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Some Structural Properties of Upper and Lower Central Series of Pairs of Groups

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Abstract

In this paper, we first present some properties of lower and upper central series of pair of groups. Then the notion of n-isoclinism for the classification of pairs of groups is introduced, and some of the structural properties of the created classes are proved. Moreover some interesting theorems such as Baer Theorem, Bioch Theorem, Hirsh Theorem for pair of groups are generalized. Finally, it is shown that each n-isoclinism family of pairs contains a quotient irreducible pair.

Keywords: n-Isoclinism, pair of groups, quotient irreducible pair, π-groups.

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1. Introduction and Motivation

It is a well known fact that classification of groups according to the isomorphism property leads to the small classes which is not necessarily applicable for the study of all groups. For this reason, P. Hall [4] introduced the notion of isoclinism for the classification of groups of prime power order. In fact, the notion of isoclinism is an equivalence relation on the class of all groups, which is weaker than isomorphism. Therefore, it causes to provide bigger classes which makes better or sometimes easier conditions to classify some kinds of groups. A generalization of isoclinism, that is n-isoclinism of groups, is implicit in a short note of P. Hall [5] on verbal and marginal subgroup. Later Hekster [6] introduced and generalized some notions

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related to n-isoclinism, such as n-stem groups, and also proved some properties concerning the internal structure of the families given by n-isoclinism.

On the other hand, pairs of groups as a tool for simultaneous study of a group and its subgroup, have been verified by many authors during these recent two decades. By a pair of groups, we mean a group G and a normal subgroup N and this notion is denoted by (G, N). For instance, Ellis introduced the Schur multiplier [3] and also capability [2] for a pair of groups and turned out some of their properties. Also a work of Salemkar, Moghaddam and Chitti [8] demonstrated some more properties of the Schur multiplier of a pair, as well as a recent work of Pourmirzaei, Hokmabadi and the last author [9] has given a criterion for characterizing the capability of a pair and also a complete classification of finitely generated abelian capable pairs.

Now, interesting in the study of pairs of groups motivates us to verify more properties of the pairs. But one of the strongest tools for the verification of each family or property of groups, is the study of classifications' tools, as a basic and fundamental notion. With this in mind, we intend to introduce the notion of n-isoclinism for pairs of groups and some other notions related to this concept. Then we will list some of the structural properties created by this type of classification. But before that, some properties of lower and upper central series of a pair will be listed. Accordingly this article actually extends some results of [10], generalizes [6] somehow, and so is a wide generalization of the basic concept of [4].

In this article all the notations are standard. For a group G and elements $x, y \in G$, the commutator of x and y is defined to be $x^{-1}y^{-1}xy$ and is denoted by $[x, y]$. A commutator of weight $n+1$ is defined recursively by $[x_1, \ldots, x_n, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$, as a left normed commutator and considering the convention $[x_1] = x_1$. Also if N is a normal subgroup of G, then the subgroup

$$< [n, g] | n \in N, g \in G >$$

of G is denoted by $[N, G]$.

2. The Upper and Lower Central Series

Let (G, N) be a pair of groups. The *center of a pair* (G, N) is defined to be the subgroup

$$Z(G, N) = \{n \in N | n^g = n, \forall g \in G\},$$

in which $n^g = g^{-1}ng$ denotes the conjugate of n by g. The *upper central series* for the pair (G, N) is the series

$$1 = Z_0(G, N) \subseteq Z_1(G, N) = Z(G, N) \subseteq Z_2(G, N) \subseteq \ldots,$$

in which each term defined by

$$\frac{Z_{n+1}(G,N)}{Z_n(G,N)} = Z(\frac{G}{Z_n(G,N)}, \frac{N}{Z_n(G,N)}).$$

One should note that $Z_n(G,N)$ is not necessarily a characteristic subgroup of N. A counterexample to this note is when we take G as the dihedral group of order 8 and N to be the Klein 4-group as the subgroup of G. The lower central series is defined as follows. First define $\gamma_1(G,N) = N$. Assume that $\gamma_i(G,N)$ is defined inductively for $i \geq 1$. Then $\gamma_{i+1}(G,N)$ is defined as the subgroup $[\gamma_i(G,N), G]$. Therefore we obtain the series

$$N = \gamma_1(G,N) \supseteq \gamma_2(G,N) \supseteq \dots$$

which is called the *lower central series* of (G,N). Also we say that (G,N) is nilpotent of class i, if $\gamma_{i+1}(G,N) = 1$ and $\gamma_i(G,N) \neq 1$. It is easy to see that $\gamma_n(G,N)/\gamma_{n+1}(G,N)$ lies in the center of $(G/\gamma_{n+1}(G,N), N/\gamma_{n+1}(G,N))$. Also the above given example which shows $Z_n(G,N)$ is not necessarily a characteristic subgroup of N, simultaneously proves that $\gamma_n(G,N)$ is not necessarily a fully invariant subgroup of N, since $Z_1(G,N) = \gamma_2(G,N) = Z(G)$.

Theorem 2.1. ***(Generalized Hirsh Theorem)*** *Let (G,N) be a pair of groups and $M \trianglelefteq G$. If $M \cap Z(G,N) = 1$, then $M \cap Z_n(G,N) = 1$, for all $n \geq 1$.*

Proof. The proof is done by induction on n. If $x \in M \cap Z_{n+1}(G,N)$ and $g \in G$, then $[g,x] \in M \cap Z_n(G,N)$, which induction hypothesis implies that $[g,x] = 1$. That is $x \in M \cap Z(G,N)$. It proves the theorem. □

Theorem 2.2. *Let (G,N) be a pair of groups and $H \leq G$ such that $G = HZ_n(G,N)$. Then*

(i) $\gamma_{n+1}(G,N) = \gamma_{n+1}(H, H \cap N)$,

(ii) $Z_n(H, H \cap N) = H \cap Z_n(G,N)$,

(iii) $\gamma_{n+1}(G,N) \cap Z_n(G,N) = \gamma_{n+1}(H, H \cap N) \cap Z_n(H, H \cap N)$.

Proof. Since the subgroup $Z_n(G,N) = Z_n(G) \cap N$ is marginal in G, therefore (i) holds. Parts (ii) and (iii) are straightforward. □

The following theorem is a wide generalization of a well-known theorem of Gaschütz to pairs of groups.

Theorem 2.3. ***(Generalized Gaschütz Theorem)*** *Let (G,N) be a pair of groups. Then for all $n \geq 0$ we have:*

$$\gamma_{n+1}(G,N) \cap Z_n(G,N) \subseteq \Phi(G) \cap N.$$

Proof. By [6, Proposition 2.6] we know that $\gamma_{n+1}(G) \cap Z_n(G) \subseteq \Phi(G)$. Since $\gamma_{n+1}(G,N) \cap Z_n(G,N) \subseteq \gamma_{n+1}(G) \cap Z_n(G)$, then the result follows. □

Theorem 2.4. ***(Generalized Baer Theorem)*** *Let (G,N) be a pair of groups and $n \geq 0$. If $\frac{G}{Z_n(G,N)}$ is finite, then $\gamma_{n+1}(G,N)$ is finite.*

Proof. Since $Z_n(G,N) = Z_n(G) \cap N$, then $[G, Z_n(G)]$ is finite. Thus from Baer Theorem [6, Theorem 2.7] we conclude that $\gamma_{n+1}(G)$ is finite. Now the fact that $\gamma_{n+1}(G,N) \subseteq \gamma_{n+1}(G)$ proves the statement. □

The next corollary is attained similar to [6, Corollary 2.8] and will be also proved analogously.

Corollary 2.5. *Let (G,N) be a pair of groups and $n \geq 1$. If $\frac{G}{Z_n(G,N)}$ is finite, then $\gamma_n(\frac{G}{Z(G,N)}, \frac{N}{Z(G,N)})$ is finite.*

Lemma 2.6. *Let (G,N) be a pair of groups and M be a normal subgroup of G such that $M \cap \gamma_{n+1}(G,N) = 1$, $n \geq 1$. Then the following properties hold:*

(i) $M \cap \gamma_{n+1-i}(G,N) \subseteq Z_i(G,N)$ for all i, $0 \leq i \leq n$,

(ii) $M \cap N \subseteq Z_n(G,N)$ and $Z_n(\frac{G}{M}, \frac{MN}{M}) = \frac{Z_n(G,N)}{M}$.

Proof. The proof is done similar to [6, Lemma 2.5]. □

Theorem 2.7. *Let (G,N) be a pair of groups and $M \trianglelefteq G$. If $[M,G] \subseteq Z_n(G,N)$, then $[\gamma_{n+1}(G,N), M] = 1$. In particular $[\gamma_{n+1}(G,N), Z_{n+1}(G,N)] = 1$.*

Proof. We know that $[M,G] \subseteq Z_n(G,N) \subseteq Z_n(G)$. Then by [7, Proposition 2.3] we have $[\gamma_{n+1}(G), M] = 1$. But $[\gamma_{n+1}(G,N), M] \subseteq [\gamma_{n+1}(G), M]$, which shows the assertion holds. □

3. n-Isoclinic Pairs of Groups

The notion of isoclinism of groups was generalized to pairs of groups in [10]. That definition can be easily stated for homoclinism and even in the general form, n-homoclinism as follows.

Definition 3.1. An n-homoclinism from (G_1, N_1) to (G_2, N_2) is a pair of homomorphisms (α, β) where $\alpha : \frac{G_1}{Z_n(G_1,N_1)} \longrightarrow \frac{G_2}{Z_n(G_2,N_2)}$ and $\beta : \gamma_{n+1}(G_1, N_1) \longrightarrow \gamma_{n+1}(G_2, N_2)$ such that $\alpha(\frac{N_1}{Z_n(G_1,N_1)}) = \frac{N_2}{Z_n(G_2,N_2)}$ and α induces β in the following sense: if $n_1 \in N_1$ and $g_{1_i} \in G_1$ for $i = 1, 2, \ldots, n$, then $\beta([n_1, g_{1_1}, \ldots, g_{1_n}]) = [n_2, g_{2_1}, \ldots, g_{2_n}]$, in which $n_2 \in \alpha(n_1 Z_n(G_1, N_1))$ and $g_{2_i} \in \alpha(g_{1_i} Z_n(G_1, N_1))$.

Now one can easily see that if normal subgroups coincide with the groups, then the notion of n-homoclinism between two pairs is the usual notion of n-homoclinism between two groups which was introduced in [6]. Therefore the introduced notion is a wide generalization of homomorphism. Also let (α, β) be an n-homoclinism from (G_1, N_1) to (G_2, N_2). Thus α induces β. One can easily see that β inherit some properties from α and vice versa. For example if α is surjective, then β is surjective, or if β is injective, then α is injective too.

Definition 3.2. Let (α,β) be an n-homoclinism from (G_1,N_1) to (G_2,N_2). We say that (G_1,N_1) and (G_2,N_2) are n-isoclinic, if α and β are isomorphisms. In this case we write $(G_1,N_1) \underset{n}{\sim} (G_2,N_2)$, and we say that the pair (α,β) is an n-isoclinism between (G_1,N_1) and (G_2,N_2). Moreover when $n=1$, for the sake of simplicity we say that these pairs are isoclinic and (α,β) is an isoclinism between (G_1,N_1) and (G_2,N_2).

Corollary 3.3. *Let (α,β) be an n-homoclinism from (G_1,N_1) to (G_2,N_2). Then (α,β) is an n-isoclinism between (G_1,N_1) and (G_2,N_2) if and only if α is surjective and β is injective.*

Let (α,β) be an n-isoclinism between (G_1,N_1) and (G_2,N_2). Since α is isomorphism and $\alpha(\frac{Z_{n+1}(G_1,N_1)}{Z_n(G_1,N_1)}) = \frac{Z_{n+1}(G_2,N_2)}{Z_n(G_2,N_2)}$, then α induces isomorphism $\bar{\alpha} : \frac{G_1}{Z_{n+1}(G_1,N_1)} \longrightarrow \frac{G_2}{Z_{n+1}(G_2,N_2)}$. Now let $\bar{\beta} = \beta \mid_{\gamma_{n+2}(G_1,N_1)}$. It is clear that $(\bar{\alpha},\bar{\beta})$ is an $(n+1)$-isoclinism between (G_1,N_1) and (G_2,N_2). Therefore each n-isoclinism induces an $(n+1)$-isoclinism. Hence the notion of n-isoclinism provides an equivalence relation on the set of all pairs of groups, which becomes more and more weak as n increases.

In the remainder of this section, we give some structural results in n-isoclinism families. These structural properties provide some interesting conclusions. For instance, let, for two pairs (G_1,N_1) and (G_2,N_2), there exist isomorphisms $\alpha : \frac{G_1}{Z_n(G_1,N_1)} \longrightarrow \frac{G_2}{Z_n(G_2,N_2)}$ and $\beta : \gamma_{n+1}(G_1,N_1) \longrightarrow \gamma_{n+1}(G_2,N_2)$ such that $\alpha(\frac{N_1}{Z_n(G_1,N_1)}) = \frac{N_2}{Z_n(G_2,N_2)}$. We know that if α does not induce β, then we can not deduce that these pairs are not n-isoclinic. In this situation, using a direct method for proving the non n-isoclinism of pairs may not be always logically, whereas applying some of the structural statements helps us to attain the claim.

Lemma 3.4. *Let (α,β) be an n-isoclinism between (G_1,N_1) and (G_2,N_2). Then the following hold:*

(i) $\alpha(xZ_n(G_1,N_1)) = \beta(x)Z_n(G_2,N_2)$, *for every* $x \in \gamma_{n+1}(G_1,N_1)$,

(ii) β *is an operator-isomorphism in the following sense: if* $g_1 \in G_1$, *then* $\beta(x^{g_1}) = \beta(x)^{g_2}$ *in which* $g_2 \in \alpha(g_1Z_n(G_1,N_1))$,

(iii) *if* $Z_n(G_1,N_1) \subseteq H_1 \subset G_1$ *and* $\alpha(\frac{H_1}{Z_n(G_1,N_1)}) = \frac{H_2}{Z_n(G_2,N_2)}$, *then* $(H_1,H_1\cap N_1) \underset{n}{\sim} (H_2,H_2\cap N_2)$,

(iv) *if* M_1 *is a normal subgroup of* G_1 *such that* $M_1 \subseteq \gamma_{n+1}(G_1,N_1)$, *then* $(\frac{G_1}{M_1},\frac{N_1}{M_1}) \underset{n}{\sim} (\frac{G_2}{\beta(M_1)},\frac{N_2}{\beta(M_1)})$.

Proof. The proof of (i) and (ii) are straightforward by the definition. For part (iii), consider $\bar{\alpha} : \frac{H_1}{Z_n(H_1,H_1\cap N_1)} \longrightarrow \frac{H_2}{Z_n(H_2,H_2\cap N_2)}$ with $\bar{\alpha}(\bar{h_1}) = \bar{h_2}$, in which $h_2 \in \alpha(h_1Z_n(G_1,N_1))$, and $\bar{\beta} : \gamma_{n+1}(H_1,H_1\cap N_1) \longrightarrow \gamma_{n+1}(H_2,H_2\cap N_2)$ with $\bar{\beta}(x) = \beta(x)$. It is easy to see that the pair $(\bar{\alpha},\bar{\beta})$ is an n-isoclinism between the pairs $(H_1, H_1 \cap N_1)$ and $(H_2, H_2 \cap N_2)$. For part (iv), put $\bar{G_1} = \frac{G_1}{M_1}$ and $\hat{G_2} = \frac{G_2}{\beta(M_1)}$. Define $\bar{\alpha} : \frac{\bar{G_1}}{Z_n(\bar{G_1},\bar{N_1})} \longrightarrow \frac{\hat{G_2}}{Z_n(\hat{G_2},\hat{N_2})}$ by $\bar{\alpha}(\bar{g_1}Z_n(\bar{G_1},\bar{N_1})) = \hat{g_2}Z_n(\hat{G_2},\hat{N_2})$, in which $g_2 \in \alpha(g_1Z_n(G_1,N_1))$, and $\bar{\beta} : \gamma_{n+1}(\bar{G_1},\bar{N_1}) \longrightarrow \gamma_{n+1}(\hat{G_2},\hat{N_2})$ with $\bar{\beta}([\bar{n}_1, \bar{g}_{1_1}, \ldots, \bar{g}_{1_n}]) = [\hat{n}_2, \hat{g}_{2_1}, \ldots, \hat{g}_{2_n}]$, where $n_2 \in \alpha(n_1Z_n(G_1,N_1))$ and $g_{2_i} \in \alpha(g_{1_i}Z_n(G_1,N_1))$ for $1 \le i \le n$. Now $(\bar{\alpha},\bar{\beta})$ is an n-isoclinism between the pairs $(\bar{G_1},\bar{N_1})$ and $(\bar{G_2},\bar{N_2})$ since (α,β) is an n-isoclinism. □

Definition 3.5. Let (G,N) be a pair of groups. We say that (H,M) is a subpair of (G,N), if $H \le G$ and $M \le N$. Also we say that the pair (K,L) is a quotient pair of (G,N) if K and L are quotient groups of G and N, respectively.

Now, we would like to verify the relation of n-isoclinism between a pair and its subpair or its quotient pair. For example it may be of interest to know that if it is possible a pair and its subpair are n-isoclinic? or which facts do we know about a pair which is n-isoclinic to one of its subpair? or what conditions does a pair need to be n-isoclinic to one of its quotient pair? and so on. The following theorem provides some answers to these questions.

Theorem 3.6. ***(Generalized Bioch Theorem)*** *Let (G,N) be a pair of groups and $n \ge 0$.*

(i) If $H \le G$, then $(H, H\cap N) \underset{n}{\sim} (HZ_n(G,N), (H\cap N)Z_n(G,N))$. In particular, if $G = HZ_n(G,N)$, then $(H, H\cap N) \underset{n}{\sim} (G,N)$. Conversely, if $|\frac{G}{Z_n(G,N)}| < \infty$ and $(H, H\cap N) \underset{n}{\sim} (G,N)$, then $G = HZ_n(G,N)$,

(ii) If M is a normal subgroup of G contained in N, then $(\frac{G}{M}, \frac{N}{M}) \underset{n}{\sim} (\frac{G}{M\cap\gamma_{n+1}(G,N)}, \frac{N}{M\cap\gamma_{n+1}(G,N)})$. In particular, if $M \cap \gamma_{n+1}(G,N) = 1$ then $(G,N) \underset{n}{\sim} (\frac{G}{M}, \frac{N}{M})$. Conversely, if $|\gamma_{n+1}(G,N)| < \infty$ and $(G,N) \underset{n}{\sim} (\frac{G}{M}, \frac{N}{M})$, then $M \cap \gamma_{n+1}(G,N) = 1$.

Proof. The proof is done similar to [1, Lemma 1.3]. □

Corollary 3.7. *Let (G,N) be a pair of groups, $H \le G$, and $M \trianglelefteq G$. Then*

(i) if $N_0 \trianglelefteq G$ and $M \cap \gamma_{n+1}(G,N) = 1$, then $(G,N) \underset{n}{\sim} (\frac{G}{N_0\cap M}, \frac{N}{N_0\cap M})$,

(ii) if $M \cap \gamma_{n+1}(G,N) = 1$, then $(H, H\cap N) \underset{n}{\sim} (\frac{HM}{M}, \frac{HM\cap N}{M})$,

(iii) if $G = HZ_n(G,N)$, then $(\frac{HM}{M}, \frac{HM\cap N}{M}) \underset{n}{\sim} (\frac{G}{M}, \frac{N}{M})$,

(iv) if $K \le G$ and $G = HZ_n(G,N)$, then $(G,N) \underset{n}{\sim} (< H, K >, < H, K > \cap N)$.

The next proposition implies that the relation of n-isoclinism between a pair and some of its subpairs is just related to the existence of α. Moreover, the relation between a pair and some of its quotients depends on the existence of β.

Proposition 3.8. *Let (G,N) be a pair of groups, $H \le G$, $M \trianglelefteq G$, and $M \le N$. Then* *(i)* $(G,N) \underset{n}{\sim} (H, H\cap N)$ *if and only if* $\frac{G}{Z_n(G,N)} \cong \frac{H}{Z_n(H,H\cap N)}$,

(ii) $(G,N) \underset{n}{\sim} (\frac{G}{M}, \frac{N}{M})$ *if and only if* $\gamma_{n+1}(G,N) \cong \gamma_{n+1}(\frac{G}{M}, \frac{N}{M})$.

Proof. In both statements the "only if" parts are trivial. The isomorphism $\frac{G}{Z_n(G,N)} \cong \frac{H}{Z_n(H,H\cap N)}$ implies $G = HZ_n(G,N)$, which proves part(i) by Theorem 3.6. On the other hand, the isomorphism $\gamma_{n+1}(G,N) \cong \gamma_{n+1}(\frac{G}{M}, \frac{N}{M})$ implies the triviality of $M \cap \gamma_{n+1}(G,N)$, which proves part (ii) by Theorem 3.6. □

In the following an equivalent condition for being n-isoclinic is stated.

Lemma 3.9. *Let* (G_1,N_1) *and* (G_2,N_2) *be pairs of groups, then* $(G_1,N_1) \underset{n}{\sim} (G_2,N_2)$ *if and only if there exist normal subgroups* A *and* B *of* G, *and isomorphisms* α *and* β *with these properties:*$A \subseteq Z_n(G_1,N_1)$, $B \subseteq Z_n(G_2,N_2), \alpha : \frac{G_1}{A} \longrightarrow \frac{G_2}{B}$ *such that* $\alpha(\frac{N_1}{A}) = \frac{N_2}{B}, \beta : \gamma_{n+1}(G_1,N_1) \longrightarrow \gamma_{n+1}(G_2,N_2)$ *and* $\beta([n_1, g_{1_1}, \ldots, g_{1_n}]) = [n_2, g_{2_1}, \ldots, g_{2_n}]$ *where for each* $1 \le i \le n$ *and* $j = 1,2$, *we have,* $g_{j_i} \in G_j$, $g_{2_i} \in \alpha(g_{1_i}A)$ *and* $n_2 \in \alpha(n_1 A)$.

Proof. It sufficient to show that $\alpha(\frac{Z_n(G_1,N_1)}{A}) = \frac{Z_n(G_2,N_2)}{B}$. Therefore, let $g_1 \in Z_n(G_1,N_1)$ and $g_2 \in \alpha(g_1 A)$. Take $g_{2_1}, \ldots, g_{2_n} \in G_2$ and $g_{1_i} \in \alpha^{-1}(g_{2_i}B)$, for $i = 1,2,\ldots,n$. Now it follows that $\alpha(\frac{Z_n(G_1,N_1)}{A}) \subseteq \frac{Z_n(G_2,N_2)}{B}$. Similar argument holds for α^{-1}, yielding $\alpha(\frac{Z_n(G_1,N_1)}{A}) \supseteq \frac{Z_n(G_2,N_2)}{B}$. This proves the statement. □

As Hekster illustrated in [6], in an n-isoclinism family of groups there might be some quantities which are the same for any two groups in that family. Following P. Hall [4], Hekster called such quantities "*family invariants*". In the following, we intend to illustrate some family invariants in n-isoclinism classes of pairs of groups. In fact we will list some properties which are the same for any two pairs in an n-isoclinism family of pairs of groups.

Theorem 3.10. *Let* (G_1,N_1) *and* (G_2,N_2) *be* n*-isoclinic pairs of groups. Then for all* $i \ge 0$,

(i) $\gamma_{i+1}(\frac{G_1}{Z_n(G_1,N_1)}, \frac{N_1}{Z_n(G_1,N_1)}) \cong \gamma_{i+1}(\frac{G_2}{Z_n(G_2,N_2)}, \frac{N_2}{Z_n(G_2,N_2)})$,

(ii) $\gamma_{n+1}(G_1,N_1) \cap Z_i(G_1,N_1) \cong \gamma_{n+1}(G_2,N_2) \cap Z_i(G_2,N_2)$.

Proof. The proof is done similar to [6, Theorem 3.12]. □

Theorem 3.11. *Let* (G_1,N_1) *and* (G_2,N_2) *be two isoclinic pairs of groups and* $n \ge 1$. *Then* $\gamma_{n+1}(G_1,N_1) \cong \gamma_{n+1}(G_2,N_2)$.

Proof. Since each n-isoclinism induces an $(n+1)$-isoclinism, the proof is clear. □

Corollary 3.12. *In the isoclinism families of pairs of groups, the nilpotency class of pairs of groups is a family invariant.*

4. Subpair and Quotient Pair Reduction

The definitions of subgroup and quotient irreducible with respect to n-isoclinism of groups can be easily defined for pairs of groups. In this section we first define these concepts and then show that each n-isoclinism family of pairs contains a quotient irreducible pair. Subsequently by considering (G,N) to be a pair of groups, $\mathcal{P}$ a property of groups and $\frac{G}{Z_n(G,N)}$ a $\mathcal{P}$-group, we will find a pair of $\mathcal{P}$-groups which is n-isoclinic to (G,N).

Definition 4.1. Let (G,N) be a pair of groups. We say that (G,N) is subpair irreducible with respect to n-isoclinism if G containing no proper subgroup H satisfying $G = HZ_n(G,N)$. Also (G,N) is called quotient irreducible with respect to n-isoclinism if G containing no non-trivial normal subgroup M such that $M \subseteq N$ and satisfying $M \cap \gamma_{n+1}(G,N) = 1$.

Definition 4.2. A pair (G,N) is called n-stem pair if $Z(G,N) \subseteq \gamma_{n+1}(G,N)$.

Proposition 4.3. *An n-stem pair is quotient irreducible with respect to n-isoclinism.*

Proof. Using Lemma 2.6 (ii) and Theorem 2.1, the proof is straightforward. □

Proposition 4.4. *Let (G,N) be a pair of groups and $Z_n(G,N) \subseteq \gamma_{n+1}(G,N)$. Then (G,N) is both subpair and quotient irreducible with respect to n-isoclinism.*

Proof. The proof is straightforward using Theorem 2.2 (i) and Proposition 4.3. □

Now let (G,N) be subpair irreducible with respect to n-isoclinism. It is easy to see that every element of $Z_n(G,N)$ is non-generator in G. More precisely, the following theorem will determine the main place of the nth term of the upper central series of such pairs of groups. It will also establish a sufficient condition under which the converse of the statement is true.

Theorem 4.5. *If (G,N) is subpair irreducible with respect to n-isoclinism, then $Z_n(G,N) \subseteq \Phi(G) \cap N$. The converse holds if $\frac{Z_n(G,N)}{Z_n(G,N)\cap\gamma_{n+1}(G,N)}$ is finitely generated.*

Proof. The proof is done similar to [6, Theorem 7.4]. □

Recall that the *socle* of G is the group generated by minimal normal subgroups of G.

Theorem 4.6. *A pair (G,N) is quotient irreducible with respect to n-isoclinism if and only if the socle of G is contained in $\gamma_{n+1}(G,N)$ and $\frac{Z_n(G,N)}{Z_n(G,N)\cap\gamma_{n+1}(G,N)}$ is a torsion group.*

The existence of a quotient irreducible pair of groups in each n- isoclinism family of pairs of groups is proved in the next theorem.

Theorem 4.7. *Let (G, N) be a pair of groups. Then there exists a normal subgroup M of G contained in N such that $(G, N) \underset{n}{\sim} (\frac{G}{M}, \frac{N}{M})$ and $(\frac{G}{M}, \frac{N}{M})$ is quotient irreducible with respect to n-isoclinism.*

Proof. Applying $\mathcal{M} = \{M | M \subseteq N, M \trianglelefteq G$ and $M \cap \gamma_{n+1}(G, N) = 1\}$ instead of $\mathcal{N}$ in [6, Theorem 7.6] and following up a similar proof, the theorem will be proved. □

Lemma 4.8. *Let (G, N) be a pair of finite groups such that $\frac{G}{Z_n(G,N)}$ is a π-group. Then there exists a subpair (H, M) of (G, N) such that (H, M) is a pair of π-groups.*

Proof. By substituting $Z_n(G, N)$ with $Z_n(G)$ and using a similar proof to [6, Lemma 7.8] the result follows. □

Corollary 4.9. *Let (G, N) be a pair of groups, such that $\frac{G}{Z_n(G,N)}$ be a finite π-group. Then there exists a pair (H, M) of finite π-group with $(G, N) \underset{n}{\sim} (H, M)$. In particular, $\gamma_{n+1}(G, N)$ is a finite π-group.*

Corollary 4.10. *Let (G, N) be a pair of finite groups. If (G, N) is a subpair irreducible with respect to n-isoclinism and $\frac{G}{Z_n(G,N)}$ is a π-group, then (G, N) is a pair of π-groups.*

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