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Ordered S-Metric Spaces and Coupled Common Fixed Point Theorems of Integral Type Contraction

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Abstract

In the present paper, we introduce the notion of integral type contractive mapping with respect to ordered S-metric space and prove some coupled common fixed point results of integral type contractive mapping in ordered S-metric space. Moreover, we give an example to support our main result.

Keywords: S-metric, ordered S-metric space, common fixed point, coupled fixed point, integral type contractive mapping, partial order, mixed g- monotone property, commuting maps.

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1. Introduction

Banach contraction principle [4], is one of the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partial order have been studied by many authors (see [1,3,5,8,12,13,17,19]). The study of metric spaces has attracted, and continued to attract the interest of many authors. There are many generalizations of metric spaces, such as 2-metric spaces [11], G-metric spaces [20], D^*-metric spaces [24], partial metric spaces [6], cone metric spaces [15], S-metric spaces [22], b-metric spaces [9] and G_b-metric spaces [2]. In 2012, Sedghi et al. [22] introduced the notion of S-metric space.

First we recall some notions, results and examples which will be useful later.

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Definition 1.1. [22] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$:*

(S1) $0 < S(x, y, z)$ *for all* $x, y, z \in X$ *with* $x \neq y \neq z$;

(S2) $S(x, y, z) = 0$ *if* $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ *for all* $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Example 1.2. [22] *Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.*

Lemma 1.3. *[21]* In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Definition 1.4. [23] *Let (X, S) be an S-metric space and $A \subseteq X$.*

(1) If for every $x \in X$ there exists $r > 0$ such that $B_s(x, r) \subseteq A$, then the subset A is called open subset of X.

(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote by $\lim_{n \longrightarrow \infty} x_n = x$.

(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.

(5) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subseteq X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_s(x, r) \subseteq A$. Then τ is a topology on X.

Lemma 1.5. *[23]* Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}, \{y_n\}$ such that $\lim_{n \longrightarrow \infty} x_n = x$ and $\lim_{n \longrightarrow \infty} y_n = y$, then $\lim_{n \longrightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.6. *[10]* Let (X, S) be an S-metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z),$$

and

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y),$$

for all $x, y, z \in X$.

Definition 1.7. [14] *Let $(X, \preceq)$ be partially ordered set. Then $a, b \in X$ are called comparable if $a \preceq b$ or $b \preceq a$ holds.*

Definition 1.8. *Let X be a nonempty set. Then $(X, S, \preceq)$ is called an ordered S-metric space if:*

(1) (X, S) is an S-metric space,

(2) $(X, \preceq)$ is a partially ordered set.

Definition 1.9. *$(X, S, \preceq)$ is said to be regular if it has the following properties:*

(i) if for a non-decreasing sequence $\{x_n\}$, $x_n \to^S x$ as $n \to \infty$, then $x_n \preceq x$ for all n;

(ii) if for a non-increasing sequence $\{x_n\}$, $x_n \to^S x$, as $n \to \infty$, then $x_n \succeq x$ for all n.

Definition 1.10. [5] *Let $(X, \preceq)$ be partially ordered set and $H : X \times X \to X$. The mapping H is said to has the mixed monotone property if H is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, i.e., for any $a, b \in X$,*

$$a_1, a_2 \in X,\ a_1 \preceq a_2 \Rightarrow H(a_1, b) \preceq H(a_2, b),$$
$$b_1, b_2 \in X,\ b_1 \preceq b_2 \Rightarrow H(a, b_1) \succeq H(a, b_2).$$

Definition 1.11. [7] *Let $(X, \preceq)$ be partially ordered set and suppose $H : X \times X \to X$ and $g : X \to X$. The mapping H is said to has the mixed g-monotone property if H is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, i.e., for any $a, b \in X$,*

$$a_1, a_2 \in X,\ g(a_1) \preceq g(a_2) \Rightarrow H(a_1, b) \preceq H(a_2, b),$$
$$b_1, b_2 \in X,\ g(b_1) \preceq g(b_2) \Rightarrow H(a, b_1) \succeq H(a, b_2).$$

Definition 1.12. [5] *An element $(a, b) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(a, b) = ga$ and $F(b, a) = gb$, and their common coupled fixed point if $F(a, b) = ga = a$ and $F(b, a) = gb = b$.*

Definition 1.13. [17] *Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(a, b) = F(ga, gb)$.*

Definition 1.14. [17] *An element $(a, b) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $F(a, b) = a$ and $F(b, a) = b$.*

Definition 1.15. *Let (X, S) and (X', S') be two S-metric spaces, and let $f : (X, S) \to (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X, $S(x_n, x_n, a) \to 0$ implies $S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous at X if and only if it is continuous at all $a \in X$.*

Definition 1.16. [16] *Let* $X, Y \subset (-\infty, +\infty)$. *The function* $\varphi : X \to Y$ *is called sub-additive integrable function if and only if for all* $c, d \in X$,

$$\int_0^{c+d} \varphi(t)dt \leq \int_0^c \varphi(t)dt + \int_0^d \varphi(t)dt.$$

Example 1.17. [16] *Let* $X = (0, \infty)$, $d(x, y) = |x - y|$, *and* $\varphi(t) = \frac{1}{t+1}$ *for all* $t > 0$. *Then for all* $c, d \in X$,

$$\int_0^{c+d} \frac{dt}{t+1} = \ln(c+d+1), \int_0^c \frac{dt}{t+1} = \ln(c+1), \int_0^d \frac{dt}{t+1} = \ln(d+1),$$

since $cd \geq 0$, *then* $c + d + 1 \leq c + d + 1 + cd = (c+1)(d+1)$. *Therefore,*

$$\ln(c + d + 1) \leq \ln((c + 1)(d + 1)) = \ln(c + 1) + \ln(d + 1).$$

So, we show that φ *is sub-additive integrable function.*

Example 1.18. *Let* $X = (1, \infty)$, *and* $\varphi(t) = e^t$. *Then the function* φ *is not sub-additive integrable function.*

Lemma 1.19. *[18]* Let $\{r_n\}_{n\in\mathbb{N}}$ be a non-negative sequence such that $\lim_{n\to\infty} r_n = a$. Then

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\, dt = \int_0^a \varphi(t)\, dt,$$

where $\varphi : [0, +\infty) \to [0, +\infty]$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$.

Lemma 1.20. *[18]* Let $\{r_n\}_{n\in\mathbb{N}}$ be a non-negative sequence. Then

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\, dt = 0,$$

if and only if $\lim_{n\to\infty} r_n = 0$, where $\varphi : [0, +\infty) \to [0, +\infty]$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$.

2. Results

Theorem 2.1. Let $(X, S, \preceq)$ be an ordered S-metric space. Let $H : X \times X \to X$ and $g : X \to X$ be mappings such that H has the mixed g-monotone property on X and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:

$$\int_0^{S(H(a,b),H(p,q),H(c,r))} \varphi(t)dt \leq k \int_0^{[S(ga,gp,gc)+S(gb,gq,gr)]} \varphi(t)dt, \qquad (1)$$

for $a, b, c, p, q, r \in X$ with $ga \succeq gp \succeq gc$ and $gb \preceq gq \preceq gr$ or $ga \preceq gp \preceq gc$ and $gb \succeq gq \succeq gr$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$. Assume the following conditions:

(a) $H(X \times X) \subset g(X)$,

(b) $g(X)$ is complete,

(c) g is continuous and commutes with H,

(d) $(X, S, \preceq)$ is regular.

Then H and g have a coupled coincidence point (a, b). If $ga \succeq gb$ or $ga \preceq gb$, then $g(a) = H(a, a) = a$.

Proof. Let a_0, b_0 be two points such that $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$. As $H(X \times X) \subset g(X)$, we may choose a_1, b_1 in a way that $g(a_1) = H(a_0, b_0)$ and $g(b_1) = H(b_0, a_0)$.
Again since $H(X \times X) \subset g(X)$, we may choose a_2, $b_2 \in X$ such that $g(a_2) = H(a_1, b_1)$ and $g(b_2) = H(b_1, a_1)$. Repeating this process, we can build two sequences $\{a_n\}$ and $\{b_n\}$ in X such that,

$$g(a_{n+1}) = H(a_n, b_n) \text{ and } g(b_{n+1}) = H(b_n, a_n), \text{ for all } n \geq 0. \tag{2}$$

Now, we claim that for all $n \geq 0$,

$$g(a_n) \preceq g(a_{n+1}), \tag{3}$$

and

$$g(b_n) \succeq g(b_{n+1}). \tag{4}$$

Now we will use mathematical induction. Suppose that $n = 0$. Since $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$, we see that $g(a_1) = H(a_0, b_0)$ and $g(b_1) = H(b_0, a_0)$, and so $g(a_0) \preceq g(a_1)$ and $g(b_0) \succeq g(b_1)$, i.e., (3) and (4) hold for $n = 0$. We now suppose that (3) and (4) are valid for some $n > 0$. As we know that H has mixed g-monotone property and also $g(a_n) \preceq g(a_{n+1})$, $g(b_n) \succeq g(b_{n+1})$, then from (2), we have

$$g(a_{n+1}) = H(a_n, b_n) \preceq H(a_{n+1}, b_n)$$

and

$$H(b_{n+1}, a_n) \preceq H(b_n, a_n) = g(b_{n+1}).$$

Also we have,

$$g(a_{n+2}) = H(a_{n+1}, b_{n+1}) \succeq H(a_{n+1}, b_n)$$

and

$$H(b_{n+1}, a_n) \succeq H(b_{n+1}, a_{n+1}) = g(b_{n+2}).$$

Then from (2) and (3), we get

$$g(a_{n+1}) \preceq g(a_{n+2}) \text{ and } g(b_{n+1}) \succeq g(b_{n+2}).$$

We conclude by mathematical induction that (3) and (4) hold for all $n \geq 0$. Continuing this process, we see clearly that

$$g(a_0) \preceq g(a_1) \preceq g(a_2) \preceq ... \preceq g(a_{n+1})...$$

and

$$g(b_0) \succeq g(b_1) \succeq g(b_2) \succeq ... \succeq g(b_{n+1})....$$

If $(a_{n+1}, b_{n+1}) = (a_n, b_n)$, then H and g have a coupled coincidence point. So we suppose that $(a_{n+1}, b_{n+1}) \neq (a_n, b_n)$ for all $n \geq 0$, i.e., we suppose that either $g(a_{n+1}) = H(a_n, b_n) \neq g(a_n)$ or $g(b_{n+1}) = H(b_n, a_n) \neq g(b_n)$.
Next, we proves that, for all $n \geq 0$,

$$\int_0^{S(ga_{n+1},ga_{n+1},ga_n)} \varphi(t)dt \leq \frac{1}{2}(2k)^n \int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]} \varphi(t)dt. \quad (5)$$

For $n = 1$, we have

$$\begin{aligned} \int_0^{S(ga_2,ga_2,ga_1)} \varphi(t)dt &= \int_0^{S(H(a_1,b_1),H(a_1,b_1),H(a_0,b_0))} \varphi(t)dt \\ &\leq k \int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]} \varphi(t)dt \\ &= \frac{1}{2}(2k)^1 \int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]} \varphi(t)dt, \end{aligned}$$

and hence (5) holds for $n = 1$. Therefore, we assume that (5) holds for $n > 0$. Since $g(a_{n+1}) \succeq g(a_n)$ and $g(b_{n+1}) \preceq g(b_n)$, by using (2) and (5), we have

$$\begin{aligned} \int_0^{S(ga_{n+1},ga_{n+1},ga_n)} \varphi(t)dt &= \int_0^{S(H(a_n,b_n),H(a_n,b_n),H(a_{n-1},b_{n-1}))} \varphi(t)dt \\ &\leq k \int_0^{[S(ga_n,ga_n,ga_{n-1})+S(gb_n,gb_n,gb_{n-1})]} \varphi(t)dt. \end{aligned} \quad (6)$$

Now,

$$\begin{aligned}\int_0^{S(ga_n,ga_n,ga_{n-1})}\varphi(t)dt &= \int_0^{S(H(a_{n-1},b_{n-1}),H(a_{n-1},b_{n-1}),H(a_{n-2},b_{n-2}))}\varphi(t)dt \\ &\le k\int_0^{[S(ga_{n-1},ga_{n-1},ga_{n-2})+S(gb_{n-1},gb_{n-1},gb_{n-2})]}\varphi(t)dt,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\int_0^{S(gb_n,gb_n,gb_{n-1})}\varphi(t)dt &= \int_0^{S(H(b_{n-1},a_{n-1}),H(b_{n-1},a_{n-1}),H(b_{n-2},a_{n-2}))}\varphi(t)dt \\ &\le k\int_0^{[S(gb_{n-1},gb_{n-1},gb_{n-2})+S(ga_{n-1},ga_{n-1},ga_{n-2})]}\varphi(t)dt.\end{aligned} \tag{8}$$

Combining (7) and (8), we get that

$$\begin{aligned}&\int_0^{S(ga_n,ga_n,ga_{n-1})}\varphi(t)dt+\int_0^{S(gb_n,gb_n,gb_{n-1})}\varphi(t)dt \\ \le\ & 2k\int_0^{[S(ga_{n-1},ga_{n-1},ga_{n-2})+S(gb_{n-1},gb_{n-1},gb_{n-2})]}\varphi(t)dt\end{aligned}$$

holds for $n\in\mathbb{N}$. From (6), we have

$$\begin{aligned}\int_0^{S(ga_{n+1},ga_{n+1},ga_n)}\varphi(t)dt &\le k\int_0^{[S(ga_n,ga_n,ga_{n-1})+S(gb_n,gb_n,gb_{n-1})]}\varphi(t)dt \\ &\le 2k^2\int_0^{[S(ga_{n-1},ga_{n-1},ga_{n-2})+S(gb_{n-1},gb_{n-1},gb_{n-2})]}\varphi(t)dt \\ &\ \vdots \\ &\le \frac{1}{2}(2k)^n\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt.\end{aligned}$$

Hence for all $n\in\mathbb{N}$, we have

$$\int_0^{S(ga_{n+1},ga_{n+1},ga_n)}\varphi(t)dt\le\frac{1}{2}(2k)^n\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt. \tag{9}$$

Suppose $m,n\in\mathbb{N}$, with $m>n$. First, let $m=2p+1$, (9) and condition that φ is sub-additive integrable function, we have

$$\begin{aligned}
\int_0^{S(ga_m,ga_m,ga_n)}\varphi(t)dt &\le 2(\int_0^{S(ga_{n+1},ga_{n+1},ga_n)}\varphi(t)dt+\cdots\\
&+\int_0^{S(ga_{m-1},ga_{m-1},ga_{m-2})}\varphi(t)dt)\\
&+\int_0^{S(ga_m,ga_m,ga_{m-1})}\varphi(t)dt\\
&\le (\sum_{i=n}^{m-2}(2k)^i+\frac{1}{2}(2k)^{m-1})\\
&\times\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt\\
&\le (\frac{(2k)^n}{1-2k}+\frac{1}{2}(2k)^{m-1})\\
&\times\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt.
\end{aligned}$$

Further, let $m = 2p$. Again, using (S3), (9) and condition that φ is sub-additive integrable function, we obtain

$$\begin{aligned}
\int_0^{S(ga_m,ga_m,ga_n)}\varphi(t)dt &\le 2(\int_0^{S(ga_{n+1},ga_{n+1},ga_n)}\varphi(t)dt+\cdots\\
&+\int_0^{S(ga_m,ga_m,ga_{m-1})}\varphi(t)dt)\\
&\le \sum_{i=n}^{m-1}(2k)^i\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt\\
&\le \frac{(2k)^n}{1-2k}\int_0^{[S(ga_1,ga_1,ga_0)+S(gb_1,gb_1,gb_0)]}\varphi(t)dt.
\end{aligned}$$

Letting $n, m \to \infty$. Since $2k < 1$, using Lemma 1.20 we conclude that

$$\lim_{n,m\longrightarrow\infty} S(ga_m, ga_m, ga_n) = 0.$$

Thus $\{ga_n\}$ is Cauchy sequence in $g(X)$. Similarly, we can show that $\{gb_n\}$ is Cauchy sequence in $g(X)$. Since $g(X)$ is complete, we have $\{ga_n\}$ and $\{gb_n\}$ are convergent to some $a \in X$ and $b \in X$ respectively. Since g is continuous, we have $\{g(ga_n)\}$ is convergent to ga and $\{g(gb_n)\}$ is convergent to gb, that is,

$$\lim_{n\to\infty} g(g(a_n)) = g(a) \text{ and } \lim_{n\to\infty} g(g(b_n)) = g(b).$$

Since, H and g are commutative, we have

$$H(g(a_n), g(b_n)) = g(H(a_n, b_n)) = g(g(a_{n+1}))$$

and

$$H(g(b_n), g(a_n)) = g(H(b_n, a_n)) = g(g(b_{n+1})).$$

Next, we claim that (a, b) is coupled coincidence point of H and g.
From (1) we have

$$\begin{aligned} \int_0^{S(H(a,b),H(a,b),gga_{n+1})} \varphi(t)dt &= \int_0^{S(H(a,b),H(a,b),H(ga_n,gb_n)} \varphi(t)dt \\ &\leq k\int_0^{[S(ga,ga,gga_n)+S(gb,gb,ggb_n)]} \varphi(t)dt. \end{aligned}$$

Letting $n \to \infty$ and also g is continuous, we get

$$\int_0^{S(H(a,b),H(a,b),ga)} \varphi(t)dt \leq k\int_0^{[S(ga,ga,ga)+S(gb,gb,gb)]} \varphi(t)dt = 0.$$

Hence $ga = H(a, b)$. Similarly, we can show that $gb = H(b, a)$.

Next we claim that $H(a, a) = g(a) = a$. Since (a, b) is a coupled coincidence point of H and g, we have $ga = H(a, b)$ and $gb = H(b, a)$. Suppose that $ga \neq gb$. Then from (1), we have

$$\begin{aligned} \int_0^{S(gb,gb,ga)} \varphi(t)dt &= \int_0^{S(H(b,a),H(b,a),H(a,b))} \varphi(t)dt \\ &\leq k\int_0^{[S(gb,gb,ga)+S(ga,ga,gb)]} \varphi(t)dt. \end{aligned}$$

Also,

$$\begin{aligned} \int_0^{S(ga,ga,gb)} \varphi(t)dt &= \int_0^{S(H(a,b),H(a,b),H(b,a))} \varphi(t)dt \\ &\leq k\int_0^{[S(ga,ga,gb)+S(gb,gb,ga)]} \varphi(t)dt. \end{aligned}$$

Therefore,

$$\int_0^{S(gb,gb,ga)} \varphi(t)dt + \int_0^{S(ga,ga,gb)} \varphi(t)dt \leq 2k\int_0^{[S(gb,gb,ga)+S(ga,ga,gb)]} \varphi(t)dt.$$

Since $2k < 1$, we get

$$\int_0^{S(gb,gb,ga)} \varphi(t)dt + \int_0^{S(ga,ga,gb)} \varphi(t)dt < \int_0^{S(gb,gb,ga)} \varphi(t)dt + \int_0^{S(ga,ga,gb)} \varphi(t)dt,$$

which is contradiction. Hence $ga = gb$ and

$$H(a,b) = ga = gb = H(b,a).$$

Since $\{ga_{n+1}\}$ is a subsequence of $\{ga_n\}$, we have $\{ga_{n+1}\}$ is convergent to a. Thus,

$$\begin{aligned}\int^{S(ga,ga,ga_{n+1})} \varphi(t)dt &= \int_0^{S(H(a,b),H(a,b),H(a_n,b_n))} \varphi(t)dt \\ &\leq k\int_0^{[S(ga,ga,ga_n)+S(gb,gb,gb_n)]} \varphi(t)dt.\end{aligned}$$

Letting $n \to \infty$ and also g is continuous, we get

$$\int_0^{S(ga,ga,a)} \varphi(t)dt \leq k\int_0^{[S(ga,ga,a)+S(gb,gb,b)]} \varphi(t)dt.$$

Similarly, we can show that

$$\int_0^{S(gb,gb,b)} \varphi(t)dt \leq k\int_0^{[S(gb,gb,b)+S(ga,ga,a)]} \varphi(t)dt.$$

Thus

$$\int_0^{S(ga,ga,a)} \varphi(t)dt + \int_0^{S(gb,gb,b)} \varphi(t)dt \leq 2k\int_0^{[S(ga,ga,a)+S(gb,gb,b)]} \varphi(t)dt.$$

Since $2k < 1$, the last inequality happens only if $S(ga, ga, a) = 0$ and $S(gb, gb, b) = 0$. Hence $a = ga$ and $b = gb$. Thus we get $ga = H(a, a) = a$. □

Corollary 2.2. Let $(X, S, \preceq)$ be an ordered S-metric space. Let $H : X \times X \to X$ and $g : X \to X$ be mappings such that H has the mixed g-monotone property on X and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:

$$\int_0^{S(H(p,q),H(p,q),H(a,b))} \varphi(t)dt \leq k\int_0^{[S(gp,gp,gc)+S(gq,gq,gr)]} \varphi(t)dt$$

for $a, b, p, q \in X$ with $ga \succeq gp$ and $gb \preceq gq$ or $ga \preceq gp$ and $gb \succeq gq$, where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Assume the following conditions:

(i) $H(X \times X) \subseteq g(X)$,

(ii) g is continuous and commutes with H,

(iii) $g(X)$ is complete,

(iv) $(X, S, \preceq)$ is regular.

Then there exists $a \in X$ such that $ga = H(a,a) = a$.

Proof. From Theorem 2.1 by taking $a = p$ and $b = q$. □

Corollary 2.3. Let $(X, S, \preceq)$ be an ordered S-metric space. Let $H : X \times X \to X$ be mapping such that H has the mixed monotone property on X and there exist two elements $a_0, b_0 \in X$ with $a_0 \preceq H(a_0, b_0)$ and $b_0 \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:

$$\int_0^{S(H(p,q),H(p,q),H(a,b))} \varphi(t)dt \leq k \int_0^{[S(p,p,a)+S(q,q,b)]} \varphi(t)dt$$

for $a, b, p, q \in X$ with $a \succeq p$ and $b \preceq q$ or $a \preceq p$ and $b \succeq q$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. If $(X, S, \preceq)$ is regular then there exists $a \in X$ such that $H(a,a) = a$.

Proof. We defined $g : X \to X$ by $ga = a$. Then the mappings H and g satisfy all the conditions of Corollary 2.2. Hence the result follows. □

Corollary 2.4. Let $(X, S, \preceq)$ be an ordered S-metric space. Let $H : X \times X \to X$ and $g : X \to X$ be mappings such that H has the mixed g-monotone property on X and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq H(a_0, b_0)$ and $g(b_0) \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:

$$S(H(p,q), H(p,q), H(a,b)) \leq k[S(gp, gp, ga) + S(gq, gq, gb)]$$

for $a, b, p, q \in X$ with $ga \succeq gp$ and $gb \preceq gq$ or $ga \preceq gp$ and $gb \succeq gq$. Assume the following conditions:

(a) $H(X \times X) \subset g(X)$,

(b) $g(X)$ is complete,

(c) g is continuous and commutes with H,

(d) $(X, S, \preceq)$ is regular.

Then there exists $a \in X$ such that $H(a,a) = ga = a$.

Proof. Put $\varphi(t) = 1$ for all $t \in [0, \infty)$, the result follows. Moreover, we get a generalization of theorem given in [5]. □

Corollary 2.5. Let $(X, S, \preceq)$ be a complete ordered S-metric space. Let $H : X \times X \to X$ be mapping has the mixed monotone property on X and there exist two elements $a_0, b_0 \in X$ with $a_0 \preceq H(a_0, b_0)$ and $b_0 \succeq H(b_0, a_0)$. Let there exists a constant $k \in (0, \frac{1}{2})$ such that the following holds:

$$S(H(p,q), H(p,q), H(a,b)) \leq k[S(p,p,a) + S(q,q,b)]$$

for $a, b, p, q \in X$ with $a \succeq p$ and $b \preceq q$ or $a \preceq p$ and $b \succeq q$. If $(X, S, \preceq)$ is regular, then there exists $a \in X$ such that $H(a,a) = a$.

Proof. Let $g : X \to X$ be defined as $g(a) = a$. Then all conditions of Corollary 2.4 are satisfied. $\square$

Example 2.6. *Suppose $X = [0,1]$ be ordered by the following relation $a \preceq b$ if and only if $a \leq b$. Let the metric S be defined by*

$$S(a,b,c) = |b+c-2a| + |b-c|.$$

Then clearly, $(X, S, \preceq)$ is a complete ordered S-metric space. Let $g : X \to X$ and $H : X \times X \to X$ be defined by

$$ga = \frac{a}{2} \text{ and } H(a,b) = \frac{a+b}{20}.$$

Let $\varphi(t) = e^t$. Then by (1), we have

$$\begin{aligned}
\int_0^{S(H(a,b),H(p,q),H(c,r))} \varphi(t)dt &= \int_0^{|H(p,q)+H(c,r)-2H(a,b)|+|H(p,q)-H(c,r)|} \varphi(t)dt \\
&= \int_0^{|\frac{p+q}{20}+\frac{c+r}{20}-\frac{2(a+b)}{20}|+|\frac{p+q}{20}-\frac{c+r}{20}|} \varphi(t)dt \\
&\leq \int_0^{|\frac{p+c-2a}{20}|+|\frac{q+r-2b}{20}|+|\frac{p-c}{20}|+|\frac{q-r}{20}|} \varphi(t)dt \\
&= \int_0^{\frac{1}{10}(|gp+gc-2ga|+|gq+gr-2gb|+|gp-gc|+|gq-gr|)} \varphi(t)dt \\
&= \int_0^{\frac{1}{10}(S(ga,gp,gc)+S(gb,gq,gr))} \varphi(t)dt \\
&\leq \frac{1}{10}\int_0^{S(ga,gp,gc)+S(gb,gq,gr)} \varphi(t)dt.
\end{aligned}$$

Hence for $k = \frac{1}{10}$, all the conditions of Theorem 2.1 are satisfied. Therefore there exists $a \in X$ such that $H(a,a) = g(a) = a$. In this example we have that $a = 0$.

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