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Sufficient Conditions for a New Class of Polynomial Analytic Functions of Reciprocal Order α

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Abstract

In this paper, we consider a new class of analytic functions in the unit disk using polynomials of order α. We give some sufficient conditions for functions belonging to this class.

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1. Introduction

Let $\sum$ denotes the class of all functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z : \ z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} - \{0\}.$$

A function $f \in \sum$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf^{'}(z)}{f(z)}\right) < -\alpha, \qquad (0 \le \alpha < 1 \ , \ z \in \mathbb{U}). \tag{2}$$

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We get, $\mathcal{MS}^*(0) = \mathcal{MS}^*$.

Furthermore, a function $f \in \mathcal{MS}^*$ is said to be in the class $\mathcal{NS}^*(\alpha)$ of meromorphic starlike of reciprocal order α if and only if

$$\Re\left(\frac{f(z)}{zf'(z)}\right) < -\alpha, \qquad (0 \le \alpha < 1\ ,\ z \in \mathbb{U}). \tag{3}$$

In recent years, several authors studied meromorphic starlike functions and starlike functions of reciprocal. We can see the earlier work by Ali and Ravichandran [1–3] and Cho et al. [5, 6] and (more recently) by Nunokawa et al. [9, 10], Silverman et al. [11], Srivastava et al. [12], Wang et al. [14–19] and the references therein.

Yong Sun et al. [13] obtained some sufficient conditions for the functions belonging to the class $\mathcal{NS}^*(\alpha)$.

In this paper, we introduce a new class of analytic starlike functions. Also we give some sufficient conditions for functions which belongs to the new class.

2. Preliminaries

Let $\mathcal{P}$ denote the class of functions $P(z)$ given by

$$P(z) = 1 + \sum_{k=1}^{\infty} P_k z^k \qquad (z \in \mathbb{U}), \tag{4}$$

which are analytic in $\mathbb{U}$.

Lemma 2.1. [[7]] *If the function $p \in \mathcal{P}$ is given by* (4) *and satisfied the condition $\Re(p(z)) > 0$, then $|p_k| \le 2$, $k \in \mathbb{N}$.*

Let $0 \le \alpha < 1$, $\mathcal{P}^*(\alpha)$ denotes the class of all functions $p(z) \in \mathcal{P}$ and satisfies the condition

$$\Re\left(\frac{zp'(z)}{p(z)}\right) < 1 - \alpha. \tag{5}$$

We set $\mathcal{P}^*(0) = \mathcal{P}^*$.

Finally, let $0 \le \alpha < 1$. We introduce the notation $\mathcal{NP}^*(\alpha)$ for the class of all mappings $p(z) \in \mathcal{P}^*$ satisfying the following condition:

$$\Re\left(\frac{p(z)}{zp'(z) - p(z)}\right) < -\alpha. \tag{6}$$

Obviously, $p(z) \in \mathcal{NP}^*(\alpha)$ if and only if $f(z) = \frac{1}{z}p(z) \in \mathcal{NS}^*(\alpha)$.

Remark 1. For $0 < \alpha < 1$, the function $p \in \mathcal{P}$ belongs to the class $\mathcal{NP}^*(\alpha)$ if and only if

$$\left|\frac{zp'(z)}{p(z)} + \frac{1 - 2\alpha}{2\alpha}\right| < \frac{1}{2\alpha}. \tag{7}$$

In the following, we give several examples of functions of belonging to the class $\mathcal{NP}^*(\alpha)$.

Example 2.2. Let $p \in \mathcal{P}$ satisfies the inequality

$$\left|\frac{zp^{'}(z)}{p(z)}\right| < 1-\alpha \qquad (0 \leq \alpha < 1\ ,\ z \in \mathbb{U}).$$

Then

$$\left|\frac{zp^{'}(z)}{p(z)} + \frac{\alpha}{2}\right| \leq \left|\frac{zp^{'}(z)}{p(z)}\right| + \frac{\alpha}{2} < 1-\alpha+\frac{\alpha}{2} \leq \frac{\alpha+2}{2},$$

therefore $p \in \mathcal{NP}^*(\frac{1}{\alpha+2})$.

Example 2.3. Let the function $p(z) \in \mathcal{P}$ be in the form

$$p(z) = e^{(1-\alpha)z} \qquad (0 < \alpha < 1\ ,\ z \in \mathbb{U}).$$

This gives us that

$$\Re\left(\frac{zp^{'}(z)}{p(z)}\right) = \Re\left((1-\alpha)z\right) < 1-\alpha.$$

Therefore, $p(z) \in \mathcal{P}^*(\alpha)$. Moreover, we have

$$\frac{p(z)}{zp^{'}(z)-p(z)} = \frac{1}{(1-\alpha)z-1}.$$

It follows that

$$\Re\left(\frac{p(z)}{zp^{'}(z)-p(z)}\right) = \Re\left(\frac{1}{(1-\alpha)e^{i\theta}-1}\right) < -\frac{1}{2-\alpha} \qquad (z = e^{i\theta}).$$

Therefore, $p(z) \in \mathcal{NP}^*(\frac{1}{2-\alpha})$.

In order to obtain our main results, we need the following lemmas.

Lemma 2.4. *(Jack's lemma [8]) Let φ be a non-constant regular function in $\mathbb{U}$. If $|\varphi|$ attains its maximum value on circle $|z| = r < 1$ at z_0, then*

$$z_0\varphi^{'}(z_0) = k\varphi(z_0),$$

where $k \geq 1$ is a real number.

Lemma 2.5. *(See, [4]) Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real numbers x, y such that $y \leq -\frac{1+x^2}{2}$. If $p(z) \in \mathcal{P}$ and $\phi(p(z), zp^{'}(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.*

3. Main Results

We begin this section by presenting the following coefficient sufficient conditions for functions belonging to the class $\mathcal{NP}^*(\alpha)$.

Theorem 3.1. *If $p \in \mathcal{P}$ satisfies*

$$\sum_{k=1}^{\infty}[1+\alpha(k-1)]|p_k| \leq \frac{1}{2}(1-|1-2\alpha|). \tag{8}$$

Then $p \in \mathcal{NP}^*(\alpha)$, for $0<\alpha<1$.

Proof. Using Remark 1 only need to show that

$$\left|\frac{2\alpha z p'(z)}{p(z)}+1-2\alpha\right|<1 \quad (z \in \mathbb{U}). \tag{9}$$

We first observe that

$$\begin{aligned}\left|\frac{2\alpha z p'(z)+(1-2\alpha)p(z)}{p(z)}\right| &= \left|\frac{(1-2\alpha)+\sum\limits_{k=1}^{\infty}[1+2\alpha(k-1)]p_k z^k}{1+\sum\limits_{k=1}^{\infty}p_k z^k}\right| \\ &\leq \frac{|1-2\alpha|+\sum\limits_{k=1}^{\infty}[1+2\alpha(k-1)]|p_k||z|^k}{1-\sum\limits_{k=1}^{\infty}|p_k||z^k} \\ &< \frac{|1-2\alpha|+\sum\limits_{k=1}^{\infty}[1+2\alpha(k-1)]|p_k|}{1-\sum\limits_{k=1}^{\infty}|p_k|}.\end{aligned}$$

Now, by using the inequality (8), we have

$$\frac{|1-2\alpha|+\sum\limits_{k=1}^{\infty}[1+2\alpha(k-1)]|p_k|}{1+\sum\limits_{k=1}^{\infty}|p_k|}<1, \tag{10}$$

which, combined with (9) and (10), completes the proof of theorem. □

Example 3.2. The function $p(z)$ given by

$$p(z)=1+\sum_{k=1}^{\infty}\frac{1-|1-2\alpha|}{k(k+1)[1+\alpha(k-1)]}z^n$$

belongs to the class $\mathcal{NP}^*(\alpha)$, for $0<\alpha<1$.

By using Jack's lemma, we now obtain the following result for the class $\mathcal{NP}^*(\alpha)$.

Theorem 3.3. *If $p \in \mathcal{P}$ satisfies*

$$\left| \frac{z^2 p^{''}(z)}{zp^{'}(z) - p(z)} - \frac{zp^{'}(z)}{p(z)} \right| < 1 - \alpha. \tag{11}$$

Then $p \in \mathcal{NP}^*(\alpha)$, for $\frac{1}{2} \leq \alpha < 1$.

Proof. Let

$$\varphi(z) = \frac{\alpha}{1-\alpha} \frac{zp^{'}(z)}{p(z)} \qquad \left(\frac{1}{2} \leq \alpha < 1; z \in \mathbb{U} \right). \tag{12}$$

Then the function $\varphi(z)$ is analytic in $\mathbb{U}$ with $\varphi(0) = 0$ and it follows from (12) that

$$\frac{z^2 p^{''}(z)}{zp^{'}(z) - p(z)} - \frac{zp^{'}(z)}{p(z)} = \frac{(1-\alpha) z\varphi^{'}(z)}{(1-\alpha)\varphi(z) - \alpha}, \tag{13}$$

therefore

$$\left| \frac{z^2 p^{''}(z)}{zp^{'}(z) - p(z)} - \frac{zp^{'}(z)}{p(z)} \right| = \left| \frac{(1-\alpha) z\varphi^{'}(z)}{(1-\alpha)\varphi(z) - \alpha} \right| < 1 - \alpha.$$

Next, we claim that $|\varphi(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\underset{|z| \leq |z_0|}{max} = |\varphi(z_0)| = 1.$$

Applying Jack's lemma to $\varphi(z)$ at the points z_0, we have

$$\varphi(z_0) = e^{i\theta}, \qquad \frac{z_0 \varphi^{'}(z_0)}{\varphi(z_0)} = k \qquad (k \geq 1).$$

This gives us

$$\left| \frac{z_0^2 p^{''}(z_0)}{z_0 p^{'}(z_0) - p(z_0)} - \frac{z_0 p^{'}(z_0)}{p(z_0)} \right|^2 = \left| \frac{k(1-\alpha)}{(1-\alpha) - \alpha e^{-i\theta}} \right| \geq \left| \frac{1-\alpha}{(1-\alpha) - \alpha e^{-i\theta}} \right|.$$

This implies that

$$\begin{aligned} \left| \frac{z_0^2 p^{''}(z_0)}{z_0 p^{'}(z_0) - p(z_0)} - \frac{z_0 p^{'}(z_0)}{p(z_0)} \right|^2 &\geq \frac{(1-\alpha)^2}{(1-\alpha)^2 + \alpha^2 - 2\alpha(1-\alpha)\cos\theta} \\ &\geq \frac{(1-\alpha)^2}{(1-\alpha)^2 + \alpha^2 + 2\alpha(1-\alpha)} \\ &= (1-\alpha)^2. \end{aligned}$$

This contradicts to the condition (11). Therefore, we conclude that $|\varphi(z)| < 1$ which shows that

$$|\varphi(z)| = \left|\frac{\alpha}{1-\alpha}\ \frac{zp^{'}(z)}{p(z)}\right| < 1$$

or

$$\left|\frac{zp^{'}(z)}{p(z)}\right| < \frac{1-\alpha}{\alpha} \qquad \left(\frac{1}{2} \le \alpha < 1; z \in \mathbb{U}\right).$$

Thun, we have

$$\left|\frac{zp^{'}(z)}{p(z)} + \frac{1-2\alpha}{2\alpha}\right| \le \left|\frac{zp^{'}(z)}{p(z)}\right| + \left|\frac{1-2\alpha}{2\alpha}\right| < \frac{1-\alpha}{\alpha} - \frac{1-2\alpha}{2\alpha} = \frac{1}{2\alpha},$$

which completes the proof. □

Example 3.4. Let us consider the function $p(z) \in \mathcal{P}$ given by

$$p(z) = 1 + p_1 z \qquad (z \in \mathbb{U}),$$

with

$$p_1 = \frac{1-\alpha}{2-\alpha}$$

for some $\frac{1}{2} \le \alpha < 1$, then we see that $0 < p_1 \le \frac{1}{3}$. According to

$$\left|\frac{z^2 p^{''}(z)}{zp^{'}(z) - p(z)} - \frac{zp^{'}(z)}{p(z)}\right| = \left|\frac{-p_1 z}{1 + p_1 z}\right| < \frac{p_1}{1-p_1} = 1-\alpha,$$

and

$$\Re\left(\frac{p(z)}{zp^{'}(z) - p(z)}\right) = \Re(-1 - p_1 z) \le p_1 - 1 = \frac{1}{\alpha - 2} < -\alpha,$$

we have, $p(z) \in \mathcal{NP}^*(\alpha)$ for $\frac{1}{2} \le \alpha < 1$.

Theorem 3.5. *If $p \in \mathcal{P}$ satisfies*

$$\mathbb{R}\left(\frac{z^2 p^{''}(z)}{zp^{'}(z) - p(z)} - \frac{zp^{'}(z)}{p(z)}\right) < \begin{cases} \dfrac{\alpha}{2(1-\alpha)} & (0 \le \alpha \le \frac{1}{2}), \\[2ex] \dfrac{1-\alpha}{2\alpha} & (\frac{1}{2} \le \alpha < 1), \end{cases} \tag{14}$$

then $p \in \mathcal{NP}^*(\alpha)$, for $0 \le \alpha < 1$.

Proof. Suppose that

$$q(z) = \frac{-\frac{p(z)}{zp^{'}(z) - p(z)} - \alpha}{1-\alpha} \qquad (0 \le \alpha < 1, z \in \mathbb{U}), \tag{15}$$

then q is analytic in $\mathbb{U}$. It follows from (15) that

$$\frac{zp'(z)}{p(z)} - \frac{z^2p'(z)}{zp'(z)-p(z)} = \frac{(1-\alpha)zq'(z)}{\alpha+(1-\alpha)q(z)} = \phi(q(z), zq'(z); z),$$

where

$$\phi(r, s; z) = \frac{(1-\alpha)s}{\alpha+(1-\alpha)r}.$$

For all real numbers x and y satisfying $y \leq -\frac{1+x^2}{2}$, we have

$$\begin{aligned}
\Re\left(\phi(ix, y; z)\right) &= \frac{(1-\alpha)\alpha y}{\alpha^2+(1-\alpha)^2x} \\
&\leq -\frac{(1-\alpha)\alpha}{2}\cdot\frac{1+x^2}{\alpha^2+(1-\alpha)^2x^2} \\
&\leq \begin{cases} -\dfrac{(1-\alpha)\alpha}{2}\cdot\dfrac{1}{(1-\alpha)^2} = -\dfrac{\alpha}{2(1-\alpha)} & (0 \leq \alpha \leq \frac{1}{2}), \\[2ex] -\dfrac{(1-\alpha)\alpha}{2}\cdot\dfrac{1}{\alpha^2} = -\dfrac{1-\alpha}{2\alpha} & (\frac{1}{2} \leq \alpha < 1). \end{cases}
\end{aligned}$$

We now put

$$\Omega = \left\{ z : \Re(z) > \begin{cases} -\frac{\alpha}{2(1-\alpha)} & (0 \leq \alpha \leq \frac{1}{2}) \\[1ex] -\frac{1-\alpha}{2\alpha} & (\frac{1}{2} \leq \alpha < 1) \end{cases} \right\},$$

then $\phi(ix, y; z) \notin \Omega$ for all x, y such that $y \leq -\frac{1+x^2}{2}$. Moreover, in view of (14), we get $\phi(q(z), zq'(z); z)$. Thus, by Lemma 2.5, we deduce that

$$\Re(q(z)) > 0 \quad (z \in \mathbb{U}),$$

which implies that $p \in \mathcal{NP}^*(\alpha)$. $\square$

Theorem 3.6. *If $p \in \mathcal{P}$ satisfies*

$$\Re\left(\frac{p(z)}{zp'(z)-p(z)}\left(1+\beta\,\frac{z^2p''(z)}{zp'(z)-p(z)}\right)\right) < \frac{1}{2}\beta(\alpha+3)-\alpha, \tag{16}$$

then $p \in \mathcal{NP}^*(\alpha)$, for $0 \leq \alpha < 1$ and $\beta \geq 0$.

Proof. Suppose that

$$q(z) = \frac{-\frac{p(z)}{zp'(z)-p(z)}-\alpha}{1-\alpha} \qquad (0 \leq \alpha < 1; z \in \mathbb{U}). \tag{17}$$

Then ϕ is analytic in $\mathbb{U}$. It follows from (17) that

$$1+\beta\,\frac{z^2p^{''}(z)}{zp^{'}(z)-p(z)}=\frac{\beta[(1-\alpha)zq^{'}(z)-1]}{(1-\alpha)q(z)+\alpha}+1-\beta. \tag{18}$$

Combining with (17) and (18), we get

$$\begin{aligned}\frac{p(z)}{zp^{'}(z)-p(z)}\left(1+\beta\,\frac{z^2p^{''}(z)}{p^{'}(z)-p(z)}\right)&=\beta(1-\alpha)zp^{'}(z)+(1-\beta)(1-\alpha)p(z)\\&+(1-\beta)\alpha-\beta\\&=\phi(q(z),zq^{'}(z);z),\end{aligned}$$

where

$$\phi(r,s;z)=\beta(1-\alpha)s+(1-\beta)(1-\alpha)r+(1-\beta)\alpha-\beta.$$

For all real numbers x and y satisfying $y\le-\frac{1+x^2}{2}$, we have

$$\begin{aligned}\Re\left(\phi(ix,y;z)\right)&=\beta(1-\alpha)y+(1-\beta)\alpha-\beta\\&\le-\frac{\beta(1-\alpha)}{2}(1+x^2)+(1-\beta)\alpha-\beta\\&\le-\frac{\beta(1-\alpha)}{2}+(1-\beta)\alpha-\beta\\&=\alpha-\frac{1}{2}\beta(\alpha+3)\quad(0\le\alpha<1).\end{aligned}$$

If we set

$$\Omega=\{z:\Re(z)>\alpha-\frac{1}{2}\beta(\alpha+3)\},$$

then, by Lemma 2.5, we conclude that

$$\Re(q(z))>0\quad(z\in\mathbb{U}),$$

which implies the assertion of theorem holds. $\square$

Theorem 3.7. *If $p\in\mathcal{P}$ satisfies*

$$\left|\left(1-2\alpha+\frac{2\alpha zp^{'}(z)}{p(z)}\right)^{'}\right|\le\beta|z|^{\gamma}, \tag{19}$$

then $p\in\mathcal{NP}^*(\alpha)$, for $0<\alpha<1$, $0<\beta\le\gamma+1$ and $\gamma\ge0$.

Proof. For $p\in\mathcal{P}$, we set

$$q(z)=z\left(1+2\alpha+\frac{2\alpha zp^{'}(z)}{p(z)}\right)\qquad(z\in\mathbb{U}),$$

then $q(z)$ is regular in $\mathbb{U}$ and $q(0) = 0$.
The condition of the theorem gives us

$$\left|\left(1 - 2\alpha + \frac{2\alpha z p'(z)}{p(z)}\right)'\right| = \left|\left(\frac{q(z)}{z}\right)'\right| \leq \beta|z|^{\gamma} \qquad (z \in \mathbb{U}).$$

It follow that

$$\left|\left(\frac{q(z)}{z}\right)'\right| = \left|\int_0^z \left(\frac{q(u)}{u}\right)' du\right| \leq \int_0^{|z|} \beta|u|^{\gamma} d|u| = \frac{\beta}{\gamma+1}|z|^{\gamma+1}.$$

This implies that

$$\left|\left(\frac{q(z)}{z}\right)'\right| \leq \frac{\beta}{\gamma+1}|z|^{\gamma+1} < 1 \qquad (1 < \beta < \gamma+1, \gamma \geq 0).$$

Therefore, by definition of $q(z)$, we have

$$\left|1 - 2\alpha + \frac{2\alpha z p'(z)}{p(z)}\right| < 1$$

or

$$\left|\frac{zp'(z)}{p(z)} + \frac{1-2\alpha}{2\alpha}\right| < \frac{1}{2\alpha}.$$

This implies that $p \in \mathcal{NP}^*(\alpha)$. □

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