Ordered $S$-Metric Spaces and Coupled Common Fixed Point Theorems of Integral Type Contraction

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Abstract

In the present paper, we introduce the notion of integral type contractive mapping with respect to ordered $S$-metric space and prove some coupled common fixed point results of integral type contractive mapping in ordered $S$-metric space. Moreover, we give an example to support our main result.

Keywords: $S$-metric, ordered $S$-metric space, common fixed point, coupled fixed point, integral type contractive mapping, partial order, mixed $g$-monotone property, commuting maps.

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1. Introduction

Banach contraction principle [4], is one of the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partial order have been studied by many authors (see [1,3,5,8,12,13,17,19]). The study of metric spaces has attracted, and continued to attract the interest of many authors. There are many generalizations of metric spaces, such as 2-metric spaces [11], $G$-metric spaces [20], $D^*$-metric spaces [24], partial metric spaces [6], cone metric spaces [15], $S$-metric spaces [22], $b$-metric spaces [9] and $G_b$-metric spaces [2]. In 2012, Sedghi et al. [22] introduced the notion of $S$-metric space.

First we recall some notions, results and examples which will be useful later.
Definition 1.1. [22] Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$:

(S1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$;
(S2) $S(x, y, z) = 0$ if $x = y = z$;
(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair $(X, S)$ is called an $S$-metric space.

Example 1.2. [22] Let $X = \mathbb{R}^2$ and $d$ be an ordinary metric on $X$. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}^2$, that is, $S$ is the perimeter of the triangle given by $x, y, z$. Then $S$ is an $S$-metric on $X$.

Lemma 1.3. [21] In an $S$-metric space, we have $S(x, x, y) = S(y, y, x)$.

Definition 1.4. [23] Let $(X, S)$ be an $S$-metric space and $A \subseteq X$.

(1) If for every $x \in X$ there exists $r > 0$ such that $B_s(x, r) \subseteq A$, then the subset $A$ is called open subset of $X$.

(2) Subset $A$ of $X$ is said to be $S$-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote by $\lim_{n \to \infty} x_n = x$.

(4) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.

(5) The $S$-metric space $(X, S)$ is said to be complete if every Cauchy sequence is convergent.

(6) Let $\tau$ be the set of all $A \subseteq X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_s(x, r) \subseteq A$. Then $\tau$ is a topology on $X$.

Lemma 1.5. [23] Let $(X, S)$ be an $S$-metric space. If there exist sequences $\{x_n\}, \{y_n\}$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then $\lim_{n \to \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.6. [10] Let $(X, S)$ be an $S$-metric space. Then
\[ S(x, x, z) \leq 2S(x, y, z) + S(y, y, z), \]
and
\[ S(x, x, z) \leq 2S(x, y, z) + S(z, z, y), \]
for all \( x, y, z \in X \).

**Definition 1.7.** [14] Let \( (X, \preceq) \) be partially ordered set. Then \( a, b \in X \) are called comparable if \( a \preceq b \) or \( b \preceq a \) holds.

**Definition 1.8.** Let \( X \) be a nonempty set. Then \( (X, S, \preceq) \) is called an ordered \( S \)-metric space if:

1. \( (X, S) \) is an \( S \)-metric space,
2. \( (X, \preceq) \) is a partially ordered set.

**Definition 1.9.** \((X, S, \preceq)\) is said to be regular if it has the following properties:

1. if for a non-decreasing sequence \( \{x_n\} \), \( x_n \rightarrow^S x \) as \( n \rightarrow \infty \), then \( x_n \preceq x \) for all \( n \);
2. if for a non-increasing sequence \( \{x_n\} \), \( x_n \rightarrow^S x \), as \( n \rightarrow \infty \), then \( x_n \succeq x \) for all \( n \).

**Definition 1.10.** [5] Let \( (X, \preceq) \) be partially ordered set and \( H : X \times X \rightarrow X \). The mapping \( H \) is said to has the mixed monotone property if \( H \) is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, i.e., for any \( a, b \in X \),

\[
\begin{align*}
a_1, a_2 &\in X, a_1 \preceq a_2 \Rightarrow H(a_1, b) \preceq H(a_2, b), \\
b_1, b_2 &\in X, b_1 \preceq b_2 \Rightarrow H(a, b_1) \succeq H(a, b_2).
\end{align*}
\]

**Definition 1.11.** [7] Let \( (X, \preceq) \) be partially ordered set and suppose \( H : X \times X \rightarrow X \) and \( g : X \rightarrow X \). The mapping \( H \) is said to has the mixed \( g \)-monotone property if \( H \) is monotone \( g \)-nondecreasing in its first argument and is monotone \( g \)-nonincreasing in its second argument, i.e., for any \( a, b \in X \),

\[
\begin{align*}
a_1, a_2 &\in X, g(a_1) \preceq g(a_2) \Rightarrow H(a_1, b) \preceq H(a_2, b), \\
b_1, b_2 &\in X, g(b_1) \preceq g(b_2) \Rightarrow H(a, b_1) \succeq H(a, b_2).
\end{align*}
\]

**Definition 1.12.** [5] An element \((a, b)\) \in \( X \times X \) is called a coupled coincidence point of the mappings \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) if \( F(a, b) = ga \) and \( F(b, a) = gb \), and their common coupled fixed point if \( F(a, b) = ga = a \) and \( F(b, a) = gb = b \).

**Definition 1.13.** [17] Let \( X \) be a nonempty set. Then we say that the mappings \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) are commutative if \( gF(a, b) = F(ga, gb) \).

**Definition 1.14.** [17] An element \((a, b)\) \in \( X \times X \) is called a coupled fixed point of mapping \( F : X \times X \rightarrow X \) if \( F(a, b) = a \) and \( F(b, a) = b \).

**Definition 1.15.** Let \( (X, S) \) and \( (X', S') \) be two \( S \)-metric spaces, and let \( f : (X, S) \rightarrow (X', S') \) be a function. Then \( f \) is said to be continuous at a point \( a \in X \) if and only if for every sequence \( x_n \) in \( X \), \( S(x_n, x_n, a) \rightarrow 0 \) implies \( S'(f(x_n), f(x_n), f(a)) \rightarrow 0 \). A function \( f \) is continuous at \( X \) if and only if it is continuous at all \( a \in X \).
Definition 1.16. [16] Let $X, Y \subset (-\infty, +\infty)$. The function $\varphi : X \to Y$ is called sub-additive integrable function if and only if for all $c, d \in X$,
\[\int_0^{c+d} \varphi(t) dt \leq \int_0^c \varphi(t) dt + \int_0^d \varphi(t) dt.\]

Example 1.17. [16] Let $X = (0, +\infty)$, $d(x, y) = |x - y|$, and $\varphi(t) = \frac{1}{t+1}$ for all $t > 0$. Then for all $c, d \in X$,
\[\int_0^{c+d} \frac{dt}{t+1} = \ln(c + d + 1), \quad \int_0^c \frac{dt}{t+1} = \ln(c + 1), \quad \int_0^d \frac{dt}{t+1} = \ln(d + 1),\]

since $cd \geq 0$, then $c + d + 1 \leq c + d + 1 + cd = (c + 1)(d + 1)$. Therefore,
\[\ln(c + d + 1) \leq \ln((c + 1)(d + 1)) = \ln(c + 1) + \ln(d + 1).\]

So, we show that $\varphi$ is sub-additive integrable function.

Example 1.18. Let $X = (1, +\infty)$, and $\varphi(t) = e^t$. Then the function $\varphi$ is not sub-additive integrable function.

Lemma 1.19. [18] Let $\{r_n\}_{n \in \mathbb{N}}$ be a non-negative sequence such that $\lim_{n \to \infty} r_n = a$. Then
\[\lim_{n \to \infty} \int_0^{r_n} \varphi(t) dt = \int_0^a \varphi(t) dt,\]
where $\varphi : [0, +\infty) \to [0, +\infty]$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^a \varphi(t) dt > 0$ for each $\varepsilon > 0$.

Lemma 1.20. [18] Let $\{r_n\}_{n \in \mathbb{N}}$ be a non-negative sequence. Then
\[\lim_{n \to \infty} \int_0^{r_n} \varphi(t) dt = 0,\]
if and only if $\lim_{n \to \infty} r_n = 0$, where $\varphi : [0, +\infty) \to [0, +\infty]$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^a \varphi(t) dt > 0$ for each $\varepsilon > 0$.

2. Results

Theorem 2.1. Let $(X, S, \preceq)$ be an ordered $S$-metric space. Let $H : X \times X \to X$ and $g : X \to X$ be mappings such that $H$ has the mixed $g$-monotone property on $X$ and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:
\[\int_0^{S(H(a,b), H(p,q), H(c,r))} \varphi(t) dt \leq k \int_0^{S(ga,gp,gc)+S(gh,gq,gr)} \varphi(t) dt,\]
for \(a, b, c, p, q, r \in X\) with \(ga \succeq gp \succeq gc\) and \(gb \preceq gq \preceq gr\) or \(ga \preceq gp \preceq gc\) and \(gb \succeq gq \succeq gr\), where \(\varphi : [0, \infty) \to [0, \infty)\) is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each \(\varepsilon > 0\), \(\int_0^\varepsilon \varphi(t)dt > 0\). Assume the following conditions:

(a) \(H(X \times X) \subset g(X)\),

(b) \(g(X)\) is complete,

(c) \(g\) is continuous and commutes with \(H\),

(d) \((X, S, \preceq)\) is regular.

Then \(H\) and \(g\) have a coupled coincidence point \((a, b)\). If \(ga \succeq gb\) or \(ga \preceq gb\), then \(g(a) = H(a, a) = a\).

**Proof.** Let \(a_0, b_0\) be two points such that \(g(a_0) \preceq H(a_0, b_0)\) and \(g(b_0) \succeq H(b_0, a_0)\).

As \(H(X \times X) \subset g(X)\), we may choose \(a_1, b_1\) in a way that \(g(a_1) = H(a_0, b_0)\) and \(g(b_1) = H(b_0, a_0)\).

Again since \(H(X \times X) \subset g(X)\), we may choose \(a_2, b_2 \in X\) such that \(g(a_2) = H(a_1, b_1)\) and \(g(b_2) = H(b_1, a_1)\). Repeating this process, we can build two sequences \(\{a_n\}\) and \(\{b_n\}\) in \(X\) such that,

\[
g(a_{n+1}) = H(a_n, b_n) \quad \text{and} \quad g(b_{n+1}) = H(b_n, a_n), \quad \text{for all } n \geq 0. \tag{2}
\]

Now, we claim that for all \(n \geq 0\),

\[
g(a_n) \preceq g(a_{n+1}), \tag{3}
\]

and

\[
g(b_n) \succeq g(b_{n+1}). \tag{4}
\]

Now we will use mathematical induction. Suppose that \(n = 0\). Since \(g(a_0) \preceq H(a_0, b_0)\) and \(g(b_0) \succeq H(b_0, a_0)\), we see that \(g(a_1) = H(a_0, b_0)\) and \(g(b_1) = H(b_0, a_0)\), so \(g(a_0) \preceq g(a_1)\) and \(g(b_0) \succeq g(b_1)\), i.e., (3) and (4) hold for \(n = 0\).

We now suppose that (3) and (4) are valid for some \(n > 0\). As we know that \(H\) has mixed \(g\)-monotone property and also \(g(a_n) \preceq g(a_{n+1})\), \(g(b_n) \succeq g(b_{n+1})\), then from (2), we have

\[
g(a_{n+1}) = H(a_n, b_n) \preceq H(a_{n+1}, b_n)
\]

and

\[
H(b_{n+1}, a_n) \preceq H(b_{n+1}, a_n) = g(b_{n+1}).
\]

Also we have,

\[
g(a_{n+2}) = H(a_{n+1}, b_{n+1}) \succeq H(a_{n+1}, b_n)
\]
and

\[ H(b_{n+1}, a_n) \geq H(b_{n+1}, a_{n+1}) = g(b_{n+2}). \]

Then from (2) and (3), we get

\[ g(a_{n+1}) \preceq g(a_{n+2}) \text{ and } g(b_{n+1}) \succeq g(b_{n+2}). \]

We conclude by mathematical induction that (3) and (4) hold for all \( n \geq 0 \). Continuing this process, we see clearly that

\[ g(a_0) \preceq g(a_1) \preceq g(a_2) \preceq ... \preceq g(a_{n+1}) \]

and

\[ g(b_0) \succeq g(b_1) \succeq g(b_2) \succeq ... \succeq g(b_{n+1}) \]

If \((a_{n+1}, b_{n+1}) = (a_n, b_n)\), then \( H \) and \( g \) have a coupled coincidence point. So we suppose that \((a_{n+1}, b_{n+1}) \neq (a_n, b_n)\) for all \( n \geq 0 \), i.e., we suppose that either \( g(a_{n+1}) = H(a_n, b_n) \neq g(a_n) \) or \( g(b_{n+1}) = H(b_n, a_n) \neq g(b_n) \).

Next, we prove that, for all \( n \geq 0 \),

\[ \int_{0}^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt \leq \frac{1}{2} (2k)^n \int_{0}^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt. \]  \((5)\)

For \( n = 1 \), we have

\[ \int_{0}^{S(ga_2, ga_2, ga_1)} \varphi(t) dt = \int_{0}^{S(H(a_1, b_1), H(a_1, b_1), H(a_0, b_0))} \varphi(t) dt \]

\[ \leq k \int_{0}^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt \]

\[ = \frac{1}{2} (2k)^1 \int_{0}^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt, \]

and hence (5) holds for \( n = 1 \). Therefore, we assume that (5) holds for \( n > 0 \).

Since \( g(a_{n+1}) \succeq g(a_n) \) and \( g(b_{n+1}) \preceq g(b_n) \), by using (2) and (5), we have

\[ \int_{0}^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt = \int_{0}^{S(H(a_n, b_n), H(a_n, b_n), H(a_{n-1}, b_{n-1}))} \varphi(t) dt \]

\[ \leq k \int_{0}^{[S(ga_n, ga_n, ga_{n-1}) + S(gb_n, gb_n, gb_{n-1})]} \varphi(t) dt. \]  \((6)\)

Now,
\[
\int_0^{S(g_{an}, ga_{n} - 1)} \varphi(t) dt = \int_0^{\rho(H(a_{n-1}, b_{n-1}), H(a_{n-2}, b_{n-2}))} \varphi(t) dt \\
\leq k \int_0^{[S(g_{an}, ga_{n-1}, ga_{n-2}) + S(g_{b_{an-1}}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt, \quad (7)
\]

and
\[
\int_0^{S(g_{bn}, gb_{n} - 1)} \varphi(t) dt = \int_0^{\rho(H(b_{n-1}, a_{n-1}), H(b_{n-2}, a_{n-2}))} \varphi(t) dt \\
\leq k \int_0^{[S(g_{bn}, gb_{n-1}, gb_{n-2}) + S(g_{an}, ga_{n-1}, ga_{n-2})]} \varphi(t) dt. \quad (8)
\]

Combining (7) and (8), we get that
\[
\int_0^{S(g_{an}, ga_{n} - 1)} \varphi(t) dt + \int_0^{S(g_{bn}, gb_{n} - 1)} \varphi(t) dt \\
\leq 2k \int_0^{[S(g_{an}, ga_{n-1}, ga_{n-2}) + S(g_{bn}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt
\]
holds for \( n \in \mathbb{N} \). From (6), we have
\[
\int_0^{S(g_{an+1}, ga_{n+1} - 1)} \varphi(t) dt \leq k \int_0^{[S(g_{an}, ga_{n}, ga_{n-1}) + S(g_{bn}, gb_{n}, gb_{n-1})]} \varphi(t) dt \\
\leq 2k^2 \int_0^{[S(g_{an}, ga_{n-1}, ga_{n-2}) + S(g_{bn}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt \\
\vdots \\
\leq \frac{1}{2} (2k)^n \int_0^{[S(g_{a1}, ga_{1}, ga_{0}) + S(g_{b1}, gb_{1}, gb_{0})]} \varphi(t) dt.
\]

Hence for all \( n \in \mathbb{N} \), we have
\[
\int_0^{S(g_{an+1}, ga_{n+1} - 1)} \varphi(t) dt \leq \frac{1}{2} (2k)^n \int_0^{[S(g_{a1}, ga_{1}, ga_{0}) + S(g_{b1}, gb_{1}, gb_{0})]} \varphi(t) dt. \quad (9)
\]

Suppose \( m, n \in \mathbb{N} \), with \( m > n \). First, let \( m = 2p + 1 \), (9) and condition that \( \varphi \) is sub-additive integrable function, we have
\[ \int_0^{S(g_{a_m}, g_{a_m}, g_{a_n})} \varphi(t) dt \leq 2 \int_0^{S(g_{a_{n+1}}, g_{a_{n+1}}, g_{a_n})} \varphi(t) dt + \cdots \]

\[ + \int_0^{S(g_{a_{m-1}}, g_{a_{m-1}}, g_{a_m} - 2)} \varphi(t) dt \]

\[ + \int_0^{S(g_{a_m}, g_{a_m}, g_{a_{m-1}})} \varphi(t) dt \]

\[ \leq \left( \sum_{i=n}^{m-2} (2k)^i + \frac{1}{2} (2k)^{m-1} \right) \]

\[ \times \int_0^{S(g_{a_1}, g_{a_1}, g_{a_0}) + S(g_{b_1}, g_{b_1}, g_{b_0})} \varphi(t) dt \]

\[ \leq \left( \frac{(2k)^n}{1 - 2k} + \frac{1}{2} (2k)^{m-1} \right) \]

\[ \times \int_0^{S(g_{a_1}, g_{a_1}, g_{a_0}) + S(g_{b_1}, g_{b_1}, g_{b_0})} \varphi(t) dt. \]

Further, let \( m = 2p. \) Again, using (S3), (9) and condition that \( \varphi \) is sub-additive integrable function, we obtain

\[ \int_0^{S(g_{a_m}, g_{a_m}, g_{a_n})} \varphi(t) dt \leq 2 \int_0^{S(g_{a_{n+1}}, g_{a_{n+1}}, g_{a_n})} \varphi(t) dt + \cdots \]

\[ + \int_0^{S(g_{a_{m-1}}, g_{a_{m-1}}, g_{a_m} - 2)} \varphi(t) dt \]

\[ \leq \sum_{i=n}^{m-1} (2k)^i \int_0^{S(g_{a_1}, g_{a_1}, g_{a_0}) + S(g_{b_1}, g_{b_1}, g_{b_0})} \varphi(t) dt \]

\[ \leq \frac{(2k)^n}{1 - 2k} \int_0^{S(g_{a_1}, g_{a_1}, g_{a_0}) + S(g_{b_1}, g_{b_1}, g_{b_0})} \varphi(t) dt. \]

Letting \( n, m \to \infty. \) Since \( 2k < 1, \) using Lemma 1.20 we conclude that

\[ \lim_{n,m \to \infty} S(g_{a_m}, g_{a_m}, g_{a_n}) = 0. \]

Thus \( \{g_{a_n}\} \) is Cauchy sequence in \( g(X). \) Similarly, we can show that \( \{g_{b_n}\} \) is Cauchy sequence in \( g(X). \) Since \( g(X) \) is complete, we have \( \{g_{a_n}\} \) and \( \{g_{b_n}\} \) are convergent to some \( a \in X \) and \( b \in X \) respectively. Since \( g \) is continuous, we have \( \{g(g_{a_n})\} \) is convergent to \( g(a) \) and \( \{g(g_{b_n})\} \) is convergent to \( g(b), \) that is,

\[ \lim_{n \to \infty} g(g_{a_n}) = g(a) \quad \text{and} \quad \lim_{n \to \infty} g(g_{b_n}) = g(b). \]
Since, $H$ and $g$ are commutative, we have
\[ H(g(a_n), g(b_n)) = g(H(a_n, b_n)) = g(a_{n+1}) \]
and
\[ H(g(b_n), g(a_n)) = g(H(b_n, a_n)) = g(b_{n+1}) \]

Next, we claim that $(a, b)$ is coupled coincidence point of $H$ and $g$.

From (1) we have
\[
\int_0^{S(H(a, b), H(a, b), g^2a_{n+1})} \varphi(t)dt = \int_0^{S(H(a, b), H(a, b), H(ga, gb))} \varphi(t)dt \leq k \int_0^{[S(ga, ga, ga_{n+1}) + S(gb, gb, gb_{n+1})]} \varphi(t)dt.
\]

Letting $n \to \infty$ and also $g$ is continuous, we get
\[
\int_0^{S(H(a, b), H(a, b), ga)} \varphi(t)dt \leq k \int_0^{[S(ga, ga, ga) + S(gb, gb, gb)]} \varphi(t)dt = 0.
\]

Hence $ga = H(a, b)$. Similarly, we can show that $gb = H(b, a)$.

Next we claim that $H(a, a) = g(a) = a$. Since $(a, b)$ is a coupled coincidence point of $H$ and $g$, we have $ga = H(a, b)$ and $gb = H(b, a)$. Suppose that $ga \neq gb$.

Then from (1), we have
\[
\int_0^{S(gb, gb, ga)} \varphi(t)dt = \int_0^{S(H(b, a), H(b, a), H(b, a))} \varphi(t)dt \leq k \int_0^{[S(gb, gb, ga) + S(ga, ga, ga)]} \varphi(t)dt.
\]

Also,
\[
\int_0^{S(ga, ga, gb)} \varphi(t)dt = \int_0^{S(H(a, b), H(a, b), H(b, a))} \varphi(t)dt \leq k \int_0^{[S(ga, ga, gb) + S(gb, gb, gb)]} \varphi(t)dt.
\]

Therefore,
\[
\int_0^{S(gb, gb, ga)} \varphi(t)dt + \int_0^{S(ga, ga, gb)} \varphi(t)dt \leq 2k \int_0^{[S(gb, gb, ga) + S(ga, ga, gb)]} \varphi(t)dt.
\]

Since $2k < 1$, we get
\[
\int_0^{S(gb, gb, ga)} \varphi(t)dt + \int_0^{S(ga, ga, gb)} \varphi(t)dt < \int_0^{S(gb, gb, ga)} \varphi(t)dt + \int_0^{S(ga, ga, gb)} \varphi(t)dt,
\]
which is contradiction. Hence $ga = gb$ and
\[ H(a, b) = ga = gb = H(b, a). \]

Since \( \{ga_{n+1}\} \) is a subsequence of \( \{ga_n\} \), we have \( \{ga_{n+1}\} \) is convergent to \( a \). Thus,

\[
\int_{S(ga, ga, ga_{n+1})} \varphi(t) dt = \int_0^{S(H(a, b), H(a, b), H(a_n, b_n))} \varphi(t) dt \\
\leq k \int_0^{[S(ga, ga, ga_n)+S(gb, gb, gb_n)]} \varphi(t) dt.
\]

Letting \( n \to \infty \) and also \( g \) is continuous, we get

\[
\int_0^{S(ga, ga, a)} \varphi(t) dt \leq k \int_0^{[S(ga, ga, a)+S(gb, gb, b)]} \varphi(t) dt.
\]

Similarly, we can show that

\[
\int_0^{S(gb, gb, b)} \varphi(t) dt \leq k \int_0^{[S(gb, gb, b)+S(ga, ga, a)]} \varphi(t) dt.
\]

Thus

\[
\int_0^{S(ga, ga, a)} \varphi(t) dt + \int_0^{S(gb, gb, b)} \varphi(t) dt \leq 2k \int_0^{[S(ga, ga, a)+S(gb, gb, b)]} \varphi(t) dt.
\]

Since \( 2k < 1 \), the last inequality happens only if \( S(ga, ga, a) = 0 \) and \( S(gb, gb, b) = 0 \). Hence \( a = ga \) and \( b = gb \). Thus we get \( ga = H(a, a) = a \). \( \square \)

**Corollary 2.2.** Let \( (X, S, \preceq) \) be an ordered S-metric space. Let \( H : X \times X \to X \) and \( g : X \to X \) be mappings such that \( H \) has the mixed \( g \)-monotone property on \( X \) and there exist two elements \( a_0, b_0 \in X \) with \( g(a_0) \preceq H(a_0, b_0) \) and \( g(b_0) \succeq H(b_0, a_0) \). Let there exists a constant \( k \in (0, \frac{1}{2}) \) such that the following holds:

\[
\int_0^{S(H(p, q), H(p, q), H(a, b))} \varphi(t) dt \leq k \int_0^{[S(gp, gp, gr)+S(gq, gq, gr)]} \varphi(t) dt
\]

for \( a, b, p, q \in X \) with \( ga \succeq gp \) and \( gb \succeq gq \) or \( ga \preceq gp \) and \( gb \succeq gq \), where \( \varphi : [0, \infty) \to [0, \infty) \) is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each \( \varepsilon > 0 \), \( \int_0^\varepsilon \varphi(t) dt > 0 \). Assume the following conditions:

(i) \( H(X \times X) \subseteq g(X) \),

(ii) \( g \) is continuous and commutes with \( H \),

(iii) \( g(X) \) is complete,

(iv) \( (X, S, \preceq) \) is regular.
Then there exists \( a \in X \) such that \( ga = H(a,a) = a \).

Proof. From Theorem 2.1 by taking \( a = p \) and \( b = q \). \( \square \)

Corollary 2.3. Let \((X, S, \preceq)\) be an ordered \( S \)-metric space. Let \( H : X \times X \to X \) be mapping such that \( H \) has the mixed monotone property on \( X \) and there exist two elements \( a_0, b_0 \in X \) with \( a_0 \preceq H(a_0, b_0) \) and \( b_0 \preceq H(b_0, a_0) \). Let there exists a constant \( k \in (0, \frac{1}{2}) \) such that the following holds:

\[
\int_0^S(H(p,q),H(p,q),H(a,b)) \varphi(t)dt \leq k \int_0^|S(p,p,a)+S(q,q,b)| \varphi(t)dt
\]

for \( a, b, p, q \in X \) with \( a \preceq p \) and \( b \preceq q \) or \( a \preceq p \) and \( b \preceq q \), where \( \varphi : [0, \infty) \to [0, \infty) \) is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each \( \epsilon > 0 \), \( \int_0^\epsilon \varphi(t)dt > 0 \). If \((X, S, \preceq)\) is regular then there exists \( a \in X \) such that \( H(a,a) = a \).

Proof. We defined \( g : X \to X \) by \( ga = a \). Then the mappings \( H \) and \( g \) satisfy all the conditions of Corollary 2.2. Hence the result follows. \( \square \)

Corollary 2.4. Let \((X, S, \preceq)\) be an ordered \( S \)-metric space. Let \( H : X \times X \to X \) and \( g : X \to X \) be mappings such that \( H \) has the mixed \( g \)-monotone property on \( X \) and there exist two elements \( a_0, b_0 \in X \) with \( g(a_0) \preceq H(a_0, b_0) \) and \( g(b_0) \preceq H(b_0, a_0) \). Let there exists a constant \( k \in (0, \frac{1}{2}) \) such that the following holds:

\[
S(H(p,q),H(p,q),H(a,b)) \leq k[S(gp, gp, ga) + S(gq, gg, gb)]
\]

for \( a, b, p, q \in X \) with \( ga \preceq gp \) and \( gb \preceq gq \) or \( ga \preceq gp \) and \( gb \preceq gq \). Assume the following conditions:

(a) \( H(X \times X) \subset g(X) \),
(b) \( g(X) \) is complete,
(c) \( g \) is continuous and commutes with \( H \),
(d) \((X, S, \preceq)\) is regular.

Then there exists \( a \in X \) such that \( H(a,a) = ga = a \).

Proof. Put \( \varphi(t) = 1 \) for all \( t \in [0, \infty) \), the result follows. Moreover, we get a generalization of theorem given in [5]. \( \square \)

Corollary 2.5. Let \((X, S, \preceq)\) be a complete ordered \( S \)-metric space. Let \( H : X \times X \to X \) be mapping has the mixed monotone property on \( X \) and there exist two elements \( a_0, b_0 \in X \) with \( a_0 \preceq H(a_0, b_0) \) and \( b_0 \preceq H(b_0, a_0) \). Let there exists a constant \( k \in (0, \frac{1}{2}) \) such that the following holds:

\[
S(H(p,q),H(p,q),H(a,b)) \leq k[S(p,p,a) + S(q,q,b)]
\]

for \( a, b, p, q \in X \) with \( a \preceq p \) and \( b \preceq q \) or \( a \preceq p \) and \( b \preceq q \). If \((X, S, \preceq)\) is regular, then there exists \( a \in X \) such that \( H(a,a) = a \).
Proof. Let \( g : X \to X \) be defined as \( g(a) = a \). Then all conditions of Corollary 2.4 are satisfied.

**Example 2.6.** Suppose \( X = [0, 1] \) be ordered by the following relation \( a \preceq b \) if and only if \( a \leq b \). Let the metric \( S \) be defined by

\[
S(a, b, c) = |b + c - 2a| + |b - c|.
\]

Then clearly, \((X, S, \preceq)\) is a complete ordered \( S \)-metric space. Let \( g : X \to X \) and \( H : X \times X \to X \) be defined by

\[
ga = \frac{a}{2} \quad \text{and} \quad H(a, b) = \frac{a + b}{20}.
\]

Let \( \varphi(t) = e^t \). Then by (1), we have

\[
\int_0^1 \varphi(t)dt = \int_0^1 |(H(p, q) + H(c, r) - 2H(a, b)) + |H(p, q) - H(c, r)|| \varphi(t)dt
\]

\[
= \int_0^1 \left| \frac{2a + 2b + 2c - 2(a + b)}{20} + \frac{a + c - a - c}{20} \right| \varphi(t)dt
\]

\[
\leq \int_0^1 \left| \frac{2a + 2b + 2c - 2(a + b)}{20} + \frac{a + c - a - c}{20} \right| \varphi(t)dt
\]

\[
= \int_0^1 \left( |gp + gc - 2ga| + |gp + gr - 2gb| + |gp - gc| + |gp - gr| \right) \varphi(t)dt
\]

\[
\leq \int_0^1 \left( S(gp, gp, gc) + S(gb, gb, gr) \right) \varphi(t)dt
\]

\[
\leq \frac{1}{10} \int_0^1 \left( S(gp, gp, gc) + S(gb, gb, gr) \right) \varphi(t)dt.
\]

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