frac{1}{2}y_1^2 - \int_0^1 u g(x) dx + \frac{1}{8}y_1.$$

Employing (21) and (23) we get

$$J \simeq \frac{1}{2}C^T \mathbf{H} C + y_1 C^T \alpha - C^T \mathbf{P}\mathbf{H}\beta + \frac{1}{2}y_1^2 + \frac{1}{8}y_1, \tag{24}$$

where $\alpha = \int_0^1 T(x)dx$. Now using (5) in (24) we have

$$J \simeq \frac{1}{2}C^T C + y_1 C^T \alpha - C^T \mathbf{P}\beta + \frac{1}{2}y_1^2 + \frac{1}{8}y_1. \tag{25}$$

The boundary conditions (17) can be expressed in terms of Hartley series as

$$C^T T(0) + y_1 = 0, \quad C^T T(1) + y_1 = 0. \tag{26}$$

We now minimize (25) subject to (26) using the Lagrange multiplier technique. Suppose

$$J^\star = J + \lambda_1 (C^T T(0) + y_1) + \lambda_2 (C^T T(1) + y_1), \tag{27}$$

where λ_1 and λ_2 are the two multipliers. Differentiating (27) with respect to C and setting the partial derivative equal to zero, we obtain

$$C + y_1 \alpha - \mathbf{P}\beta + \lambda_1 T(0) + \lambda_2 T(1) = 0. \tag{28}$$

Equations (26) and (28) define a set of $(2n + 4)$ linear algebraic equations from which the coefficient vector C and the multipliers λ_1 and λ_2 and value of y_1 can be found. Using this method with $n = 10$ and $n = 12$, the approximate solution is calculated and the absolute errors $|y - y_{exact}|$ are plotted in figures 1 and 2 respectively.

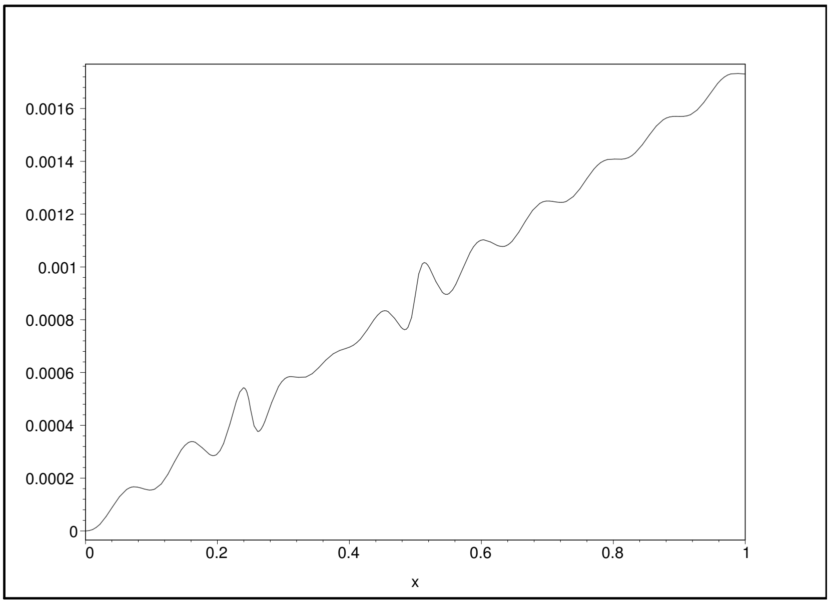

Figure 1: Plot of the absolute error with $n = 10$ for example 2.

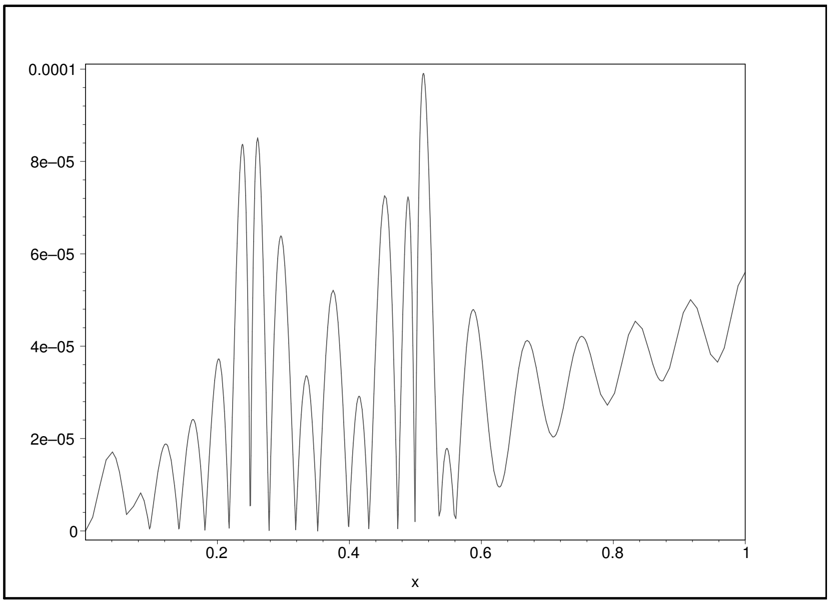

Figure 2: Plot of the absolute error with $n = 12$ for example 2.

4. Conclusion

This paper described an efficient method for solving variational problem. The Hartley series operational matrix $\mathbf{P}$, together with the integration of the product of two *cas* vectors, are used to solve the variational problem. Our approach reduces a variational problem into a set of algebraic equations. The obtained results showed that this approach can solve the problem effectively.

References

[1] R. N. Bracewell, *The Hartley Transform*, Oxford University Press, New York, 1986.

[2] C. Hwang and Y. P. Shih, Laguerre series direct method for variational problems, *J. Optim. Theory Appl.* **39**(1) (1983) 143–149.

[3] H. R. Marzban, H. R. Tabrizidooz and M. Razzaghi, Solution of variational problems via hybrid of block-pulse and Lagrange interpolating, *IET Control Theory Appl.* **3**(10) (2009) 1363–1369.

[4] J. J. R. Melgoza, G. T. Heydt, A. Keyhani, B. L. Agrawal and D. Selin, Synchronous machine parameter estimation using the Hartley series, *IEEE Trans. Energy Conversion* **16**(1) (2001) 49–54.

[5] K. J. Olejniczak and G. T. Heydt, Scanning the special section on the Hartley transform, *Proc. IEEE* **82**(3) (1994) 372–380.

[6] M. Razzaghi and Y. Ordokhani, An application of rationalized Haar functions for variational problems, *Appl. Math. Comput.* **122**(3) (2001) 353–364.

[7] M. Razzaghi and S. Yousefi, Legendre wavelets direct method for variational problems, *Math. Comput. Simulation* **53**(3) (2000) 185–192.

[8] A. Saadatmandi and T. Abdolahi-Niasar, An analytic study on the Euler-Lagrange equation arising in calculus of variations, *Comput. Methods Differ. Equ.* **2**(3) (2014) 140–152.

[9] A. Saadatmandi and M. Dehghan, The numerical solution of problems in calculus of variation using Chebyshev finite difference method, *Phys. Lett. A* **372**(22) (2008) 4037–4040.

[10] A. Saadatmandi, M. Razzaghi and M. Dehghan, Hartley series approximations for the parabolic equations, *Int. J. Comput. Math.* **82**(9) (2005) 1149–1156.

[11] R. S. Schechter, *The Variational Method in Engineering*, Mc Graw-Hill, NewYork, 1967.

[12] M. Tatari and M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, *Phys. Lett. A* **362**(5–6) (2007) 401–406.

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