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On the Configurations with n Points and Two Distances

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Abstract

In this paper we investigate the geometric structures of $M(n, 2)$ containing n points in $\mathbb{R}^3$ having two distinct distances. We will show that up to pseudo-equivalence there are 5 constructible models for $M(4, 2)$ and 17 constructible models for $M(5, 2)$.

Keywords: Constructible models, distinct distances, isomorphic graphs, pseudo-equivalent models.

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1. Introduction

The study of the geometric structures $M(n, k)$ containing n points in $\mathbb{R}^3$ having k distinct distances is the 3 dimensional version of the Erdos's problem [5]: What is the minimum number of distinct distances determined by n points in the plane? Moser [7] and Chung [2] found some lower bounds for this number. We refer to $[1, 3, 6]$ for further study on the problem.

In [8], we introduced the structures $M(n, k)$. Our motivation is the study of the geometric structure of the molecules in terms of the number of distinct distances between atoms. Here, the points are used as atoms. For example a molecule of methane, CH_4, is structured with four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid, so this molecule

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is a model for $M(5,2)$, namely five points(atoms) with 2 distinct distances. Our goal is to obtain a *necessary* (not sufficient) condition to determine which of the $M(n,k)$ structures are eligible for a molecule. For example we can't find 5 points in $\mathbb{R}^3$ whose pair-wise distances are equal, so it is impossible to find a molecule with structure $M(5,1)$. In this case, we say $M(5,1)$ is not *constructible.* We also recognize when two structures for $M(n,k)$ are equivalent. In [8], we have defined the concepts of *similarity* and *pseudo equivalence* for the constructible models and have shown that up to similarity there exist uncountable models $M(n,k)$, and up to pseudo equivalence the number of possible models for $M(n,k)$ is finite. Here, we investigate the case $k=2$, namely structures containing n points having two distinct distances.

2. Constructible Models and Pseudo Equivalence

The number of real numbers which can be attributed to the pairwise distances of n points in $\mathbb{R}^3$ is at most $n(n-1)/2$, the number of edges.

Definition 2.1. Let $p_1, p_2, \ldots, p_n$ be points in $\mathbb{R}^3$ such that

$$card\{d(p_i,p_j)|i \neq j,\ i,j = 1,2,\ldots,n\} = k,$$

(d is the Euclidean distance). Then the set $\{p_1, p_2, \ldots, p_n\}$ is called a model for $M(n,k)$. We say $M(n,k)$ is constructible if there exists a model for it.

For example there is at most four points in $\mathbb{R}^3$ such that their pairwise distances are equal (the vertices of a regular tetrahedron), and any other points connect to this model, make an additional distances. So $M(n,1)$ *is not constructible for* $n > 4$.

The following statements have been proved in [8]:

- For each $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $M(n,k)$ is constructible.
- $M(2k,k)$, $M(2k+1,k)$, and $M(2k+2,k)$ are constructible for all $k \in \mathbb{N}$.
- For each $k \in \mathbb{N}$ there exists a maximum $N \in \mathbb{N}$ such that $M(N,k)$ is constructible.

Now we prove some facts for $k=2$.

Proposition 2.2. Let $n > 4$. In every constructible model for $M(n,2)$, there is a point p_j such that $card\{d(p_i,p_j)|i \neq j,\ i,j = 1,2,\ldots,n\} = 2$.

Proof. If $card\{d(p_i,p_j)|i \neq j,\ i,j = 1,2,\ldots,n\} = 1$ for all j, then we have only one distance. So it is a model for $M(n,1)$. □

Proposition 2.3. Let $n \geq 3$. If $M(n,2)$ is not constructible then $M(n+1,2)$ is not constructible too.

Proof. Suppose $M(n+1,2)$ is constructible. Let $p_1, p_2, \ldots, p_{n+1}$ be the points of this model and p_1 be the point which has two distances from other points. Define an $(n+1)\times(n+1)$ matrix (a_{ij}) correspond to this model by taking $a_{ij} = d(p_i, p_j)$, (a symmetric matrix which the elements of the main diagonal are zero). In the first column we have n nonzero elements which are equal to two distances. Since $n \geq 3$, at least one of the distances appears in two rows. Assume that one distance is repeated in the jth row. By deleting this row and the jth column we obtain an $n \times n$ matrix which represents a model for $M(n,2)$. Note that $\{p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_{n+1}\}$ is a model for $M(n,2)$. So, $M(n,2)$ is constructible. □

Corollary 2.4. For $n \geq 3$, each model of $M(n+1,2)$ is obtained by adding a point to a model of $M(n,2)$.

Definition 2.5. Let $m, m_1, m_2, \ldots, m_k$ are natural numbers such that

$$m = m_1 + m_2 + \cdots + m_k \ , \quad 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k.$$

Then the summand $m_1 + m_2 + \cdots + m_k$ is called a k-partition for m.

Notation. We correspond to each model $M(n,k)$, a k-partition of $m = n(n-1)/2$ (the number of edges) as follow. Let $d_1, d_2, \ldots, d_k$ be the distances in this model and m_j be the number of edge with length d_j. Without loss of generality we can assume $m_1 \leq m_2 \leq \cdots \leq m_k$. Then the number of all edges is

$$m = m_1 + m_2 + \cdots + m_k.$$

We also correspond to each model $M(n,k)$, a graph with n vertices in which the edges with same length have same color.

Definition 2.6. Let $n, k \in \mathbb{N}$, and M_1 and M_2 be models for $M(n,k)$ whose points are $\{p_1, p_2, \ldots, p_n\}$ and $\{q_1, q_2, \ldots, q_n\}$ respectively. We say M_1 and M_2 are pseudo- equivalent if their corresponding graphs are isomorphic.

Note that if the partitions of two models are not identical, they can't be pseudo-equivalent. We also emphasize that it is possible that two pseudo equivalent models have different shape geometrically, but we can obtain one from the other by moving points. So, the equality of two edges saved by this *moving*. It is easy to check that pseudo equivalence is an equivalence relation.

Example 2.7. Consider the models in Figure 1 for $M(4,2)$. They are the vertices of a pyramid, a square, and a kite.

These models are not pseudo-equivalent. The partitions of (a) and (b) are 3+3 and 2+4, respectively. So, they are not pseudo-equivalent. Both (b) and (c) have same partition 2+4. But in (b) the same length edges are not adjacent while in (c) they are. In fact (c) has a point with the same distance to other vertices (upper point) and (b) has no such point.

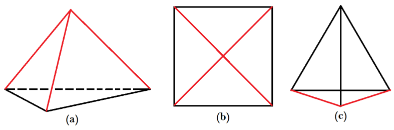


Figure 1: Some shapes for $M(4,2)$.

Proposition 2.8. Up to pseudo-equivalence, there is only one model For $M(3,2)$.

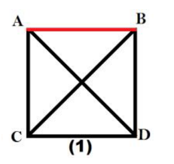
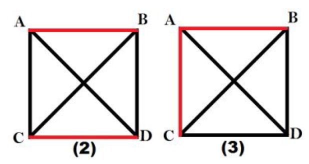
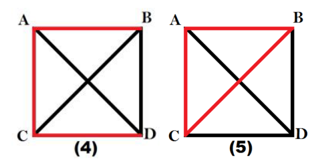
Proof. The number of edge for such model is $\binom{3}{2} = 3$, the only 2 partition for 3 is $1+2$, and the only possible geometric shape is an isosceles triangle, $d_1 = 1, 0 < d_2 \neq 1$. $\square$

In the following two sections, we'll determine all possible constructible models for $M(4,2)$ and $M(5,2)$ up to pseudo-equivalence.

3. Pseudo-Equivalent Models for $M(4,2)$

The number of edges in a $M(4,2)$ model is $\binom{4}{2} = 6$. The $2-$partitions for 6 are $1+5$, $2+4$, and $3+3$. The graphs correspond to these partitions have been shown in Table 1.

Table 1: $2-$Partitions of 6 with corresponding graphs.

1+5	2+4	3+3
		

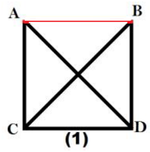
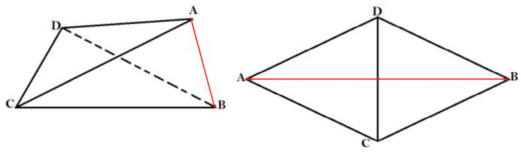
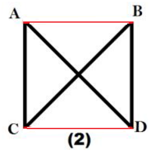
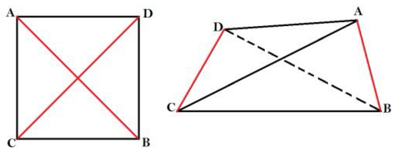
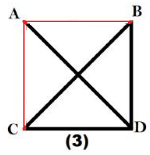
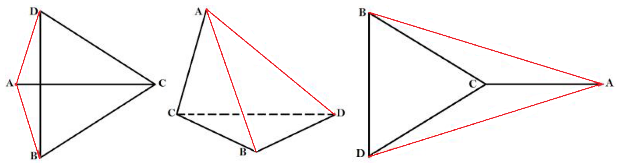
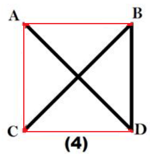
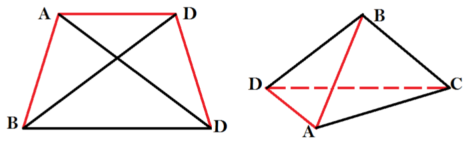
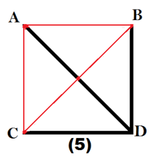
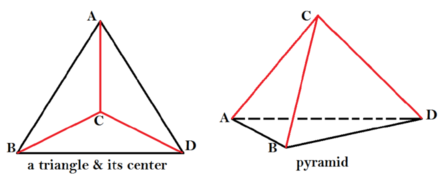
All possible shapes for these models have been shown in Table 2. We computed the distances carefully by taking $d_1 = 1$. As you see in the Table 2, some shapes are planar (rhombus, triangle, square, kite, and trapezoid) and the rest are several pyramids. For example the vertices of a pyramid or a rhombus are the geometrical configurations for model (1), and the vertices of a square and a pyramid indicate shapes for model (2). The shapes for models (3), (4), and (5) are obtained similarly. The geometric shapes in each row are pseudo equivalent.

Thus, we have proved the following theorem.

Theorem 3.1. Up to pseudo-equivalence, there are exactly five models for $M(4,2)$.

Corollary 3.2. The number of pseudo-equivalence models for $k = 2$ is finite.

Table 2: All possible shapes for $M(4,2)$.

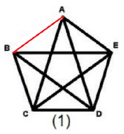
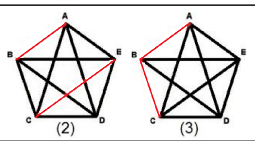
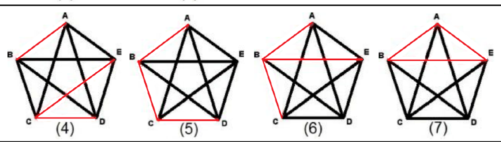
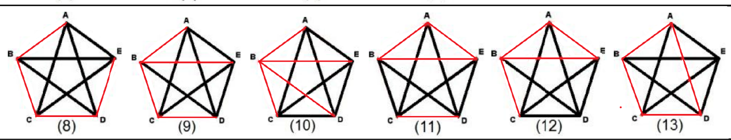
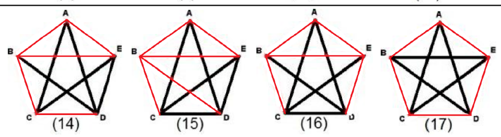
Partitions	Graphs	Geometric Shapes (Constructible Models)
1+5		 $(d_1 = 1, 1 \neq d_2 \leq \sqrt{3})$
2+4		 $(d_1 = 1, 1 \neq d_2 \leq \sqrt{2})$
		 $\left(d_1 = 1, \sqrt{2-\sqrt{3}} \leq d_2 \leq \sqrt{2+\sqrt{3}}, d_2 \neq 1\right)$
3+3		 $\left(d_1 = 1, 1 \neq d_2 \leq \sqrt{2 - 2\cos\frac{3\pi}{5}}\right)$
		 $(d_1 = 1, \sqrt{3}/3 \leq d_2 \neq 1)$

Proof. Up to pseudo-equivalence, $M(4,2)$ has five models. According to the Corollary 2.4, the models of $M(n,2)$ are obtained by adding $n-4$ points to the models of $M(4,2)$. We can add a finite number of points to the models of $M(4,2)$ (pyramids and planar shapes) such that the number of distances remains constant, so the number of constructible models for $M(n,2)$ is finite. □

4. Pseudo-Equivalent Models for $M(5,2)$

The number of edges in an $M(5,2)$ model is $\binom{5}{2} = 10$. The 2−partitions for 10 are $1+9$, $2+8$, $3+7$, $4+6$, and $5+5$. These partitions and their corresponding graphs have been shown in Table 3.

Table 3: 2−Partitions of 10 with corresponding graphs.

Partitions	Graphs
1+9	 (1)
2+8	 (2) (3)
3+7	 (4) (5) (6) (7)
4+6	 (8) (9) (10) (11) (12) (13)
5+5	 (14) (15) (16) (17)

We investigate each graph to find some possible geometrical model for it. This task needs some calculations and computations. We have done the calculations exactly. The models (9), (12), and (14) need analytical calculations, so we describe them separately, the rest models have been shown in Table 4 with a brief description.

Table 4: Models for $M(5,2)$.

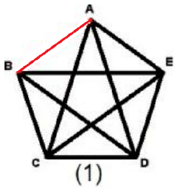
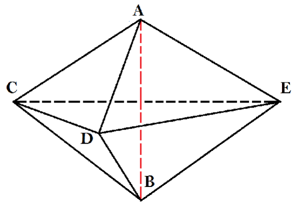
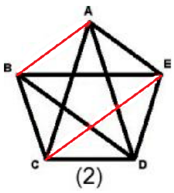
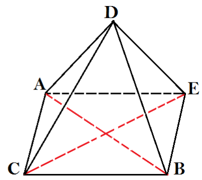
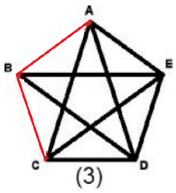
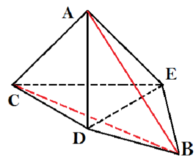
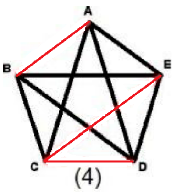
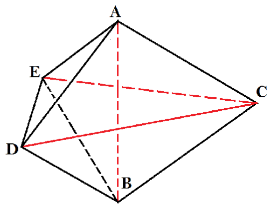
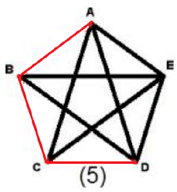
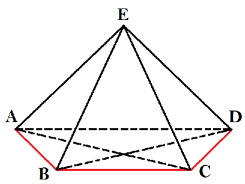
Graph	Geometrical model
	
$ACDE$ and $BCDE$ are regular pyramids with common face CDE. $d_1 = 1$, $d_2 = 2\sqrt{2/3}$.	
	
$ACBE$ is a square with side length 1, and D is a point whose distance from vertex of square is 1. $d_1 = 1$, $d_2 = \sqrt{2}$.	
	
$ACDE$ is a regular pyramid and EDB is a equilateral triangle whose plane is perpendicular to AC. $d_1 = 1$, $d_2 = \sqrt{6+2\sqrt{6}}/2$.	
	
$AEDC$ and $BEDC$ are two pyramids with common face EDC. $d_1 = 1$, $d_2 = \sqrt{17 \pm \sqrt{161}}/4$.	
	
A, B, C, and D are consecutive vertices of a regular pentagon and E is a point whose distance from each vertex is equal to the diagonal of the pentagon. $d_1 = 1$, $d_2 = \sqrt{2 - 2\cos 3\pi/5}$.	

Table 4: Continued

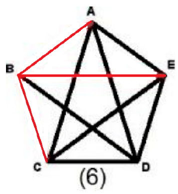
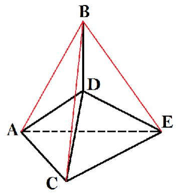
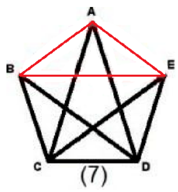
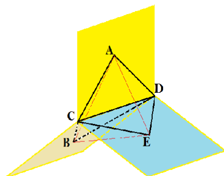
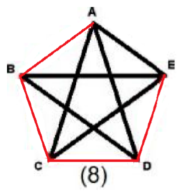
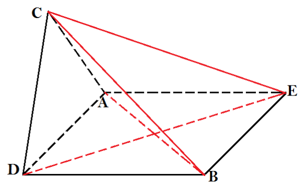
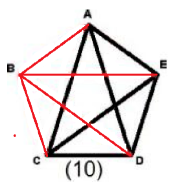
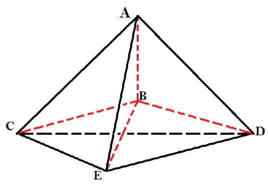
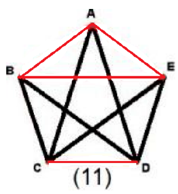
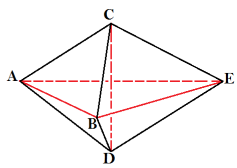
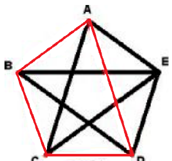
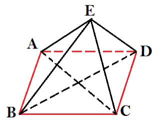
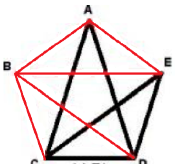
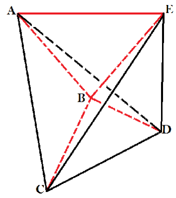
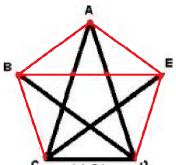
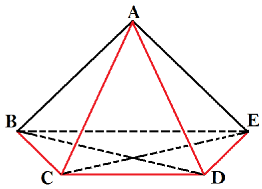
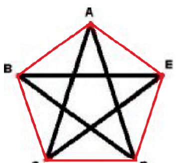
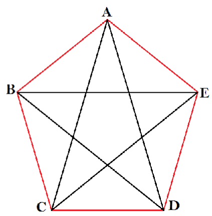
Graph	Geometrical model
	
$ACDE$ is a regular pyramid and B lies on the line through D perpendicular to $\triangle ACE$, and $d(D,B)=d(A,D)$. $d_1=1$, $d_2=\sqrt{2\pm 2\sqrt{2/3}}$.	
	
ACD, BCD, and ECD are equilateral triangles which lie on three halfplans with common boundary and pairwise angle equal to $2\pi/3$. $d_1=1$, $d_2=3/2$.	
	
$ADBE$ is a square in the xy-plan, and CAD is an equilateral triangle in the xz-plan (i.e. the square is perpendicular to the triangle). $d_1=1$, $d_2=\sqrt{2}$.	
	
$ACDE$ is a regular pyramid and B is its center. $d_1=1$, $d_2=\sqrt{6}/4$.	
	
ABE is an equilateral triangle and CD is a line-segment perpendicular to the plane contain ABE. $d_1=1$, $d_2=\sqrt{7/12}$.	

Table 4: Continued

Graph	Geometrical model
 (13)	
$ABCD$ is a square and E is a point whose distance from each vertex is equal to the diagonal of square. $d_1 = 1$, $d_2 = \sqrt{2}$.	
 (15)	
B is the center of a sphere, and $ABCD$ is a pyramid whose vertices lie on the sphere. $d_1 = 1$, $d_2 = 2\sqrt{1 - \left(\frac{\sqrt{3}\pm\sqrt{35}}{8}\right)^2}$.	
 (16)	
A, C, D, and E are consecutive vertices of a regular pentagon, and ADC and ABE are equilateral triangles. $d_1 = 1$, $d_2 = \sqrt{2 - 2\cos 3\pi/5}$.	
 (17)	
$ABCDE$ is a regular pentagon. $d_1 = 1$, $d_2 = \sqrt{2 - 2\cos 3\pi/5}$.	

Description of model (9): We show that this model is constructible.

Figure 2: Graph and geometrical shape for model(9).

In Figure 2, AEC is an equilateral triangle, so one can take $A = (-1/2, 0, 0)$, $E = (1/2, 0, 0)$, and $C = (0, \sqrt{3}/2, 0)$. Since AED is an equilateral triangle, the coordinate of D will be as $D = (0, \sqrt{3}/2\cos\theta, \sqrt{3}/2\sin\theta)$ for some θ (we rotate $\triangle AEC$ through θ about AE to obtain $\triangle AED$). We prove that there exist such θ and so this model is constructible.

Let $B = (x, y, z)$. Since $d(A, B) = d(E, B)$, we have

$$(x + 1/2)^2 + y^2 + z^2 = (x - 1/2)^2 + y^2 + z^2,$$

so $x = 0$. Similarly, from $d(E, B) = d(B, C)$ we have $(-1/2)^2 + y^2 + z^2 = (y - \sqrt{3}/2)^2 + z^2$, which implies $y = 1/(2\sqrt{3})$. On the other hand $d(A, B) = d(C, D)$, so one can write $z^2 = 7/6 - 3/2\cos\theta$. In a similar way, from $d(B, D) = 1$ we have $z^2 = 2/12 - 1/2cos + \sqrt{3}z\sin\theta$. Thus

$$(1 - 2\cos\theta)^2 = 3(1 - \cos^2\theta)(7/6 - 3/2\cos\theta).$$

By taking $t := \cos\theta$, the last equation is reduced to $9t^3 - 15t^2 - t + 5 = 0$. This equation has a root in the interval $(-1, 0)$ and another one in $(0, 1)$. So, there exist such θ, and the coordinates of B are $\big(0, 1/(2\sqrt{3}), (1 - 2\cos\theta)/(\sqrt{3}\sin\theta)\big)$, where θ satisfies the following equation

$$9\cos^3\theta - 15\cos^2\theta - \cos\theta + 5 = 0.$$

The above shape has been drown by assuming, $-1 < \cos\theta < 0$. If $0 < \cos\theta < 1$, one can obtain another shape for this model. $d_1 = 1$ and $d_2 = \sqrt{3/2 - 3/2\cos\theta}$.

Description of model (12): We find five points which make a geometrical model for model (12). This model has been shown in Figure 3.

Let $B = (0, 0, 0)$, $A = (1/2, \sqrt{3}/2, 0)$, $E = (-1/2, \sqrt{3}/2, 0)$ and $C = (0, \cos\theta, \sin\theta)$. Since D has same distance from each of the other points, by taking $D = (x, y, z)$ a simple calculation shows that $x = 0$, $y = \sqrt{3}/3$, $z^2 = 5/3 - \sqrt{3}\cos\theta$, and

$$\sqrt{3}/3\cos\theta + \sqrt{1 - \cos^2\theta}\sqrt{5/3 - \sqrt{3}\cos\theta} - 1/2 = 0.$$

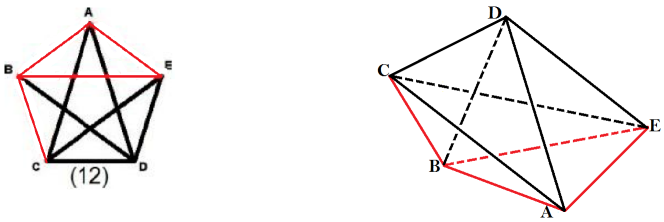


Figure 3: Graph and geometrical shape for model(12).

Let $t = \cos\theta$, then the above equation has a root in the interval $(-1, 0)$, i.e. there is a $\pi/2 < \theta < \pi$ which satisfies in the above equation. $d_1 = 1$, $d_2 = \sqrt{2 - \sqrt{3}\cos\theta}$.
Description of model (14): As previous models, we make this model using coordinate system. Figure 4 shows the geometrical shape for this model.

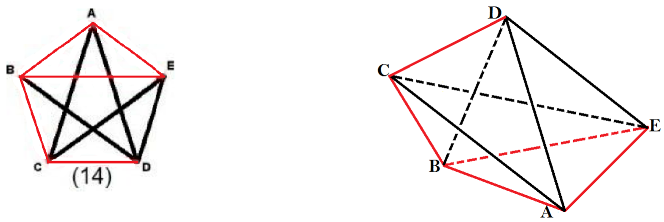


Figure 4: Graph and geometrical shape for model(14).

Take $B = (0, 0, 0)$, $A = (1/2, \sqrt{3}/2, 0)$, and $E = (-1/2, \sqrt{3}/2, 0)$. Let D and C be the points in the yz-plane such that $D = (0, \sqrt{3}/3, t)$, $C = (0, y, z)$, and $BC = 1 = CD$, $BD = CD$. Then we have $1 = d(B, C) = y^2 + z^2$, $1 = (\sqrt{3}/3 - y)^2 + (t - z)^2$, $1/3 + t^2 = 1/4 + (y - \sqrt{3}/2)^2 + z^2$. By compounding these equations one can write

$$12\sqrt{3}y^3 - 45y^2 + 8\sqrt{3}y + 8 = 0.$$

This equation has a root in the interval $(-1, 0)$, so t and z are obtained as $t = \sqrt{1 - y^2 + (y - \sqrt{3}/2)^2 - 1/2}$ and $z = \sqrt{1 - y^2}$. $d_1 = 1$, and $d_2 = \sqrt{1/3 + t^2}$.

By considering all above models we have proved the following theorem.

Theorem 4.1. Up to pseudo-equivalence, there are exactly seventeen models for $M(5, 2)$.

5. Summary and Conclusions

According to the previous sections, up to pseudo-equivalence, the number of constructible models for $M(3,2)$, $M(4,2)$, and $M(5,2)$ is 1, 5, and 17 respectively. Because of the variety of possible models, it is more difficult to obtain the structures of $M(6,2)$. We know that $M(6,2)$ is constructible, since $M(2k+2,2)$ is constructible. Croft [4] proved that no configuration of 7 points with 2 distinct distances exists, that is $M(7,2)$ is not constructible. It follows from the Proposition 2.3 that $M(n,2)$ is not constructible for all $n > 7$. So, the next step is to complete the Table 5.

Table 5: The number of constrictible models.

Models	Number of constructible models
$M(3,2)$	1
$M(4,2)$	5
$M(5,2)$	17
$M(6,2)$	?

To realize the difficulty of investigating the possible models for $M(6,2)$, notice that such model has 15 edges, and 2−partitions of 15 are $1+14$, $2+13$, $3+12$, $4+11$, $5+10$, $6+9$, and $7+8$. Each partition has several cases (graphs). But not all graphs are constructible. For example the partition $1+14$ is not constructible. This argument can be continued for $k = 3, 4, \ldots$.

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