Sufficient Conditions for a New Class of Polynomial Analytic Functions of Reciprocal Order $\alpha$

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Abstract

In this paper, we consider a new class of analytic functions in the unit disk using polynomials of order $\alpha$. We give some sufficient conditions for functions belonging to this class.

Keywords: Polynomial analytic functions, starlike functions, meromorphic functions.

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1. Introduction

Let $\Sigma$ denotes the class of all functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the punctured open unit disk

$$U^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} = \mathbb{U} - \{0\}.$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphic starlike functions of order $\alpha$ if it satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < -\alpha, \quad (0 \leq \alpha < 1, \ z \in U).$$

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We get, $\mathcal{MS}^*(0) = \mathcal{MS}^*$.

Furthermore, a function $f \in \mathcal{MS}^*$ is said to be in the class $\mathcal{NS}^*(\alpha)$ of meromorphic starlike of reciprocal order $\alpha$ if and only if
\[
\Re\left(\frac{f(z)}{zf'(z)}\right) < -\alpha, \quad (0 \leq \alpha < 1, \ z \in U).
\] (3)

In recent years, several authors studied meromorphic starlike functions and starlike functions of reciprocal. We can see the earlier work by Ali and Ravichandran [1–3] and Cho et al. [5, 6] and (more recently) by Nunokawa et al. [9, 10], Silverman et al. [11], Srivastava et al. [12], Wang et al. [14–19] and the references therein.

Yong Sun et al. [13] obtained some sufficient conditions for the functions belonging to the class $\mathcal{NS}^*(\alpha)$.

In this paper, we introduce a new class of analytic starlike functions. Also we give some sufficient conditions for functions which belongs to the new class.

2. Preliminaries

Let $\mathcal{P}$ denote the class of functions $P(z)$ given by
\[
P(z) = 1 + \sum_{k=1}^{\infty} P_k z^k \quad (z \in U),
\] (4)

which are analytic in U.

Lemma 2.1. [7] If the function $p \in \mathcal{P}$ is given by (4) and satisfied the condition $\Re(p(z)) > 0$, then $|p_k| \leq 2$, $k \in \mathbb{N}$.

Let $0 \leq \alpha < 1$, $\mathcal{P}^*(\alpha)$ denotes the class of all functions $p(z) \in \mathcal{P}$ and satisfies the condition
\[
\Re\left(\frac{zp'(z)}{p(z)}\right) < 1 - \alpha.
\] (5)

We set $\mathcal{P}^*(0) = \mathcal{P}^*$.

Finally, let $0 \leq \alpha < 1$. We introduce the notation $\mathcal{NP}^*(\alpha)$ for the class of all mappings $p(z) \in \mathcal{P}^*$ satisfying the following condition:
\[
\Re\left(\frac{p(z)}{zp'(z) - p(z)}\right) < -\alpha.
\] (6)

Obviously, $p(z) \in \mathcal{NP}^*(\alpha)$ if and only if $f(z) = \frac{1}{z}p(z) \in \mathcal{NS}^*(\alpha)$.

Remark 1. For $0 < \alpha < 1$, the function $p \in \mathcal{P}$ belongs to the class $\mathcal{NP}^*(\alpha)$ if and only if
\[
\left|\frac{zp'(z)}{p(z)} + \frac{1 - 2\alpha}{2\alpha}\right| < \frac{1}{2\alpha}.
\] (7)
In the following, we give several examples of functions of belonging to the class $NP^*(\alpha)$.

**Example 2.2.** Let $p \in P$ satisfies the inequality
\[
\left| \frac{zp'(z)}{p(z)} \right| < 1 - \alpha \quad (0 \leq \alpha < 1, \ z \in \mathbb{U}).
\]
Then
\[
\left| \frac{zp'(z) + \alpha}{p(z)} \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \frac{\alpha}{2} < 1 - \alpha + \frac{\alpha}{2} \leq \frac{\alpha + 2}{2},
\]
therefore $p \in NP^*(\frac{\alpha + 2}{2})$.

**Example 2.3.** Let the function $p(z) \in P$ be in the form
\[
p(z) = e^{(1-\alpha)z} \quad (0 < \alpha < 1, \ z \in \mathbb{U}).
\]
This gives us that
\[
\Re \left( \frac{zp'(z)}{p(z)} \right) = \Re ((1 - \alpha)z) < 1 - \alpha.
\]
Therefore, $p(z) \in P^*(\alpha)$. Moreover, we have
\[
P(z) = \frac{zp'(z) - p(z)}{(1-\alpha)z - 1}.
\]
It follows that
\[
\Re \left( \frac{p(z)}{zp'(z) - p(z)} \right) = \Re \left( \frac{1}{(1-\alpha)e^{i\theta} - 1} \right) < -\frac{1}{2-\alpha} \quad (z = e^{i\theta}).
\]
Therefore, $p(z) \in NP^*(\frac{1}{2-\alpha})$.

In order to obtain our main results, we need the following lemmas.

**Lemma 2.4.** (Jack’s lemma [8]) Let $\varphi$ be a non-constant regular function in $\mathbb{U}$. If $|\varphi|$ attains its maximum value on circle $|z| = r < 1$ at $z_0$, then
\[
z_0 \varphi'(z_0) = k \varphi(z_0),
\]
where $k \geq 1$ is a real number.

**Lemma 2.5.** (See, [4]) Let $\Omega$ be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi$ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real numbers $x, y$ such that $y \leq -\frac{1+x^2}{2}$. If $p(z) \in P$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$. 

3. Main Results

We begin this section by presenting the following coefficient sufficient conditions for functions belonging to the class $NP^*(\alpha)$.

**Theorem 3.1.** If $p \in P$ satisfies

$$\sum_{k=1}^{\infty} |1 + \alpha(k-1)||p_k| \leq \frac{1}{2} (1 - |1 - 2\alpha|).$$

(8)

Then $p \in NP^*(\alpha)$, for $0 < \alpha < 1$.

**Proof.** Using Remark 1 only need to show that

$$\left| \frac{2\alpha z p'(z)}{p(z)} + 1 - 2\alpha \right| < 1 \quad (z \in U).$$

(9)

We first observe that

$$\left| \frac{2\alpha z p'(z)}{p(z)} + 1 - 2\alpha \right| = \frac{|1 - 2\alpha| + \sum_{k=1}^{\infty} [1 + 2\alpha(k-1)]|p_k|z^k}{1 + \sum_{k=1}^{\infty} p_k z^k}.$$

$$\leq \frac{|1 - 2\alpha| + \sum_{k=1}^{\infty} [1 + 2\alpha(k-1)]|p_k||z|^k}{1 - \sum_{k=1}^{\infty} |p_k||z|^k}.$$

$$< \frac{|1 - 2\alpha| + \sum_{k=1}^{\infty} [1 + 2\alpha(k-1)]|p_k|}{1 - \sum_{k=1}^{\infty} |p_k|}.$$

Now, by using the inequality (8), we have

$$\frac{|1 - 2\alpha| + \sum_{k=1}^{\infty} [1 + 2\alpha(k-1)]|p_k|}{1 + \sum_{k=1}^{\infty} |p_k|} < 1,$$

(10)

which, combined with (9) and (10), completes the proof of theorem. \qed

**Example 3.2.** The function $p(z)$ given by

$$p(z) = 1 + \sum_{k=1}^{\infty} \frac{1 - |1 - 2\alpha|}{k(k + 1)[1 + \alpha(k-1)]} z^n$$

belongs to the class $NP^*(\alpha)$, for $0 < \alpha < 1$. 
By using Jack’s lemma, we now obtain the following result for the class \( NP^*(\alpha) \).

**Theorem 3.3.** If \( p \in \mathcal{P} \) satisfies

\[
\left| \frac{z^2 p''(z)}{zp'(z) - p(z)} - \frac{zp'(z)}{p(z)} \right| < 1 - \alpha. \tag{11}
\]

Then \( p \in NP^*(\alpha) \), for \( \frac{1}{2} \leq \alpha < 1 \).

**Proof.** Let

\[
\varphi(z) = \frac{\alpha}{1 - \alpha} \frac{zp'(z)}{p(z)} \quad \left( \frac{1}{2} \leq \alpha < 1; z \in U \right). \tag{12}
\]

Then the function \( \varphi(z) \) is analytic in \( U \) with \( \varphi(0) = 0 \) and it follows from (12) that

\[
\frac{z^2 p''(z)}{zp'(z) - p(z)} - \frac{zp'(z)}{p(z)} = \frac{(1 - \alpha)z \varphi'(z)}{(1 - \alpha) \varphi(z) - \alpha}, \tag{13}
\]

therefore

\[
\left| \frac{z^2 p''(z)}{zp'(z) - p(z)} - \frac{zp'(z)}{p(z)} \right| = \left| \frac{(1 - \alpha)z \varphi'(z)}{(1 - \alpha) \varphi(z) - \alpha} \right| < 1 - \alpha.
\]

Next, we claim that \( |\varphi(z)| < 1 \). Indeed, if not, there exists a point \( z_0 \in U \) such that

\[
\max_{|z| \leq |z_0|} |\varphi(z)| = 1.
\]

Applying Jack’s lemma to \( \varphi(z) \) at the points \( z_0 \), we have

\[
\varphi(z_0) = e^{i\theta}, \quad \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = k \quad (k \geq 1).
\]

This gives us

\[
\left| \frac{z_0^2 p''(z_0)}{z_0 p'(z_0) - p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} \right|^2 = \left| \frac{k(1 - \alpha)}{(1 - \alpha) - \alpha e^{-i\theta}} \right|^2 \geq \frac{1 - \alpha}{(1 - \alpha) - \alpha e^{-i\theta}}.
\]

This implies that

\[
\left| \frac{z_0^2 p''(z_0)}{z_0 p'(z_0) - p(z_0)} - \frac{z_0 p'(z_0)}{p(z_0)} \right|^2 \geq \frac{(1 - \alpha)^2}{(1 - \alpha)^2 + \alpha^2 - 2\alpha(1 - \alpha) \cos \theta} \geq \frac{(1 - \alpha)^2}{(1 - \alpha)^2 + \alpha^2 + 2\alpha(1 - \alpha)} = (1 - \alpha)^2.
\]
This contradicts to the condition (11). Therefore, we conclude that $|\varphi(z)| < 1$ which shows that

$$|\varphi(z)| = \left| \frac{\alpha}{1 - \alpha} \frac{zp'(z)}{p(z)} \right| < 1$$

or

$$\left| \frac{zp'(z)}{p(z)} \right| < \frac{1 - \alpha}{\alpha}, \quad \left( \frac{1}{2} \leq \alpha < 1; z \in U \right).$$

Thus, we have

$$\left| \frac{zp'(z)}{p(z)} + \frac{1 - 2\alpha}{2\alpha} \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{1 - 2\alpha}{2\alpha} \right| < \frac{1 - \alpha}{\alpha} - \frac{1 - 2\alpha}{2\alpha} = \frac{1}{2\alpha},$$

which completes the proof. \qed

**Example 3.4.** Let us consider the function $p(z) \in \mathcal{P}$ given by

$$p(z) = 1 + p_1 z \quad (z \in U),$$

with

$$p_1 = \frac{1 - \alpha}{2 - \alpha},$$

for some $\frac{1}{2} \leq \alpha < 1$, then we see that $0 < p_1 \leq \frac{1}{2}$. According to

$$\left| \frac{z^2 p''(z)}{zp'(z) - p(z)} \frac{zp'(z)}{p(z)} - \frac{zp'(z)}{p(z)} \right| = \left| \frac{-p_1 z}{1 + p_1 z} \right| < \frac{p_1}{1 - p_1} = 1 - \alpha,$$

and

$$\Re \left( \frac{p(z)}{zp'(z) - p(z)} \right) = \Re(-1 - p_1 z) \leq p_1 - 1 = \frac{1}{\alpha - 2} < -\alpha,$$

we have, $p(z) \in \mathcal{NP}^*(\alpha)$ for $\frac{1}{2} \leq \alpha < 1$.

**Theorem 3.5.** If $p \in \mathcal{P}$ satisfies

$$\Re \left( \frac{z^2 p''(z)}{zp'(z) - p(z)} \right) \leq \begin{cases} \frac{\alpha}{2(1 - \alpha)} & (0 \leq \alpha \leq \frac{1}{2}), \\ \frac{1 - \alpha}{2\alpha} & (\frac{1}{2} \leq \alpha < 1), \end{cases}$$

(14)

then $p \in \mathcal{NP}^*(\alpha)$, for $0 \leq \alpha < 1$.

**Proof.** Suppose that

$$q(z) = \frac{-\frac{p(z)}{zp'(z) - p(z)} - \alpha}{1 - \alpha} \quad (0 \leq \alpha < 1, z \in U),$$

(15)
then \( q \) is analytic in \( U \). It follows from (15) that
\[
\frac{zp'(z)}{p(z)} - \frac{z^2p'(z)}{zp(z) - p(z)} = \frac{(1 - \alpha)zq'(z)}{\alpha + (1 - \alpha)q(z)} = \phi(q(z), zq'(z); z),
\]
where
\[
\phi(r, s; z) = \frac{(1 - \alpha)s}{\alpha + (1 - \alpha)r}.
\]
For all real numbers \( x \) and \( y \) satisfying \( y \leq -\frac{1 + x^2}{2} \), we have
\[
\Re(\phi(ix, y; z)) = \frac{(1 - \alpha)\alpha y}{\alpha^2 + (1 - \alpha)^2x^2} \leq \frac{(1 - \alpha)\alpha}{2} \cdot \frac{1 + x^2}{\alpha^2 + (1 - \alpha)^2x^2} \leq \begin{cases} 
-\frac{(1 - \alpha)\alpha}{2} \cdot \frac{1}{\alpha^2} = -\frac{\alpha}{2(1 - \alpha)} & (0 \leq \alpha \leq \frac{1}{2}) \\
-\frac{(1 - \alpha)\alpha}{2} \cdot \frac{1}{\alpha^2} = -\frac{1 - \alpha}{2\alpha} & \left(\frac{1}{2} \leq \alpha < 1\right)
\end{cases}
\]
We now put
\[
\Omega = \left\{ z : \Re(z) > \begin{cases} 
-\frac{\alpha}{2(1 - \alpha)} & (0 \leq \alpha \leq \frac{1}{2}) \\
-\frac{1 - \alpha}{2\alpha} & \left(\frac{1}{2} \leq \alpha < 1\right)
\end{cases} \right\},
\]
then \( \phi(ix, y; z) \notin \Omega \) for all \( x, y \) such that \( y \leq -\frac{1 + x^2}{2} \). Moreover, in view of (14), we get \( \phi(q(z), zq'(z); z) \). Thus, by Lemma 2.5, we deduce that
\[
\Re(q(z)) > 0 \quad (z \in U),
\]
which implies that \( p \in \mathcal{N}\mathcal{P}^*(\alpha) \). \( \square \)

**Theorem 3.6.** If \( p \in \mathcal{P} \) satisfies
\[
\Re\left( \frac{p(z)}{zp(z) - p(z)} \left( 1 + \beta \frac{z^2p''(z)}{zp(z) - p(z)} \right) \right) < \frac{1}{2} \beta (\alpha + 3) - \alpha,
\]
then \( p \in \mathcal{N}\mathcal{P}^*(\alpha) \), for \( 0 \leq \alpha < 1 \) and \( \beta \geq 0 \).

**Proof.** Suppose that
\[
q(z) = \frac{zp(z) - p(z)}{1 - \alpha} - \alpha \quad (0 \leq \alpha < 1; z \in U).
\]
Then $\phi$ is analytic in $U$. It follows from (17) that
\[
1 + \beta \frac{z^2 p''(z)}{zp'(z) - p(z)} = \frac{\beta[(1 - \alpha)zq(z) - 1]}{(1 - \alpha)q(z) + \alpha} + 1 - \beta.
\]
(18)
Combining with (17) and (18), we get
\[
\frac{p(z)}{zp'(z) - p(z)} \left(1 + \beta \frac{z^2 p''(z)}{p'(z) - p(z)}\right) = \beta(1 - \alpha)zp'(z) + (1 - \beta)(1 - \alpha)p(z) + (1 - \beta)\alpha - \beta
\]
\[
= \phi(q(z), zq'(z); z),
\]
where
\[
\phi(r, s; z) = \beta(1 - \alpha)s + (1 - \beta)(1 - \alpha)r + (1 - \beta)\alpha - \beta.
\]
For all real numbers $x$ and $y$ satisfying $y \leq -1 + x^2$, we have
\[
\Re(\phi(ix, y; z)) = \beta(1 - \alpha)y + (1 - \beta)\alpha - \beta
\]
\[
\leq -\frac{\beta(1 - \alpha)}{2}(1 + x^2) + (1 - \beta)\alpha - \beta
\]
\[
\leq -\frac{\beta(1 - \alpha)}{2} + (1 - \beta)\alpha - \beta
\]
\[
= \alpha - \frac{1}{2}\beta(\alpha + 3) \quad (0 \leq \alpha < 1).
\]
If we set
\[
\Omega = \{z : \Re(z) > \alpha - \frac{1}{2}\beta(\alpha + 3)\},
\]
then, by Lemma 2.5, we conclude that
\[
\Re(q(z)) > 0 \quad (z \in U),
\]
which implies the assertion of theorem holds.

**Theorem 3.7.** If $p \in \mathcal{P}$ satisfies
\[
\left|\left(1 - 2\alpha + \frac{2\alpha zp'(z)}{p(z)}\right)\right| \leq \beta|z|^{\gamma},
\]
(19)
then $p \in \mathcal{NP}^*(\alpha)$, for $0 < \alpha < 1$, $0 < \beta \leq \gamma + 1$ and $\gamma \geq 0$.

**Proof.** For $p \in \mathcal{P}$, we set
\[
q(z) = z \left(1 + 2\alpha + \frac{2\alpha zp'(z)}{p(z)}\right) \quad (z \in U),
\]
then $q(z)$ is regular in $U$ and $q(0) = 0$.

The condition of the theorem gives us

$$\left| 1 - 2\alpha + \frac{2\alpha z p'(z)}{p(z)} \right| = \left| \left( \frac{q(z)}{z} \right)' \right| \leq \beta |z|^\gamma \quad (z \in U).$$

It follow that

$$\left| \left( \frac{q(z)}{z} \right)' \right| = \left| \int_0^z \left( \frac{q(u)}{u} \right)' \, du \right| \leq \int_0^{|z|} \beta |u|\gamma \, |d|u| = \frac{\beta}{\gamma + 1} |z|^{\gamma + 1}.$$ 

This implies that

$$\left| \left( \frac{q(z)}{z} \right)' \right| \leq \frac{\beta}{\gamma + 1} |z|^{\gamma + 1} < 1 \quad (1 < \beta < \gamma + 1, \gamma \geq 0).$$

Therefore, by definition of $q(z)$, we have

$$\left| 1 - 2\alpha + \frac{2\alpha z p'(z)}{p(z)} \right| < 1$$

or

$$\left| \frac{z p'(z)}{p(z)} + \frac{1 - 2\alpha}{2\alpha} \right| < \frac{1}{2\alpha}. $$

This implies that $p \in N^P_\alpha^*$. 

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