

On Eccentricity Version of Laplacian Energy of a Graph

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Abstract

The energy of a graph G is equal to the sum of absolute values of the eigenvalues of the adjacency matrix of G , whereas the Laplacian energy of a graph G is equal to the sum of the absolute value of the difference between the eigenvalues of the Laplacian matrix of G and average degree of the vertices of G . Motivated by the work from Sharafadini et al. [R. Sharafadini, H. Panahbar, Vertex weighted Laplacian graph energy and other topological indices, *J. Math. Nanosci.* 6 (2016) 57 – 65], in this paper we investigate the eccentricity version of Laplacian energy of a graph G .

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1. Introduction

Let G be a simple graph with n vertices and m edges. Let the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$ respectively. The degree of a vertex v , denoted by $d_G(v)$, is the number of vertices adjacent to v . For any two vertices $u, v \in V(G)$, the distance between u and v is denoted by $d_G(u, v)$ and is given by the number of edges in the shortest path connecting u and v . Also we denote the sum of distances between $v \in V(G)$ and every other vertices in G by $d(x|G)$, i.e., $d(x|G) = \sum_{v \in V(G)} d_G(x, v)$. The eccentricity of a vertex v , denoted by $\varepsilon_G(v)$, is the largest distance from v to any other vertex u of G . The total eccentricity of a graph is denoted by $\zeta(G)$ and is equal to sum of eccentricities of all the vertices of the graph. Let $A = [a_{ij}]$ be the adjacency matrix of G and let $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A which are the eigenvalues of the graph G . The energy of a graph

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is introduced by Ivan Gutman in 1978 [7] and defined as the sum of the absolute values of its eigenvalues and is denoted by $E(G)$. Thus

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

A large number of results on the graph energy have been reported, see for instance [2, 8, 10, 13]. Motivated by the success of the theory of graph energy, other different energy like quantities have been proposed and studied by different researcher. Let $D(G) = [d_{ij}]$ be the diagonal matrix associated with the graph G , where $d_{ii} = d_G(v_i)$ and $d_{ij} = 0$ if $i \neq j$. Define $L(G) = D(G) - A(G)$, where $L(G)$ is called the Laplacian matrix of G . Let $\mu_1, \mu_2, \dots, \mu_n$ be the Laplacian eigenvalues of G . Then the Laplacian energy of G is defined as [9]

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|.$$

Various studies on Laplacian energy of graphs were reported in the literature [1, 3, 6, 12, 18]. Analogues to Laplacian energy of a graph, a different new type of graph energy was introduced and in this present study, inspired by the work in [16], we investigate the eccentricity version of Laplacian energy of a graph denoted by $LE_\varepsilon(G)$. In this case, we define the Laplacian eccentricity matrix as $L_\varepsilon(G) = \varepsilon(G) - A(G)$, where $\varepsilon(G) = [e_{ij}]$ is the $(n \times n)$ diagonal matrix of G with $e_{ii} = \varepsilon_G(v_i)$ and $e_{ij} = 0$ if $i \neq j$. Here, $\varepsilon_G(v_i)$ is the eccentricity of the vertex $v_i, i = 1, 2, \dots, n$. Let $\mu'_1, \mu'_2, \dots, \mu'_n$ be the eigenvalues of the matrix $L_\varepsilon(G)$. Then the eccentricity version of Laplacian energy of G is defined as

$$LE_\varepsilon(G) = \sum_{i=1}^n \left| \mu'_i - \frac{\zeta(G)}{n} \right|.$$

Recall that $\zeta(G)/n$ is the average vertex eccentricity. In this paper, we first calculate some basic properties and then establish some upper and lower bounds for $LE_\varepsilon(G)$.

2. Main Results

We know that, the ordinary and Laplacian graph eigenvalues obey the following relations:

$$\sum_{i=1}^n \lambda_i = 0; \quad \sum_{i=1}^n \lambda_i^2 = 2m$$

$$\sum_{i=1}^n \mu_i = 2m; \quad \sum_{i=1}^n \mu_i^2 = 2m + \sum_{i=1}^n d(v_i)^2.$$

As the Laplacian spectrum is denoted by $\mu'_1, \mu'_2, \dots, \mu'_n$, let $\nu'_i = \left| \mu'_i - \frac{\zeta(G)}{n} \right|$. Recall that the first Zagreb eccentricity index of a graph is denoted by $E_1(G)$ and

is equal to the sum of square of all the vertices of the graph G . Thus, we have $E_1(G) = \sum_{i=1}^n \varepsilon(v_i)^2$ (for details see [4, 11, 17]). In the following, now we investigate some basic properties of μ'_i and ν'_i .

Lemma 2.1. *The eigenvalues $\mu'_1, \mu'_2, \dots, \mu'_n$ satisfy the following relations*

$$(i) \sum_{i=1}^n \mu'_i = \zeta(G) \text{ and } (ii) \sum_{i=1}^n \mu_i'^2 = E_1(G) + 2m.$$

Proof. (i) Since, the trace of a square matrix is equal to the sum of its eigenvalues, we have

$$\sum_{i=1}^n \mu'_i = \text{tr}(EL_\varepsilon(G)) = \sum_{i=1}^n [\varepsilon(v_i) - a_{ii}] = \zeta(G).$$

(ii) Again we have,

$$\begin{aligned} \sum_{i=1}^n \mu_i'^2 &= \text{tr}[(\varepsilon(G) - A(G))(\varepsilon(G) - A(G))^T] \\ &= \text{tr}[(\varepsilon(G) - A(G))(\varepsilon(G)^T - A(G)^T)] \\ &= \text{tr}[\varepsilon(G)\varepsilon(G)^T - A(G)\varepsilon(G)^T - A(G)^T\varepsilon(G) + A(G)A(G)^T] \\ &= \sum_{i=1}^n \varepsilon(v_i)^2 + \sum_{i=1}^n \lambda_i^2 \\ &= E_1(G) + 2m. \end{aligned}$$

□

Lemma 2.2. *The eigenvalues $\nu'_1, \nu'_2, \dots, \nu'_n$ satisfy the following relations*

$$(i) \sum_{i=1}^n \nu'_i = 0 \text{ and } (ii) \sum_{i=1}^n \nu_i'^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m.$$

Proof. (i) We have from definition,

$$\sum_{i=1}^n \nu'_i = \sum_{i=1}^n \left(\mu'_i - \frac{\zeta(G)}{n} \right) = \sum_{i=1}^n \mu'_i - \zeta(G) = 0.$$

(ii) Again, similarly we have

$$\begin{aligned} \sum_{i=1}^n \nu_i'^2 &= \sum_{i=1}^n \left(\mu'_i - \frac{\zeta(G)}{n} \right)^2 \\ &= \sum_{i=1}^n \left[\mu_i'^2 + \frac{\zeta(G)^2}{n^2} - 2\mu'_i \frac{\zeta(G)}{n} \right] \\ &= E_1(G) + 2m + \frac{\zeta(G)^2}{n} - 2 \frac{\zeta(G)^2}{n}, \end{aligned}$$

which proves the desired result. \square

Lemma 2.3. *The eigenvalues $\nu'_1, \nu'_2, \dots, \nu'_n$ satisfy the following relations*

$$\left| \sum_{i < j} \nu'_i \nu'_j \right| = \frac{1}{2} \left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right].$$

Proof. Since $\sum_{i=1}^n \nu'_i = 0$, so we can write $\sum_{i=1}^n \nu_i'^2 = -2 \sum_{i < j} \nu'_i \nu'_j$. Thus,

$$2 \left| \sum_{i < j} \nu'_i \nu'_j \right| = \sum_{i=1}^n \nu_i'^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m.$$

Hence the desired result follows. \square

As an example, in the following, we now calculate the eccentric version of Laplacian energy and corresponding spectrum of two particular type of graphs, namely, complete graph and complete bipartite graph.

Example 2.4. Let K_n be the complete graph with n vertices, then

$$L_\varepsilon(K_n) = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ -1 & 1 & -1 & \dots & -1 \\ -1 & -1 & 1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & 1 \end{pmatrix}.$$

Its characteristic equation is

$$(\mu' - 2)^{(n-1)}(\mu' - (2 - n)) = 0.$$

Thus the eccentricity Laplacian spectrum of K_n is given by

$$\text{spec}_\varepsilon(G) = \begin{pmatrix} (1 - n) & 1 \\ 1 & (n - 1) \end{pmatrix}$$

and hence $EL_\varepsilon(G) = 2(n - 1)$.

Example 2.5. Let $K_{m,n}$ be the complete graph with $(m + n)$ vertices and mn

edges, then

$$L_\varepsilon(K_{n,n}) = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 2 & 0 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 0 & 2 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 2 & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 2 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & -1 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

Its characteristic equation is

$$(\mu' - 2)^{2(n-1)} (\mu' - (2 + n)) (\mu' - (2 - n)) = 0.$$

So, the eccentricity Laplacian spectrum of $K_{n,n}$ is given by

$$spec_\varepsilon(G) = \begin{pmatrix} -n & n & 0 \\ 1 & 1 & 2(n-1) \end{pmatrix}$$

and hence $EL_\varepsilon(K_{n,n}) = 2n$.

Note that, from the above two examples, the properties of the eigenvalues $\nu'_1, \nu'_2, \dots, \nu'_n$ can be verified easily. In the following, next we investigate some upper and lower bounds of eccentricity version of Laplacian energy of a graph G .

Theorem 2.6. *Let G be a connected graph of order n and size m , then*

$$EL_\varepsilon(G) \geq 2\sqrt{m + \frac{1}{2}(E_1(G) - \frac{1}{n}\zeta(G)^2)}.$$

Proof. We have from definition, $EL_\varepsilon(G) = \sum_{i=1}^n |\nu'_i|$. So we can write,

$$EL_\varepsilon(G)^2 = \sum_{i=1}^n \nu'^2_i + 2 \sum_{i < j} |\nu'_i \nu'_j| \geq \left(E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right) + 2 \sum_{i < j} |\nu'_i \nu'_j|.$$

Hence using Lemma 2.3, the desired result follows. □

Theorem 2.7. *Let G be a connected graph of order n and size m ; and ν'_1 and ν'_n are maximum and minimum absolute values of ν'_i s, then*

$$EL_\varepsilon(G) \geq \sqrt{nE_1(G) - \zeta(G)^2 + 2mn - \frac{n^2}{4}(\nu'_1 - \nu'_n)^2}.$$

Proof. Let a_i and b_i , $1 \leq i \leq n$ are non-negative real numbers, then using the Ozeki's inequality [14], we have

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

where $M_1 = \max(a_i)$, $m_1 = \min(a_i)$, $M_2 = \max(b_i)$ and $m_2 = \min(b_i)$. Let $a_i = |\nu'_i|$ and $b_i = 1$, then from above we have

$$\sum_{i=1}^n |\nu'_i|^2 \sum_{i=1}^n 1^2 - \left(\sum_{i=1}^n |\nu'_i| \right)^2 \leq \frac{n^2}{4} (\nu'_1 - \nu'_n)^2.$$

Thus, we can write

$$\begin{aligned} EL_\varepsilon(G)^2 &\geq n \sum_{i=1}^n |\nu'_i|^2 - \frac{n^2}{4} (\nu'_1 - \nu'_n)^2 \\ &= n \left(E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right) - \frac{n^2}{4} (\nu'_1 - \nu'_n)^2, \end{aligned}$$

from where the desired result follows. \square

Theorem 2.8. Let G be a connected graph of order n and size m ; and ν'_1 and ν'_n are maximum and minimum absolute values of ν'_i s, then

$$EL_\varepsilon(G) \geq \frac{1}{(\nu'_1 + \nu'_n)} \left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m - n\nu'_1\nu'_n \right].$$

Proof. Let a_i and b_i , $1 \leq i \leq n$ are non-negative real numbers, then from Diaz-Metcalf inequality [5], we have

$$\sum_{i=1}^n b_i^2 + mM \sum_{i=1}^n a_i^2 \leq (m + M) \left(\sum_{i=1}^n a_i b_i \right),$$

where, $ma_i \leq b_i \leq Ma_i$. Let $a_i = 1$ and $b_i = |\nu'_i|$, then from above we have

$$\sum_{i=1}^n |\nu'_i|^2 + \nu'_1 \nu'_n \sum_{i=1}^n 1^2 \leq (\nu'_1 + \nu'_n) \left(\sum_{i=1}^n |\nu'_i| \right).$$

Now, since

$$EL_\varepsilon(G) = \sum_{i=1}^n |\nu'_i|$$

and $\sum_{i=1}^n \nu_i'^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m$, we get

$$E_1(G) - \frac{\zeta(G)^2}{n} + 2m + n\nu_1'\nu_n' \leq (\nu_1' + \nu_n')EL_\varepsilon(G),$$

from where the desired result follows. □

Theorem 2.9. *Let G be a connected graph of order n and size m , then*

$$EL_\varepsilon(G) \leq \sqrt{\left[nE_1(G) - \zeta(G)^2 + 2mn \right]}.$$

Proof. Using the Cauchy-Schwarz inequality to the vectors $(|\nu_1'|, |\nu_2'|, \dots, |\nu_n'|)$ and $(1, 1, \dots, 1)$, we have

$$\sum_{i=1}^n |\nu_i'| \leq \sqrt{n} \sqrt{\sum_{i=1}^n |\nu_i'|^2}$$

Thus from definition, we have

$$\begin{aligned} EL_\varepsilon(G) = \sum_{i=1}^n |\nu_i'| &\leq \sqrt{n} \sqrt{\sum_{i=1}^n |\nu_i'|^2} \\ &= \sqrt{n} \sqrt{\sum_{i=1}^n \left| \mu_i' - \frac{\zeta(G)}{n} \right|^2} = \sqrt{n} \sqrt{\left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right]}. \end{aligned}$$

Hence the desired result follows. □

Theorem 2.10. *Let G be a connected graph of order n and size m , then*

$$EL_\varepsilon(G) \geq \frac{2\sqrt{\nu_1'\nu_n'}}{\nu_1' + \nu_n'} \sqrt{\left[nE_1(G) - \zeta(G)^2 + 2mn \right]}.$$

Proof. We have, from Polya-Szego inequality [15] for non-negative real numbers a_i and b_i , $1 \leq i \leq n$

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^n a_i b_i \right)^2$$

where $M_1 = \max(a_i)$, $m_1 = \min(a_i)$, $M_2 = \max(b_i)$ and $m_2 = \min(b_i)$. Let $a_i = |\nu_i'|$ and $b_i = 1$, then from above we have

$$\sum_{i=1}^n |\nu_i'|^2 \sum_{i=1}^n 1^2 \leq \frac{1}{4} \left(\sqrt{\frac{\nu_n'}{\nu_1'}} + \sqrt{\frac{\nu_1'}{\nu_n'}} \right)^2 \left(\sum_{i=1}^n |\nu_i'| \right)^2.$$

That is,

$$\begin{aligned} n \sum_{i=1}^n |\nu'_i|^2 &\leq \frac{1}{4} \left(\sqrt{\frac{\nu'_n}{\nu'_1}} + \sqrt{\frac{\nu'_1}{\nu'_n}} \right)^2 (EL_\varepsilon(G))^2 \\ &= \frac{1}{4} \frac{(\nu'_n + \nu'_1)^2}{\nu'_n \nu'_1} (EL_\varepsilon(G))^2. \end{aligned}$$

Since, $\sum_{i=1}^n \nu'_i{}^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m$, we get the desired result from above. \square

3. Conclusion

In this paper, we study different properties and bounds of eccentricity version of Laplacian energy of a graph G . It is found that, there is great analogy between the original Laplacian energy and eccentricity version of Laplacian energy, where as also have some distinct difference.

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