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More Equienergetic Signed Graphs

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Dedicated to Professor Ivan Gutman

Abstract

The energy of signed graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. Two signed graphs are said to be equienergetic if they have same energy. In the literature the construction of equienergetic signed graphs are reported. In this paper we obtain the characteristic polynomial and energy of the join of two signed graphs and thereby we give another construction of unbalanced, noncospectral equieneregtic signed graphs on $n \geq 8$ vertices.

Keywords: Signed graph, energy of a graph, equienergetic graphs.

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1. Introduction

Let G be a simple undirected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set be $E(G) = \{e_1, e_2, \ldots, e_m\}$. A *signed graph* (or *sigraph*) is an ordered pair $G^s = (G, f)$ where G is the underlying graph of G^s and $f : E(G) \to \{+1, -1\}$ is the signing function from the edge set $E(G)$ into the set $\{+1, -1\}$. Thus the signed graph G^s is a graph obtained from the graph G by assigning positive sign or negative sign to the edges of G. A signed graph G^s is called *homogeneous* if its edges are all positive or all negative, otherwise it is called *heterogeneous*. The sign of a cycle is the product of the signs of its edges. A signed graph is said to be *balanced* if all its cycles are positive.

The *adjacency matrix* of a signed graph G^s is an $n \times n$ real symmetric matrix

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$A(G^s) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if the edge between the vertices } v_i \text{ and } v_j \text{ is positive} \\ -1, & \text{if the edge between the vertices } v_i \text{ and } v_j \text{ is negative} \\ 0, & \text{otherwise.} \end{cases}$$

The *characteristic polynomial* of $A(G^s)$ is $\phi(G^s : \xi) = \det(\xi I - A(G^s))$, where I is the identity matrix of order n. The eigenvalues of the matrix $A(G^s)$ are denoted by $\xi_1, \xi_2, \ldots, \xi_n$ and their collection is called the *spectra* of G^s. Since the adjacency matrix of the signed graph G^s is real and symmetric, its eigenvalues are real. Two signed graphs are said to be *cospectral* if they have same spectra. A signed graph is balanced if and only if it is cospectral with its underlying unsigned graph [1].

The *energy* of a signed graph G^s denoted by $\mathscr{E}(G^s)$ is defined as [6]

$$\mathscr{E}(G^s) = \sum_{i=1}^{n} |\xi_i| \ . \tag{1}$$

The Eq. (1) is in full analogy to the *ordinary graph energy* [7] defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. For more details about the graph energy one can see the book [10].

The signed graphs G_1^s and G_2^s are said to be *equienergetic* if $\mathscr{E}(G_1^s) = \mathscr{E}(G_2^s)$. In trivial manner, the cospectral signed graphs are equienergetic. Nayak [11] constructed pairs of equienergetic signed graphs on $2n$ vertices, $n \geq 4$. For odd $n \geq 5$ and for even $n \geq 6$, Bhat and Pirzada [3] constructed the pairs of unbalanced, noncospectral equienergetic signed graphs. If G is an odd unicyclic graph, then any two signed graphs on G have the same energy [3]. There are several results on equienergetic graphs [2, 4, 8, 9, 12–14]. In this paper we obtain the characteristic polynomial and energy of the join of two signed graphs and give another construction for unbalanced, noncospectral equieneregtic signed graphs on $n \geq 8$ vertices.

2. Spectra and Energy of the Join of Signed Graphs

In Figures 1 and 2, the dotted line represents negative edge and thick line represents positive edge.

Definition 2.1. The positive join of two signed graphs G_1^s and G_2^s denoted by $G_1^s \oplus G_2^s$ is a graph obtained from G_1^s and G_2^s by joining each vertex of G_1^s to all vertices of G_2^s with positive edges.

Definition 2.2. The negative join of two signed graphs G_1^s and G_2^s denoted by $G_1^s \ominus G_2^s$ is a graph obtained from G_1^s and G_2^s by joining each vertex of G_1^s to all vertices of G_2^s with negative edges.

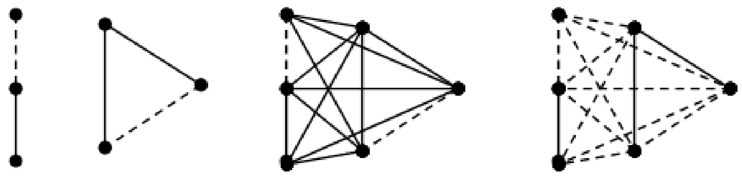

Figure 1: Signed graphs G_1^s and G_2^s and their joins $G_1^s \oplus G_2^s$ and $G_1^s \ominus G_2^s$.

Let W be a subset of the vertex set $V(G^s)$ of a signed graph G^s and $\overline{W} = V(G^s) \setminus W$. Let G'^s be the signed graph obtained from G^s by changing the positive edges to negative edges and negative edges to positive edges between W and $\overline{W}$. We say that G'^s has been obtained from G^s by switching with respect to W. Two signed graphs are said to be switching equivalent if one is obtained from the other by a sequence of switching. The signed graphs $G_1^s \oplus G_2^s$ and $G_1^s \ominus G_2^s$ are switching equivalent.

Theorem 2.3. $\phi(G_1^s \oplus G_2^s : \xi) = \phi(G_1^s \ominus G_2^s : \xi)$.

Proof.

$$\begin{aligned}
\phi(G_1^s \oplus G_2^s : \xi) &= \det(\xi I - A(G_1^s \oplus G_2^s)) \\
&= \begin{vmatrix} \xi I_{n_1} - A(G_1^s) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & \xi I_{n_2} - A(G_2^s) \end{vmatrix}, \qquad (2)
\end{aligned}$$

where J is the matrix whose all entries are equal to 1. In determinant (2), taking -1 common from first n_1 rows and then from the first n_1 columns we get

$$\begin{aligned}
\phi(G_1^s \oplus G_2^s : \xi) &= (-1)^{n_1}(-1)^{n_1} \begin{vmatrix} \xi I_{n_1} - A(G_1^s) & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & \xi I_{n_2} - A(G_2^s) \end{vmatrix} \\
&= \det(\xi I - A(G_1^s \ominus G_2^s)) \\
&= \phi(G_1^s \ominus G_2^s : \xi).
\end{aligned}$$

□

By Theorem 2.3, the signed graphs $G_1^s \oplus G_2^s$ and $G_1^s \ominus G_2^s$ are cospectral. Hence $\mathscr{E}(G_1^s \oplus G_2^s) = \mathscr{E}(G_1^s \ominus G_2^s)$.

The *positive* (*negative*) *degree* of a vertex v_i in a signed graph G^s is the number of positive (negative) edges incident to v_i and is denoted by d_i^+ (d_i^-). The degree of a vertex v_i in G^s is $deg(v_i) = d_i^+ + d_i^-$. The *net degree* of a vertex v_i in G^s is

$d_i^+ - d_i^-$. A signed graph G^s is said to be k-*net regular* if all its vertices have same net degree equal to k, that is $k = d_i^+ - d_i^-$ for $i = 1, 2, \ldots, n$. The net regularity of signed graph can be either positive or negative or zero.

Theorem 2.4. *Let G_i^s be a k_i-net regular signed graph on n_i vertices, $i = 1, 2$. Then the characteristic polynomial of the adjacency matrix of positive join $G_1^s \oplus G_2^s$ is*

$$\phi(G_1^s \oplus G_2^s : \xi) = \frac{[(\xi - k_1)(\xi - k_2) - n_1 n_2]}{(\xi - k_1)(\xi - k_2)} \phi(G_1^s : \xi)\phi(G_2^s : \xi). \tag{3}$$

Proof.

$$\begin{aligned} \phi(G_1^s \oplus G_2^s : \xi) &= \det(\xi I - A(G_1^s \oplus G_2^s)) \\ &= \begin{vmatrix} \xi I_{n_1} - A(G_1^s) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & \xi I_{n_2} - A(G_2^s) \end{vmatrix} \end{aligned} \tag{4}$$

where J is the matrix whose all entries are equal to unity.

The determinant (4) can be written as

$$\begin{vmatrix} \xi & -a_{12} & \cdots & -a_{1n_1} & -1 & -1 & \cdots & -1 \\ -a_{21} & \xi & \cdots & -a_{2n_1} & -1 & -1 & \cdots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -a_{n_11} & -a_{n_12} & \cdots & \mu & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \xi & -a'_{12} & \cdots & -a'_{1n_2} \\ -1 & -1 & \cdots & -1 & -a'_{21} & \xi & \cdots & -a'_{2n_2} \\ \vdots & & \vdots & & & & \vdots & \\ -1 & -1 & \cdots & -1 & -a'_{n_21} & -a'_{n_22} & \cdots & \xi \end{vmatrix} \tag{5}$$

where a_{ij} is the (i, j)-th entry in $A(G_1^s)$, $i, j = 1, 2, \ldots, n_1$ and a'_{ij} is the (i, j)-th entry in the matrix $A(G_2^s)$, $i, j = 1, 2, \ldots, n_2$. Since G_i^s is a k_i-net regular signed graph

$$\sum_{j=1}^{n_1} a_{ij} = k_1 \qquad \text{for} \quad i = 1, 2, \ldots, n_1 \tag{6}$$

and

$$\sum_{j=1}^{n_2} a'_{ij} = k_2 \qquad \text{for} \quad i = 1, 2, \ldots, n_2. \tag{7}$$

We now perform the number of operations on the determinant (5).

Subtract the row $(n_1 + 1)$ from the rows $(n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)$ of (5) to obtain (8)

$$\begin{vmatrix} \xi & -a_{12} & \cdots & -a_{1n_1} & -1 & -1 & \cdots & -1 \\ -a_{21} & \xi & \cdots & -a_{2n_1} & -1 & -1 & \cdots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -a_{n_11} & -a_{n_12} & \cdots & \mu & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \xi & -a'_{12} & \cdots & -a'_{1n_2} \\ 0 & 0 & \cdots & 0 & -a'_{21}-\xi & \xi+a'_{12} & \cdots & -a'_{2n_2}+a'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & -a'_{n_21}-\xi & -a'_{n_22}+a'_{12} & \cdots & \xi+a'_{1n_2} \end{vmatrix} . \tag{8}$$

Adding the columns $(n_1+2),(n_1+3),\ldots,(n_1+n_2)$ to the column (n_1+1) of (8) and using Eq. (7) we arrive at the determinant (9):

$$\begin{vmatrix} \xi & -a_{12} & \cdots & -a_{1n_1} & -n_2 & -1 & \cdots & -1 \\ -a_{21} & \xi & \cdots & -a_{2n_1} & -n_2 & -1 & \cdots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -a_{n_11} & -a_{n_12} & \cdots & \xi & -n_2 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \xi-k_2 & -a'_{12} & \cdots & -a'_{1n_2} \\ 0 & 0 & \cdots & 0 & 0 & \xi+a'_{12} & \cdots & -a'_{2n_2}+a'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & -a'_{n_22}+a'_{12} & \cdots & \xi+a'_{1n_2} \end{vmatrix} \tag{9}$$

which is equal to (10):

$$\begin{vmatrix} \xi & -a_{12} & \cdots & -a_{1n_1} & -n_2 \\ -a_{21} & \xi & \cdots & -a_{2n_1} & -n_2 \\ \vdots & & \vdots & & \\ -a_{n_11} & -a_{n_12} & \cdots & \xi & -n_2 \\ -1 & -1 & \cdots & -1 & \xi-k_2 \end{vmatrix} |B| \tag{10}$$

where

$$|B| = \begin{vmatrix} \xi+a'_{12} & -a'_{23}+a'_{13} & \cdots & -a'_{2n_2}+a'_{1n_2} \\ -a'_{32}+a'_{12} & \xi+a'_{13} & \cdots & -a'_{3n_2}+a'_{1n_2} \\ \vdots & & \vdots & \\ -a'_{n_22}+a'_{12} & -a'_{n_23}+a'_{13} & \cdots & \xi+a'_{1n_2} \end{vmatrix} . \tag{11}$$

The first determinant in (10) is of order (n_1+1). Subtract the first row from the rows $2, 3, \ldots, n_1$, in (10) to obtain (12):

$$\begin{vmatrix} \xi & -a_{12} & \cdots & -a_{1n_1} & -n_2 \\ -a_{21}-\xi & \xi+a_{12} & \cdots & -a_{2n_1}+a_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -a_{n_11}-\xi & -a_{n_12}+a_{12} & \cdots & \xi+a_{1n_1} & 0 \\ -1 & -1 & \cdots & -1 & \xi-k_2 \end{vmatrix} |B| \, . \tag{12}$$

Adding columns $2, 3, \ldots, n_1$ to the first column of (12) and using Eq. (6) we get (13):

$$\begin{vmatrix} \xi-k_1 & -a_{12} & \cdots & -a_{1n_1} & -n_2 \\ 0 & \xi+a_{12} & \cdots & -a_{2n_1}+a_{1n_1} & 0 \\ \vdots & & \vdots & & \\ 0 & -a_{n_12}+a_{12} & \cdots & \xi+a_{1n_1} & 0 \\ -n_1 & -1 & \cdots & -1 & \xi-k_2 \end{vmatrix} |B| \, . \tag{13}$$

Expand it along the first column to obtain (14):

$$\{(\xi-k_1)\,\Delta_1 - (-1)^{n_1}(n_1)\Delta_2\}\,|B| \tag{14}$$

where

$$\Delta_1 = \begin{vmatrix} \xi+a_{12} & -a_{23}+a_{13} & \cdots & -a_{2n_1}+a_{1n_1} & 0 \\ -a_{32}+a_{12} & \xi+a_{13} & \cdots & -a_{3n_1}+a_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -a_{n_12}+a_{12} & -a_{n_13}+a_{13} & \cdots & \xi+a_{1n_1} & 0 \\ -1 & -1 & \cdots & -1 & \xi-k_2 \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n_1} & -n_2 \\ \xi+a_{12} & -a_{23}+a_{13} & \cdots & -a_{2n_1}+a_{1n_1} & 0 \\ -a_{32}+a_{12} & \xi+a_{13} & \cdots & -a_{3n_1}+a_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -a_{n_12}+a_{12} & -a_{n_13}+a_{13} & \cdots & \xi+a_{1n_1} & 0 \end{vmatrix} .$$

The expression (14) can be rewritten as

$$\begin{aligned} & \left\{(\xi-k_1)(\xi-k_2)\,|A| - (-1)^{n_1}(n_1)(-1)^{1+n_1}(-n_2)|A|\right\}|B| \\ = \; & \{(\xi-k_1)(\xi-k_2) - n_1n_2\}\,|A||B| \end{aligned} \tag{15}$$

where

$$|A| = \begin{vmatrix} \xi+a_{12} & -a_{23}+a_{13} & \cdots & -a_{2n_1}+a_{1n_1} \\ -a_{32}+a_{12} & \xi+a_{13} & \cdots & -a_{3n_1}+a_{1n_1} \\ \vdots & & \vdots & \\ -a_{n_12}+a_{12} & -a_{n_13}+a_{13} & \cdots & \xi+a_{1n_1} \end{vmatrix} . \tag{16}$$

The determinant (16) can be written as

$$|A| = \frac{1}{(\xi - k_1)} \begin{vmatrix} \xi - k_1 & -a_{12} & -a_{13} & \cdots & -a_{1n_1} \\ 0 & \xi + a_{12} & -a_{23} + a_{13} & \cdots & -a_{2n_1} + a_{1n_1} \\ 0 & -a_{32} + a_{12} & \xi + a_{13} & \cdots & -a_{3n_1} + a_{1n_1} \\ \vdots & & & \vdots & \\ 0 & -a_{n_12} + a_{12} & -a_{n_13} + a_{13} & \cdots & \xi + a_{1n_1} \end{vmatrix} . \quad (17)$$

From Eq. (6) the sum of the i-th row in (17) is $\xi + a_{i1}$ for $i = 2, 3, \ldots, n_1$. Therefore, by subtracting the columns $2, 3, \ldots, n_1$ of (17) from the first column, we obtain (18):

$$|A| = \frac{1}{(\xi - k_1)} \begin{vmatrix} \xi & -a_{12} & -a_{13} & \cdots & -a_{1n_1} \\ -\xi - a_{21} & \xi + a_{12} & -a_{23} + a_{13} & \cdots & -a_{2n_1} + a_{1n_1} \\ -\xi - a_{31} & -a_{32} + a_{12} & \xi + a_{13} & \cdots & -a_{3n_1} + a_{1n_1} \\ \vdots & & & \vdots & \\ -\xi - a_{n_11} & -a_{n_12} + a_{12} & -a_{n_13} + a_{13} & \cdots & \xi + a_{1n_1} \end{vmatrix} . \quad (18)$$

Add the first row of (18) to the rows $2, 3, \ldots, n_1$ to obtain (19):

$$\begin{aligned} |A| &= \frac{1}{(\xi - k_1)} \begin{vmatrix} \xi & -a_{12} & -a_{13} & \cdots & -a_{1n_1} \\ -a_{21} & \xi & -a_{23} & \cdots & -a_{2n_1} \\ -a_{31} & -a_{32} & \xi & \cdots & -a_{3n_1} \\ \vdots & & & \vdots & \\ -a_{n_11} & -a_{n_12} & -a_{n_13} & \cdots & \xi \end{vmatrix} \\ &= \frac{1}{(\xi - k_1)} \, \phi(G_1^s : \xi) \ . \end{aligned} \quad (19)$$

In a similar manner we can show that from (11),

$$|B| = \frac{1}{(\xi - k_2)} \, \phi(G_2^s : \xi) \ . \quad (20)$$

Substituting (19) and (20) back into (15) gives Eq. (3). □

Theorem 2.5. *Let G_i^s be a k_i-net regular signed graph on n_i vertices, $i = 1, 2$. Then*

$$\mathscr{E}(G_1^s \oplus G_2^s) = \mathscr{E}(G_1^s) + \mathscr{E}(G_2^s) - (k_1 + k_2) + \sqrt{(k_1 - k_2)^2 + 4n_1n_2}.$$

Proof. From Theorem 2.4,

$$\phi(G_1^s \oplus G_2^s : \xi) = \frac{[(\xi - k_1)(\xi - k_2) - n_1n_2]}{(\xi - k_1)(\xi - k_2)} \phi(G_1^s : \xi)\phi(G_2^s : \xi),$$

which gives that

$$(\xi - k_1)(\xi - k_2)\phi(G_1^s \oplus G_2^s : \xi) = [(\xi - k_1)(\xi - k_2) - n_1 n_2]\,\phi(G_1^s : \xi)\phi(G_2^s : \xi).$$

Let

$$P_1(\xi) = (\xi - k_1)(\xi - k_2)\,\phi(G_1^s \oplus G_2^s : \xi)$$

and

$$P_2(\xi) = [(\xi - k_1)(\xi - k_2) - n_1 n_2]\ \phi(G_1^s : \xi)\phi(G_2^s : \xi).$$

The roots of $P_1(\xi) = 0$ are k_1, k_2 and the eigenvalues of $G_1^s \oplus G_2^s$. Therefore the sum of the absolute values of the roots of $P_1(\xi) = 0$ is

$$k_1 + k_2 + \mathscr{E}(G_1^s \oplus G_2^s). \tag{21}$$

The roots of $P_2(\xi) = 0$ are the eigenvalues of G_1^s, eigenvalues of G_2^s and

$$\frac{1}{2}\left[(k_1 + k_2) \pm \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)}\,\right].$$

Therefore the sum of the absolute values of the roots of $P_2(\xi) = 0$ is

$$\begin{aligned}\mathscr{E}(G_1^s) \quad + \quad & \mathscr{E}(G_2^s) + \left|\frac{1}{2}\left[(k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)}\right]\right| \\ + \quad & \left|\frac{1}{2}\left[(k_1 + k_2) - \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)}\right]\right|. \end{aligned}\tag{22}$$

Since $P_1(\xi) = P_2(\xi)$, equating (21) and (22), we get

$$\begin{aligned}\mathscr{E}(G_1^s \oplus G_2^s) &= \mathscr{E}(G_1^s) + \mathscr{E}(G_2^s) - (k_1 + k_2) \\ &+ \left|\frac{1}{2}\left[(k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)}\right]\right| \\ &+ \left|\frac{1}{2}\left[(k_1 + k_2) - \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)}\right]\right|. \end{aligned}\tag{23}$$

Since $k_1 k_2 \leq (n_1 - 1)(n_2 - 1) < n_1 n_2$, the Eq. (23) reduces to

$$\begin{aligned}\mathscr{E}(G_1^s \oplus G_2^s) &= \mathscr{E}(G_1^s) + \mathscr{E}(G_2^s) - (k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 4(k_1 k_2 - n_1 n_2)} \\ &= \mathscr{E}(G_1^s) + \mathscr{E}(G_2^s) - (k_1 + k_2) + \sqrt{(k_1 - k_2)^2 + 4 n_1 n_2}\,. \end{aligned}$$

□

Corollary 2.6. *If H_1^s and H_2^s are non cospectral, equienergetic net regular signed graphs on n vertices and of same net regularity, then for any net regular signed graph G^s, $\mathscr{E}(H_1^s \oplus G^s) = \mathscr{E}(H_2^s \oplus G^s)$.*

The complete graph K_p is a net regular signed graph on p vertices with net regularity $p-1$ and $\mathscr{E}(K_p) = 2(p-1)$ [5].

Corollary 2.7. *Let G^s be a k-net regular signed graph on n vertices. Then*

$$\mathscr{E}(G^s \oplus K_p) = \mathscr{E}(G^s) + p - 1 - k + \sqrt{(p-1-k)^2 + 4np}.$$

The totally disconnected graph $\overline{K_p}$ is a net regular signed graph on p vertices with net regularity 0 and $\mathscr{E}(\overline{K_p}) = 0$ [5].

Corollary 2.8. *Let G^s be a k-net regular signed graph on n vertices. Then*

$$\mathscr{E}(G^s \oplus \overline{K_p}) = \mathscr{E}(G^s) - k + \sqrt{k^2 + 4np}.$$

Any ordinary regular graph H of degree r is a net regular signed graph with net regularity r.

Corollary 2.9. *Let H be an r-regular ordinary graph on n_1 vertices and G^s be a k-net regular signed graph on n_2 vertices. Then*

$$\mathscr{E}(H \oplus G^s) = \mathscr{E}(H) + \mathscr{E}(G^s) - (r+k) + \sqrt{(r-k)^2 + 4n_1n_2}.$$

3. Construction of Equienergetic Signed Graphs

Consider the signed graphs H_a^s and H_b^s as shown in the Figure 2.

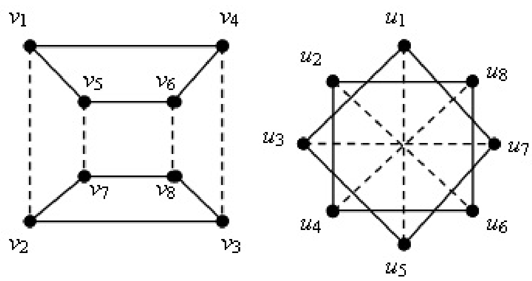

Figure 2: Signed graphs H_a^s and H_b^s.

By direct computation,

$$\phi(H_a^s : \xi) = (\xi - 1)^6(\xi + 3)^2 \tag{24}$$

and

$$\phi(H_b^s : \xi) = (\xi - 3)(\xi - 1)^3(\xi + 1)^3(\xi + 3). \tag{25}$$

Both H_a^s and H_b^s are net regular signed graphs on 8 vertices and of net regularity 1 and $\mathscr{E}(H_a^s) = \mathscr{E}(H_b^s) = 12$.

Let H^s be any k-net regular signed graph on $p \geq 1$ vertices. Then by Theorem 2.5,

$$\mathscr{E}(H_a^s \oplus H^s) = \mathscr{E}(H_b^s \oplus H^s) = \mathscr{E}(H^s) + 11 - k + \sqrt{(1-k)^2 + 32p}.$$

Thus, $H_a^s \oplus H^s$ and $H_b^s \oplus H^s$ are equienergetic signed graphs. By Eqs. (24) and (25), H_a^s and H_b^s are non cospectral, so by Theorem 2.4, $H_a^s \oplus H^s$ and $H_b^s \oplus H^s$ are also non cospectral. Further $H_a^s \oplus H^s$ and $H_b^s \oplus H^s$ are heterogeneous, unbalanced signed graphs possessing equal number of vertices $n = 8 + p, \;\; p = 0, 1, \ldots$.

4. Conclusion

From Corollary 2.6 it is easy to construct a pair of non cospectral, equienergetic signed graphs. In particular by the construction given in Section 3, it is easy to obtain a pair of non cospectral, heterogeneous, unblanced, equienergetic n-vertex signed graphs for $n \geq 8$.

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