

# Ordered $S$ -metric Spaces and Coupled Common Fixed Point Theorems of Integral Type Contraction

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## Abstract

In the present paper, we introduce the notion of integral type contractive mapping with respect to ordered  $S$ -metric space and prove some coupled common fixed point results of integral type contractive mapping in ordered  $S$ -metric space. Moreover, we give an example to support our main result.

**Keywords:**  $S$ -metric, ordered  $S$ -metric space, common fixed point, coupled fixed point, integral type contractive mapping, partial order, mixed  $g$ - monotone property, commuting maps.

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## 1. Introduction

Banach contraction principle [4], is one of the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partial order have been studied by many authors (see [1, 3, 5, 8, 12, 13, 17, 19]). The study of metric spaces has attracted, and continued to attract the interest of many authors. There are many generalizations of metric spaces, such as 2-metric spaces [11],  $G$ -metric spaces [20],  $D^*$ -metric spaces [24], partial metric spaces [6], cone metric spaces [15],  $S$ -metric spaces [22],  $b$ -metric spaces [9] and  $G_b$ -metric spaces [2]. In 2012, Sedghi et al. [22] introduced the notion of  $S$ -metric space.

First we recall some notions, results and examples which will be useful later.

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**Definition 1.1.** [22] Let  $X$  be a nonempty set. An  $S$ -metric on  $X$  is a function  $S : X^3 \rightarrow [0, \infty)$  that satisfies the following conditions for all  $x, y, z, a \in X$ :

- (S1)  $0 < S(x, y, z)$  for all  $x, y, z \in X$  with  $x \neq y \neq z$ ;
- (S2)  $S(x, y, z) = 0$  if  $x = y = z$ ;
- (S3)  $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$  for all  $x, y, z, a \in X$ .

The pair  $(X, S)$  is called an  $S$ -metric space.

**Example 1.2.** [22] Let  $X = \mathbb{R}^2$  and  $d$  be an ordinary metric on  $X$ . Put  $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$  for all  $x, y, z \in \mathbb{R}^2$ , that is,  $S$  is the perimeter of the triangle given by  $x, y, z$ . Then  $S$  is an  $S$ -metric on  $X$ .

**Lemma 1.3.** [21] In an  $S$ -metric space, we have  $S(x, x, y) = S(y, y, x)$ .

**Definition 1.4.** [23] Let  $(X, S)$  be an  $S$ -metric space and  $A \subseteq X$ .

- (1) If for every  $x \in X$  there exists  $r > 0$  such that  $B_s(x, r) \subseteq A$ , then the subset  $A$  is called open subset of  $X$ .
- (2) Subset  $A$  of  $X$  is said to be  $S$ -bounded if there exists  $r > 0$  such that  $S(x, x, y) < r$  for all  $x, y \in A$ .
- (3) A sequence  $\{x_n\}$  in  $X$  converges to  $x$  if and only if  $S(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . That is, for each  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for each  $n \geq n_0$ ,  $S(x_n, x_n, x) < \varepsilon$  and we denote by  $\lim_{n \rightarrow \infty} x_n = x$ .
- (4) A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if for each  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for each  $n, m \geq n_0$ ,  $S(x_n, x_n, x_m) < \varepsilon$ .
- (5) The  $S$ -metric space  $(X, S)$  is said to be complete if every Cauchy sequence is convergent.
- (6) Let  $\tau$  be the set of all  $A \subseteq X$  with  $x \in A$  if and only if there exists  $r > 0$  such that  $B_s(x, r) \subseteq A$ . Then  $\tau$  is a topology on  $X$ .

**Lemma 1.5.** [23] Let  $(X, S)$  be an  $S$ -metric space. If there exist sequences  $\{x_n\}, \{y_n\}$  such that  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y$ , then  $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$ .

**Lemma 1.6.** [10] Let  $(X, S)$  be an  $S$ -metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z),$$

and

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y),$$

for all  $x, y, z \in X$ .

**Definition 1.7.** [14] Let  $(X, \preceq)$  be partially ordered set. Then  $a, b \in X$  are called comparable if  $a \preceq b$  or  $b \preceq a$  holds.

**Definition 1.8.** Let  $X$  be a nonempty set. Then  $(X, S, \preceq)$  is called an ordered  $S$ -metric space if:

- (1)  $(X, S)$  is an  $S$ -metric space,
- (2)  $(X, \preceq)$  is a partially ordered set.

**Definition 1.9.**  $(X, S, \preceq)$  is said to be regular if it has the following properties:

- (i) if for a non-decreasing sequence  $\{x_n\}$ ,  $x_n \rightarrow^S x$  as  $n \rightarrow \infty$ , then  $x_n \preceq x$  for all  $n$ ;
- (ii) if for a non-increasing sequence  $\{x_n\}$ ,  $x_n \rightarrow^S x$ , as  $n \rightarrow \infty$ , then  $x_n \succeq x$  for all  $n$ .

**Definition 1.10.** [5] Let  $(X, \preceq)$  be partially ordered set and  $H : X \times X \rightarrow X$ . The mapping  $H$  is said to has the mixed monotone property if  $H$  is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, i.e., for any  $a, b \in X$ ,

$$\begin{aligned} a_1, a_2 \in X, a_1 \preceq a_2 &\Rightarrow H(a_1, b) \preceq H(a_2, b), \\ b_1, b_2 \in X, b_1 \preceq b_2 &\Rightarrow H(a, b_1) \succeq H(a, b_2). \end{aligned}$$

**Definition 1.11.** [7] Let  $(X, \preceq)$  be partially ordered set and suppose  $H : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . The mapping  $H$  is said to has the mixed  $g$ -monotone property if  $H$  is monotone  $g$ -nondecreasing in its first argument and is monotone  $g$ -nonincreasing in its second argument, i.e., for any  $a, b \in X$ ,

$$\begin{aligned} a_1, a_2 \in X, g(a_1) \preceq g(a_2) &\Rightarrow H(a_1, b) \preceq H(a_2, b), \\ b_1, b_2 \in X, g(b_1) \preceq g(b_2) &\Rightarrow H(a, b_1) \succeq H(a, b_2). \end{aligned}$$

**Definition 1.12.** [5] An element  $(a, b) \in X \times X$  is called a coupled coincidence point of the mappings  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $F(a, b) = ga$  and  $F(b, a) = gb$ , and their common coupled fixed point if  $F(a, b) = ga = a$  and  $F(b, a) = gb = b$ .

**Definition 1.13.** [17] Let  $X$  be a nonempty set. Then we say that the mappings  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are commutative if  $gF(a, b) = F(ga, gb)$ .

**Definition 1.14.** [17] An element  $(a, b) \in X \times X$  is called a coupled fixed point of mapping  $F : X \times X \rightarrow X$  if  $F(a, b) = a$  and  $F(b, a) = b$ .

**Definition 1.15.** Let  $(X, S)$  and  $(X', S')$  be two  $S$ -metric spaces, and let  $f : (X, S) \rightarrow (X', S')$  be a function. Then  $f$  is said to be continuous at a point  $a \in X$  if and only if for every sequence  $x_n$  in  $X$ ,  $S(x_n, x_n, a) \rightarrow 0$  implies  $S'(f(x_n), f(x_n), f(a)) \rightarrow 0$ . A function  $f$  is continuous at  $X$  if and only if it is continuous at all  $a \in X$ .

**Definition 1.16.** [16] Let  $X, Y \subset (-\infty, +\infty)$ . The function  $\varphi : X \rightarrow Y$  is called sub-additive integrable function if and only if for all  $c, d \in X$ ,

$$\int_0^{c+d} \varphi(t) dt \leq \int_0^c \varphi(t) dt + \int_0^d \varphi(t) dt.$$

**Example 1.17.** [16] Let  $X = (0, \infty)$ ,  $d(x, y) = |x - y|$ , and  $\varphi(t) = \frac{1}{t+1}$  for all  $t > 0$ . Then for all  $c, d \in X$ ,

$$\int_0^{c+d} \frac{dt}{t+1} = \ln(c+d+1), \int_0^c \frac{dt}{t+1} = \ln(c+1), \int_0^d \frac{dt}{t+1} = \ln(d+1),$$

since  $cd \geq 0$ , then  $c+d+1 \leq c+d+1+cd = (c+1)(d+1)$ . Therefore,

$$\ln(c+d+1) \leq \ln((c+1)(d+1)) = \ln(c+1) + \ln(d+1).$$

So, we show that  $\varphi$  is sub-additive integrable function.

**Example 1.18.** Let  $X = (1, \infty)$ , and  $\varphi(t) = e^t$ . Then the function  $\varphi$  is not sub-additive integrable function.

**Lemma 1.19.** [18] Let  $\{r_n\}_{n \in \mathbb{N}}$  be a non-negative sequence such that  $\lim_{n \rightarrow \infty} r_n = a$ . Then

$$\lim_{n \rightarrow \infty} \int_0^{r_n} \varphi(t) dt = \int_0^a \varphi(t) dt,$$

where  $\varphi : [0, +\infty) \rightarrow [0, +\infty]$  is Lebesgue integrable, summable on each compact subset of  $[0, +\infty)$  and  $\int_0^\varepsilon \varphi(t) dt > 0$  for each  $\varepsilon > 0$ .

**Lemma 1.20.** [18] Let  $\{r_n\}_{n \in \mathbb{N}}$  be a non-negative sequence. Then

$$\lim_{n \rightarrow \infty} \int_0^{r_n} \varphi(t) dt = 0,$$

if and only if  $\lim_{n \rightarrow \infty} r_n = 0$ , where  $\varphi : [0, +\infty) \rightarrow [0, +\infty]$  is Lebesgue integrable, summable on each compact subset of  $[0, +\infty)$  and  $\int_0^\varepsilon \varphi(t) dt > 0$  for each  $\varepsilon > 0$ .

## 2. Results

**Theorem 2.1.** Let  $(X, S, \preceq)$  be an ordered  $S$ -metric space. Let  $H : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $H$  has the mixed  $g$ -monotone property on  $X$  and there exist two elements  $a_0, b_0 \in X$  with  $g(a_0) \preceq H(a_0, b_0)$  and  $g(b_0) \succeq H(b_0, a_0)$ . Let there exists a constant  $k \in (0, \frac{1}{2})$  such that the following holds:

$$\int_0^{S(H(a,b), H(p,q), H(c,r))} \varphi(t) dt \leq k \int_0^{[S(ga, gp, gc) + S(gb, gq, gr)]} \varphi(t) dt, \quad (1)$$

for  $a, b, c, p, q, r \in X$  with  $ga \succeq gp \succeq gc$  and  $gb \preceq gq \preceq gr$  or  $ga \preceq gp \preceq gc$  and  $gb \succeq gq \succeq gr$ , where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t)dt > 0$ . Assume the following conditions:

- (a)  $H(X \times X) \subset g(X)$ ,
- (b)  $g(X)$  is complete,
- (c)  $g$  is continuous and commutes with  $H$ ,
- (d)  $(X, S, \preceq)$  is regular.

Then  $H$  and  $g$  have a coupled coincidence point  $(a, b)$ . If  $ga \succeq gb$  or  $ga \preceq gb$ , then  $g(a) = H(a, a) = a$ .

*Proof.* Let  $a_0, b_0$  be two points such that  $g(a_0) \preceq H(a_0, b_0)$  and  $g(b_0) \succeq H(b_0, a_0)$ . As  $H(X \times X) \subset g(X)$ , we may choose  $a_1, b_1$  in a way that  $g(a_1) = H(a_0, b_0)$  and  $g(b_1) = H(b_0, a_0)$ . Again since  $H(X \times X) \subset g(X)$ , we may choose  $a_2, b_2 \in X$  such that  $g(a_2) = H(a_1, b_1)$  and  $g(b_2) = H(b_1, a_1)$ . Repeating this process, we can build two sequences  $\{a_n\}$  and  $\{b_n\}$  in  $X$  such that,

$$g(a_{n+1}) = H(a_n, b_n) \text{ and } g(b_{n+1}) = H(b_n, a_n), \text{ for all } n \geq 0. \tag{2}$$

Now, we claim that for all  $n \geq 0$ ,

$$g(a_n) \preceq g(a_{n+1}), \tag{3}$$

and

$$g(b_n) \succeq g(b_{n+1}). \tag{4}$$

Now we will use mathematical induction. Suppose that  $n = 0$ . Since  $g(a_0) \preceq H(a_0, b_0)$  and  $g(b_0) \succeq H(b_0, a_0)$ , we see that  $g(a_1) = H(a_0, b_0)$  and  $g(b_1) = H(b_0, a_0)$ , and so  $g(a_0) \preceq g(a_1)$  and  $g(b_0) \succeq g(b_1)$ , i.e., (3) and (4) hold for  $n = 0$ . We now suppose that (3) and (4) are valid for some  $n > 0$ . As we know that  $H$  has mixed  $g$ -monotone property and also  $g(a_n) \preceq g(a_{n+1})$ ,  $g(b_n) \succeq g(b_{n+1})$ , then from (2), we have

$$g(a_{n+1}) = H(a_n, b_n) \preceq H(a_{n+1}, b_n)$$

and

$$H(b_{n+1}, a_n) \preceq H(b_n, a_n) = g(b_{n+1}).$$

Also we have,

$$g(a_{n+2}) = H(a_{n+1}, b_{n+1}) \succeq H(a_{n+1}, b_n)$$

and

$$H(b_{n+1}, a_n) \succeq H(b_{n+1}, a_{n+1}) = g(b_{n+2}).$$

Then from (2) and (3), we get

$$g(a_{n+1}) \preceq g(a_{n+2}) \text{ and } g(b_{n+1}) \succeq g(b_{n+2}).$$

We conclude by mathematical induction that (3) and (4) hold for all  $n \geq 0$ . Continuing this process, we see clearly that

$$g(a_0) \preceq g(a_1) \preceq g(a_2) \preceq \dots \preceq g(a_{n+1}) \dots$$

and

$$g(b_0) \succeq g(b_1) \succeq g(b_2) \succeq \dots \succeq g(b_{n+1}) \dots$$

If  $(a_{n+1}, b_{n+1}) = (a_n, b_n)$ , then  $H$  and  $g$  have a coupled coincidence point. So we suppose that  $(a_{n+1}, b_{n+1}) \neq (a_n, b_n)$  for all  $n \geq 0$ , i.e., we suppose that either  $g(a_{n+1}) = H(a_n, b_n) \neq g(a_n)$  or  $g(b_{n+1}) = H(b_n, a_n) \neq g(b_n)$ . Next, we prove that, for all  $n \geq 0$ ,

$$\int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt \leq \frac{1}{2}(2k)^n \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt. \quad (5)$$

For  $n = 1$ , we have

$$\begin{aligned} \int_0^{S(ga_2, ga_2, ga_1)} \varphi(t) dt &= \int_0^{S(H(a_1, b_1), H(a_1, b_1), H(a_0, b_0))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt \\ &= \frac{1}{2}(2k)^1 \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt, \end{aligned}$$

and hence (5) holds for  $n = 1$ . Therefore, we assume that (5) holds for  $n > 0$ . Since  $g(a_{n+1}) \succeq g(a_n)$  and  $g(b_{n+1}) \preceq g(b_n)$ , by using (2) and (5), we have

$$\begin{aligned} \int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt &= \int_0^{S(H(a_n, b_n), H(a_n, b_n), H(a_{n-1}, b_{n-1}))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga_n, ga_n, ga_{n-1}) + S(gb_n, gb_n, gb_{n-1})]} \varphi(t) dt. \end{aligned} \quad (6)$$

Now,

$$\begin{aligned} \int_0^{S(ga_n, ga_n, ga_{n-1})} \varphi(t) dt &= \int_0^{S(H(a_{n-1}, b_{n-1}), H(a_{n-1}, b_{n-1}), H(a_{n-2}, b_{n-2}))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga_{n-1}, ga_{n-1}, ga_{n-2}) + S(gb_{n-1}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \int_0^{S(gb_n, gb_n, gb_{n-1})} \varphi(t) dt &= \int_0^{S(H(b_{n-1}, a_{n-1}), H(b_{n-1}, a_{n-1}), H(b_{n-2}, a_{n-2}))} \varphi(t) dt \\ &\leq k \int_0^{[S(gb_{n-1}, gb_{n-1}, gb_{n-2}) + S(ga_{n-1}, ga_{n-1}, ga_{n-2})]} \varphi(t) dt. \end{aligned} \tag{8}$$

Combining (7) and (8), we get that

$$\begin{aligned} &\int_0^{S(ga_n, ga_n, ga_{n-1})} \varphi(t) dt + \int_0^{S(gb_n, gb_n, gb_{n-1})} \varphi(t) dt \\ &\leq 2k \int_0^{[S(ga_{n-1}, ga_{n-1}, ga_{n-2}) + S(gb_{n-1}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt \end{aligned}$$

holds for  $n \in \mathbb{N}$ . From (6), we have

$$\begin{aligned} \int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt &\leq k \int_0^{[S(ga_n, ga_n, ga_{n-1}) + S(gb_n, gb_n, gb_{n-1})]} \varphi(t) dt \\ &\leq 2k^2 \int_0^{[S(ga_{n-1}, ga_{n-1}, ga_{n-2}) + S(gb_{n-1}, gb_{n-1}, gb_{n-2})]} \varphi(t) dt \\ &\vdots \\ &\leq \frac{1}{2} (2k)^n \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt. \end{aligned}$$

Hence for all  $n \in \mathbb{N}$ , we have

$$\int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt \leq \frac{1}{2} (2k)^n \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt. \tag{9}$$

Suppose  $m, n \in \mathbb{N}$ , with  $m > n$ . First, let  $m = 2p + 1$ , (9) and condition that  $\varphi$  is sub-additive integrable function, we have

$$\begin{aligned}
\int_0^{S(ga_m, ga_m, ga_n)} \varphi(t) dt &\leq 2 \left( \int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt + \dots \right. \\
&+ \int_0^{S(ga_{m-1}, ga_{m-1}, ga_{m-2})} \varphi(t) dt \\
&+ \left. \int_0^{S(ga_m, ga_m, ga_{m-1})} \varphi(t) dt \right) \\
&\leq \left( \sum_{i=n}^{m-2} (2k)^i + \frac{1}{2} (2k)^{m-1} \right) \\
&\times \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt \\
&\leq \left( \frac{(2k)^n}{1-2k} + \frac{1}{2} (2k)^{m-1} \right) \\
&\times \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt.
\end{aligned}$$

Further, let  $m = 2p$ . Again, using (S3), (9) and condition that  $\varphi$  is sub-additive integrable function, we obtain

$$\begin{aligned}
\int_0^{S(ga_m, ga_m, ga_n)} \varphi(t) dt &\leq 2 \left( \int_0^{S(ga_{n+1}, ga_{n+1}, ga_n)} \varphi(t) dt + \dots \right. \\
&+ \left. \int_0^{S(ga_m, ga_m, ga_{m-1})} \varphi(t) dt \right) \\
&\leq \sum_{i=n}^{m-1} (2k)^i \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt \\
&\leq \frac{(2k)^n}{1-2k} \int_0^{[S(ga_1, ga_1, ga_0) + S(gb_1, gb_1, gb_0)]} \varphi(t) dt.
\end{aligned}$$

Letting  $n, m \rightarrow \infty$ . Since  $2k < 1$ , using Lemma 1.20 we conclude that

$$\lim_{n, m \rightarrow \infty} S(ga_m, ga_m, ga_n) = 0.$$

Thus  $\{ga_n\}$  is Cauchy sequence in  $g(X)$ . Similarly, we can show that  $\{gb_n\}$  is Cauchy sequence in  $g(X)$ . Since  $g(X)$  is complete, we have  $\{ga_n\}$  and  $\{gb_n\}$  are convergent to some  $a \in X$  and  $b \in X$  respectively. Since  $g$  is continuous, we have  $\{g(ga_n)\}$  is convergent to  $ga$  and  $\{g(gb_n)\}$  is convergent to  $gb$ , that is,

$$\lim_{n \rightarrow \infty} g(ga_n) = g(a) \quad \text{and} \quad \lim_{n \rightarrow \infty} g(gb_n) = g(b).$$



Since,  $H$  and  $g$  are commutative, we have

$$H(g(a_n), g(b_n)) = g(H(a_n, b_n)) = g(g(a_{n+1}))$$

and

$$H(g(b_n), g(a_n)) = g(H(b_n, a_n)) = g(g(b_{n+1})).$$

Next, we claim that  $(a, b)$  is coupled coincidence point of  $H$  and  $g$ . From (1) we have

$$\begin{aligned} \int_0^{S(H(a,b), H(a,b), gga_{n+1})} \varphi(t) dt &= \int_0^{S(H(a,b), H(a,b), H(ga_n, gb_n))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga, ga, gga_n) + S(gb, gb, ggb_n)]} \varphi(t) dt. \end{aligned}$$

Letting  $n \rightarrow \infty$  and also  $g$  is continuous, we get

$$\int_0^{S(H(a,b), H(a,b), ga)} \varphi(t) dt \leq k \int_0^{[S(ga, ga, ga) + S(gb, gb, gb)]} \varphi(t) dt = 0.$$

Hence  $ga = H(a, b)$ . Similarly, we can show that  $gb = H(b, a)$ .

Next we claim that  $H(a, a) = g(a) = a$ . Since  $(a, b)$  is a coupled coincidence point of  $H$  and  $g$ , we have  $ga = H(a, b)$  and  $gb = H(b, a)$ . Suppose that  $ga \neq gb$ . Then from (1), we have

$$\begin{aligned} \int_0^{S(gb, gb, ga)} \varphi(t) dt &= \int_0^{S(H(b,a), H(b,a), H(a,b))} \varphi(t) dt \\ &\leq k \int_0^{[S(gb, gb, ga) + S(ga, ga, gb)]} \varphi(t) dt. \end{aligned}$$

Also,

$$\begin{aligned} \int_0^{S(ga, ga, gb)} \varphi(t) dt &= \int_0^{S(H(a,b), H(a,b), H(b,a))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga, ga, gb) + S(gb, gb, ga)]} \varphi(t) dt. \end{aligned}$$

Therefore,

$$\int_0^{S(gb, gb, ga)} \varphi(t) dt + \int_0^{S(ga, ga, gb)} \varphi(t) dt \leq 2k \int_0^{[S(gb, gb, ga) + S(ga, ga, gb)]} \varphi(t) dt.$$

Since  $2k < 1$ , we get

$$\int_0^{S(gb, gb, ga)} \varphi(t) dt + \int_0^{S(ga, ga, gb)} \varphi(t) dt < \int_0^{S(gb, gb, ga)} \varphi(t) dt + \int_0^{S(ga, ga, gb)} \varphi(t) dt,$$

which is contradiction. Hence  $ga = gb$  and

$$H(a, b) = ga = gb = H(b, a).$$

Since  $\{ga_{n+1}\}$  is a subsequence of  $\{ga_n\}$ , we have  $\{ga_{n+1}\}$  is convergent to  $a$ . Thus,

$$\begin{aligned} \int_0^{S(ga, ga, ga_{n+1})} \varphi(t) dt &= \int_0^{S(H(a, b), H(a, b), H(a_n, b_n))} \varphi(t) dt \\ &\leq k \int_0^{[S(ga, ga, ga_n) + S(gb, gb, gb_n)]} \varphi(t) dt. \end{aligned}$$

Letting  $n \rightarrow \infty$  and also  $g$  is continuous, we get

$$\int_0^{S(ga, ga, a)} \varphi(t) dt \leq k \int_0^{[S(ga, ga, a) + S(gb, gb, b)]} \varphi(t) dt.$$

Similarly, we can show that

$$\int_0^{S(gb, gb, b)} \varphi(t) dt \leq k \int_0^{[S(gb, gb, b) + S(ga, ga, a)]} \varphi(t) dt.$$

Thus

$$\int_0^{S(ga, ga, a)} \varphi(t) dt + \int_0^{S(gb, gb, b)} \varphi(t) dt \leq 2k \int_0^{[S(ga, ga, a) + S(gb, gb, b)]} \varphi(t) dt.$$

Since  $2k < 1$ , the last inequality happens only if  $S(ga, ga, a) = 0$  and  $S(gb, gb, b) = 0$ . Hence  $a = ga$  and  $b = gb$ . Thus we get  $ga = H(a, a) = a$ .  $\square$

**Corollary 2.2.** Let  $(X, S, \preceq)$  be an ordered  $S$ -metric space. Let  $H : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $H$  has the mixed  $g$ -monotone property on  $X$  and there exist two elements  $a_0, b_0 \in X$  with  $g(a_0) \preceq H(a_0, b_0)$  and  $g(b_0) \succeq H(b_0, a_0)$ . Let there exists a constant  $k \in (0, \frac{1}{2})$  such that the following holds:

$$\int_0^{S(H(p, q), H(p, q), H(a, b))} \varphi(t) dt \leq k \int_0^{[S(gp, gp, gc) + S(gq, gq, gr)]} \varphi(t) dt$$

for  $a, b, p, q \in X$  with  $ga \succeq gp$  and  $gb \preceq gq$  or  $ga \preceq gp$  and  $gb \succeq gq$ , where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t) dt > 0$ . Assume the following conditions:

- (i)  $H(X \times X) \subseteq g(X)$ ,
- (ii)  $g$  is continuous and commutes with  $H$ ,
- (iii)  $g(X)$  is complete,
- (iv)  $(X, S, \preceq)$  is regular.

Then there exists  $a \in X$  such that  $ga = H(a, a) = a$ .

*Proof.* From Theorem 2.1 by taking  $a = p$  and  $b = q$ . □

**Corollary 2.3.** Let  $(X, S, \preceq)$  be an ordered  $S$ -metric space. Let  $H : X \times X \rightarrow X$  be mapping such that  $H$  has the mixed monotone property on  $X$  and there exist two elements  $a_0, b_0 \in X$  with  $a_0 \preceq H(a_0, b_0)$  and  $b_0 \succeq H(b_0, a_0)$ . Let there exists a constant  $k \in (0, \frac{1}{2})$  such that the following holds:

$$\int_0^{S(H(p,q), H(p,q), H(a,b))} \varphi(t) dt \leq k \int_0^{[S(p,p,a)+S(q,q,b)]} \varphi(t) dt$$

for  $a, b, p, q \in X$  with  $a \succeq p$  and  $b \preceq q$  or  $a \preceq p$  and  $b \succeq q$ , where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t) dt > 0$ . If  $(X, S, \preceq)$  is regular then there exists  $a \in X$  such that  $H(a, a) = a$ .

*Proof.* We defined  $g : X \rightarrow X$  by  $ga = a$ . Then the mappings  $H$  and  $g$  satisfy all the conditions of Corollary 2.2. Hence the result follows. □

**Corollary 2.4.** Let  $(X, S, \preceq)$  be an ordered  $S$ -metric space. Let  $H : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $H$  has the mixed  $g$ -monotone property on  $X$  and there exist two elements  $a_0, b_0 \in X$  with  $g(a_0) \preceq H(a_0, b_0)$  and  $g(b_0) \succeq H(b_0, a_0)$ . Let there exists a constant  $k \in (0, \frac{1}{2})$  such that the following holds:

$$S(H(p, q), H(p, q), H(a, b)) \leq k[S(gp, gp, ga) + S(gq, gq, gb)]$$

for  $a, b, p, q \in X$  with  $ga \succeq gp$  and  $gb \preceq gq$  or  $ga \preceq gp$  and  $gb \succeq gq$ . Assume the following conditions:

- (a)  $H(X \times X) \subset g(X)$ ,
- (b)  $g(X)$  is complete,
- (c)  $g$  is continuous and commutes with  $H$ ,
- (d)  $(X, S, \preceq)$  is regular.

Then there exists  $a \in X$  such that  $H(a, a) = ga = a$ .

*Proof.* Put  $\varphi(t) = 1$  for all  $t \in [0, \infty)$ , the result follows. Moreover, we get a generalization of theorem given in [5]. □

**Corollary 2.5.** Let  $(X, S, \preceq)$  be a complete ordered  $S$ -metric space. Let  $H : X \times X \rightarrow X$  be mapping has the mixed monotone property on  $X$  and there exist two elements  $a_0, b_0 \in X$  with  $a_0 \preceq H(a_0, b_0)$  and  $b_0 \succeq H(b_0, a_0)$ . Let there exists a constant  $k \in (0, \frac{1}{2})$  such that the following holds:

$$S(H(p, q), H(p, q), H(a, b)) \leq k[S(p, p, a) + S(q, q, b)]$$

for  $a, b, p, q \in X$  with  $a \succeq p$  and  $b \preceq q$  or  $a \preceq p$  and  $b \succeq q$ . If  $(X, S, \preceq)$  is regular, then there exists  $a \in X$  such that  $H(a, a) = a$ .

*Proof.* Let  $g : X \rightarrow X$  be defined as  $g(a) = a$ . Then all conditions of Corollary 2.4 are satisfied.  $\square$

**Example 2.6.** Suppose  $X = [0, 1]$  be ordered by the following relation  $a \preceq b$  if and only if  $a \leq b$ . Let the metric  $S$  be defined by

$$S(a, b, c) = |b + c - 2a| + |b - c|.$$

Then clearly,  $(X, S, \preceq)$  is a complete ordered  $S$ -metric space. Let  $g : X \rightarrow X$  and  $H : X \times X \rightarrow X$  be defined by

$$ga = \frac{a}{2} \text{ and } H(a, b) = \frac{a + b}{20}.$$

Let  $\varphi(t) = e^t$ . Then by (1), we have

$$\begin{aligned} \int_0^{S(H(a,b), H(p,q), H(c,r))} \varphi(t) dt &= \int_0^{|H(p,q)+H(c,r)-2H(a,b)|+|H(p,q)-H(c,r)|} \varphi(t) dt \\ &= \int_0^{|\frac{p+q}{20} + \frac{c+r}{20} - \frac{2(a+b)}{20}| + |\frac{p+q}{20} - \frac{c+r}{20}|} \varphi(t) dt \\ &\leq \int_0^{|\frac{p+c-2a}{20}| + |\frac{q+r-2b}{20}| + |\frac{p-c}{20}| + |\frac{q-r}{20}|} \varphi(t) dt \\ &= \int_0^{\frac{1}{10}(|gp+gc-2ga| + |gq+gr-2gb| + |gp-gc| + |gq-gr|)} \varphi(t) dt \\ &= \int_0^{\frac{1}{10}(S(ga, gp, gc) + S(gb, gq, gr))} \varphi(t) dt \\ &\leq \frac{1}{10} \int_0^{S(ga, gp, gc) + S(gb, gq, gr)} \varphi(t) dt. \end{aligned}$$

Hence for  $k = \frac{1}{10}$ , all the conditions of Theorem 2.1 are satisfied. Therefore there exists  $a \in X$  such that  $H(a, a) = g(a) = a$ . In this example we have that  $a = 0$ .

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