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Seidel Signless Laplacian Energy of Graphs

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Abstract

Let $S(G)$ be the Seidel matrix of a graph G of order n and let $D_S(G) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be the diagonal matrix with d_i denoting the degree of a vertex v_i in G. The Seidel Laplacian matrix of G is defined as $SL(G) = D_S(G) - S(G)$ and the Seidel signless Laplacian matrix as $SL^+(G) = D_S(G) + S(G)$. The Seidel signless Laplacian energy $E_{SL^+}(G)$ is defined as the sum of the absolute deviations of the eigenvalues of $SL^+(G)$ from their mean. In this paper, we establish the main properties of the eigenvalues of $SL^+(G)$ and of $E_{SL^+}(G)$.

Keywords: Seidel Laplacian eigenvalues, Seidel Laplacian energy, Seidel signless Laplacian matrix, Seidel signless Laplacian eigenvalues, Seidel signless Laplacian energy.

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1. Introduction

Let G be a simple, undirected graph with n vertices and m edges. We say that G is an (n, m)-graph.

Let $v_1, v_2, \ldots, v_n$ be the vertices of G. The degree of a vertex v_i is the number of edges incident to it and is denoted by d_i. If $d_i = r$ for all $i = 1, 2, \ldots, n$, then G is said to be an r-regular graph.

By $\overline{G}$ will be denoted the complement of the graph G.

The *adjacency matrix* of a graph G is the square matrix $A(G) = (a_{ij})$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are

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the eigenvalues of $A(G)$, then the *ordinary energy* of a graph G is defined as [14,19]

$$E_A(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1}$$

The spectral graph theory based on the eigenvalues of the adjacency matrix is the most extensively elaborated part of algebraic graph theory [4, 6]. Also the graph energy, based on the eigenvalues of the adjacency matrix, attracted much attention and has been studied to a great extent [19]. Recently, a number of other graph energies have been introduced; for details see the recent survey [16]. Among these, the Laplacian and Seidel energies are of importance for the following considerations.

Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal degree matrix. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$. This matrix and its spectral properties have been extensively studied in the past [12, 13, 20–22]. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $L(G)$. Then the *Laplacian energy* of G is defined as [17]

$$E_L(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{2}$$

In 2007, Cvetković et al. [5] noticed that if the definition of the Laplacian matrix is changed into $D(G) + A(G)$, then interesting and non-trivial spectral properties are obtained. Eventually, an entire theory of the so-called signless Laplacian spectra has been developed [7–9].

Thus, the *signless Laplacian matrix* of G is defined as $L^+(G) = D(G) + A(G)$. Let $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ be its eigenvalues. Then the *signless Laplacian energy* of G is defined as [1]

$$E_{L^+}(G) = \sum_{i=1}^{n} \left| \mu_i^+ - \frac{2m}{n} \right| . \tag{3}$$

The *Seidel matrix* of a graph G is the $n \times n$ real symmetric matrix $S(G) = (s_{ij})$, where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent, and $s_{ij} = 0$ if $i = j$. It is easy to see that $S(G) = A(\overline{G}) - A(G)$, where $\overline{G}$ is the complement of G.

The eigenvalues of the Seidel matrix, labeled as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, are said to be the *Seidel eigenvalues* of G and form the Seidel spectrum. In analogy to Eq. (1), the *Seidel energy* of a graph G is defined as [18]

$$E_S(G) = \sum_{i=1}^{n} |\sigma_i| \,. \tag{4}$$

More results on Seidel energy are reported in [11, 23, 24, 27–29].

The Seidel Laplacian matrix of a graph was introduced in [30], where the main properties of its eigenvalues and of Seidel Laplacian energy were established.

Let $D_S(G) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be a diagonal matrix in which d_i stands for degree of a vertex v_i in G. Then, in analogy with the ordinary Laplacian matrix, the *Seidel Laplacian matrix* of G is defined as

$$SL(G) = D_S(G) - S(G)\,.$$

Note that $D_S(G) = D(\overline{G}) - D(G)$ and $SL(G) = L(\overline{G}) - L(G)$.

Let $\sigma_1^L, \sigma_2^L, \ldots, \sigma_n^L$, be the eigenvalues of $SL(G)$. In analogy to Eq. (2), the *Seidel Laplacian energy* of G is defined as [30]

$$E_{SL}(G) = \sum_{i=1}^{n} \left| \sigma_i^L - \frac{n(n-1)-4m}{n} \right|.$$

Bearing in mind the concept of signless Laplacian matrix, we introduce here the Seidel signless Laplacian matrix of a graph and study the basic properties of its eigenvalues and energy.

Thus, the *Seidel signless Laplacian matrix* of G is defined as

$$SL^+(G) = D_S(G) + S(G)\,.$$

In the case of Laplacian matrices, the adjective "signless" is fully appropriate, since no element of $L^+(G)$ is negative–valued. In the case of Seidel Laplacians, we use the same adjective by analogy, aware of the fact that all Seidel signless Laplacian matrices, with the exception of $SL^+(\overline{K_n})$, possess negative–valued elements, where K_n denotes the complete graph on n vertices.

Note that

$$\begin{aligned} SL^+(G) &= D(\overline{G}) - D(G) + A(\overline{G}) - A(G) \\ &= D(\overline{G}) + A(\overline{G}) - D(G) - A(G) \\ &= L^+(\overline{G}) - L^+(G)\,. \end{aligned}$$

Let $\sigma_1^{L^+}, \sigma_2^{L^+}, \ldots, \sigma_n^{L^+}$, be the eigenvalues of $SL^+(G)$. In analogy to Eq. (3), the *Seidel signless Laplacian energy* of G is defined as

$$E_{SL^+}(G) = \sum_{i=1}^{n} \left| \sigma_i^{L^+} - \frac{n(n-1)-4m}{n} \right|.$$

If we introduce the auxiliary quantities

$$\xi_i = \sigma_i^{L^+} - \frac{n(n-1)-4m}{n} \quad , \quad i = 1, 2, \ldots, n$$

then the expression for Seidel signless Laplacian energy becomes analogous to the formula for ordinary graph energy, Eq. (1), namely:

$$E_{SL^+}(G) = \sum_{i=1}^{n} |\xi_i|\,.$$

2. Seidel Signless Laplacian Eigenvalues

The adjacency eigenvalues, Laplacian eigenvalues and Seidel eigenvalues satisfy the relations

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = 2m\,;$$
$$\sum_{i=1}^{n} \mu_i = 2m \quad \text{and} \quad \sum_{i=1}^{n} \mu_i^2 = 2m + Z_1(G)\,;$$
$$\sum_{i=1}^{n} \sigma_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \sigma_i^2 = n(n-1)$$

where $Z_1(G) = \sum_{i=1}^{n} d_i^2$ is the well-known graph invariant called first Zagreb index [3, 15].

The respective formulas for the Seidel Laplacian eigenvalues read [30]:

$$\sum_{i=1}^{n} \sigma_i^L = n(n-1) - 4m \quad \text{and} \quad \sum_{i=1}^{n} (\sigma_i^L)^2 = (n-1)(n^2-8m) + 4Z_1(G)\,. \tag{5}$$

In the next lemma, we show that the Seidel signless Laplacian eigenvalues have properties fully analogous to those of the ordinary Seidel Laplacians, Eqs. (5).

Lemma 2.1. *Let G be an (n,m)-graph. Then the eigenvalues of the Seidel signless Laplacian matrix satisfy the relations:*

$$\sum_{i=1}^{n} \sigma_i^{L^+} = n(n-1) - 4m \quad \textit{and} \quad \sum_{i=1}^{n} (\sigma_i^{L^+})^2 = (n-1)(n^2-8m) + 4Z_1(G)\,.$$

Proof.

$$\sum_{i=1}^{n} \sigma_i^{L^+} = \text{trace}[SL^+(G)] = \sum_{i=1}^{n} (n-1-2d_i) = n(n-1) - 4m\,.$$

$$\begin{aligned} \sum_{i=1}^{n} (\sigma_i^{L^+})^2 &= \text{trace}[(SL^+(G))^2] \\ &= \sum_{i=1}^{n} [(n-1-2d_i)^2 + (n-1)] \\ &= n^2(n-1) - 8m(n-1) + 4Z_1(G) \\ &= (n-1)(n^2-8m) + 4Z_1(G)\,. \end{aligned}$$

□

Lemma 2.2. *Let $\sigma_1^{L^+}, \sigma_2^{L^+}, \dots, \sigma_n^{L^+}$ be the eigenvalues of $SL^+(G)$. Let $\xi_i = \sigma_i^{L^+} - \frac{n(n-1)-4m}{n}$, $i = 1, 2, \dots, n$. Then*

$$\sum_{i=1}^{n} \xi_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \xi_i^2 = M$$

where $M = n(n-1) + 4Z_1(G) - \frac{16m^2}{n}$.

Proof.

$$\begin{aligned} \sum_{i=1}^{n} \xi_i &= \sum_{i=1}^{n} \left(\sigma_i^{L^+} - \frac{n(n-1)-4m}{n} \right) \\ &= \sum_{i=1}^{n} \sigma_i^{L^+} - [n(n-1)-4m] = n(n-1) - 4m - [n(n-1)-4m] = 0\,. \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n} \xi_i^2 &= \sum_{i=1}^{n} \left(\sigma_i^{L^+} - \frac{n(n-1)-4m}{n} \right)^2 \\ &= \sum_{i=1}^{n} (\sigma_i^{L^+})^2 - 2\left(\frac{n(n-1)-4m}{n} \right) \sum_{i=1}^{n} \sigma_i^{L^+} + \sum_{i=1}^{n} \left(\frac{n(n-1)-4m}{n} \right)^2 \\ &= (n-1)(n^2-8m) + 4Z_1(G) - 2\left(\frac{n(n-1)-4m}{n} \right)(n(n-1)-4m) \\ &\quad + n\left[\frac{(n(n-1)-4m)^2}{n^2} \right] = n(n-1) + 4Z_1(G) - \frac{16m^2}{n}\,. \end{aligned}$$

□

As a consequence of Lemmas 2.1 and 2.2, the lower and upper bounds for the Seidel signless Laplacian energy stated below in Theorems 3.1, 3.2, 3.7–3.11 are fully analogous to the results earlier obtained for the ordinary Seidel Laplacian energy [30].

The next proposition follows from $SL^+(G) = L^+(\overline{G}) - L^+(G) = -SL^+(\overline{G})$.

Proposition 2.3. *If $\sigma_i^{L^+}$, $i = 1, 2, \dots, n$, are the Seidel signless Laplacian eigenvalues of G, then $-\sigma_i^{L^+}$, $i = 1, 2, \dots, n$, are the Seidel signless Laplacian eigenvalues of $\overline{G}$.*

Theorem 2.4. *If $\sigma_1, \sigma_2, \dots, \sigma_n$ are the Seidel eigenvalues of an r-regular graph G, then the Seidel signless Laplacian eigenvalues of G are $n-1-2r+\sigma_i$, $i = 1, 2, \dots, n$.*

Proof. Let the characteristic polynomial of the Seidel matrix $S(G)$ be denoted by $\phi_S(G, \lambda)$ and the characteristic polynomial of the Seidel signless Laplacian matrix $SL^+(G)$ by $\phi_{SL^+}(G, \lambda)$.

If G is an r-regular graph, then $SL^+(G) = (n-1-2r)I + S(G)$, where I is the identity matrix of order n. Therefore,

$$\begin{aligned}\phi_{SL^+}(G,\lambda) &= \det(\lambda I - SL^+(G)) \\ &= \det\left[(\lambda - n + 1 + 2r)I - S(G)\right] \\ &= \phi_S(G, \lambda - n + 1 + 2r)\,.\end{aligned}$$

□

Lemma 2.5. *Let the notation be same as in Lemma 2.2. Then*

$$\left|\sum_{i<j} \xi_i \xi_j\right| = \frac{1}{2} M\,.$$

Proof. Since $\sum_{i=1}^{n} \xi_i = 0$, we get $\sum_{i=1}^{n} \xi_i^2 = -2 \sum_{i<j} \xi_i\,\xi_j$. Therefore

$$2\left|\sum_{i<j} \xi_i\,\xi_j\right| = \sum_{i=1}^{n} \xi_i^2 = M\,.$$

□

3. Seidel Signless Laplacian Energy

Theorem 3.1. *Let G be an (n,m)-graph. Then*

$$E_{SL^+}(G) \le \sqrt{n\left[n(n-1) + 4Z_1(G) - \frac{16m^2}{n}\right]}\,. \tag{6}$$

Proof. The Cauchy–Schwartz inequality states that,

$$\left(\sum_{i=1}^{n} a_i\,b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Set $a_i = 1$ and $b_i = |\xi_i|$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}\left(\sum_{i=1}^{n} |\xi_i|\right)^2 &\le n \sum_{i=1}^{n} |\xi_i|^2 \\ \left[E_{SL^+}(G)\right]^2 &\le n\left[n(n-1) + 4Z_1(G) - \frac{16m^2}{n}\right]\end{aligned}$$

and inequality (6) follows. □

Theorem 3.2. *Let G be an (n,m)-graph. Then*

$$E_{SL^+}(G) \geq \sqrt{2\left[n(n-1)+4Z_1(G)-\frac{16m^2}{n}\right]}.$$

Proof. $E_{SL^+}(G)=\sum_{i=1}^{n}|\xi_i|$. Therefore

$$\begin{aligned}\left[E_{SL^+}(G)\right]^2 &= \sum_{i=1}^{n}|\xi_i|^2+2\sum_{1\le i<j\le n}|\xi_i||\xi_j| \geq \sum_{i=1}^{n}|\xi_i|^2+2\left|\sum_{1\le i<j\le n}\xi_i\xi_j\right| \\ &= M+M=2\left[n(n-1)+4Z_1(G)-\frac{16m^2}{n}\right].\end{aligned}$$

□

Theorem 3.3. [25] *Suppose that a_i and b_i , $1\le i\le n$, are positive real numbers. Then*

$$\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1\,M_2}{m_1\,m_2}}+\sqrt{\frac{m_1\,m_2}{M_1\,M_2}}\right)^2\left(\sum_{i=1}^{n}a_i\,b_i\right)^2$$

where $M_1=\max_{1\le i\le n}(a_i)$, $M_2=\max_{1\le i\le n}(b_i)$, $m_1=\min_{1\le i\le n}(a_i)$, and $m_2=\min_{1\le i\le n}(b_i)$.

Theorem 3.4. [26] *Let a_i and b_i , $1\le i\le n$, be non-negative real numbers. Then*

$$\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_i\,b_i\right)^2 \leq \frac{n^2}{4}\left(M_1\,M_2-m_1\,m_2\right)^2$$

where M_i and m_i, $i=1,2$, are defined similarly as in Theorem 3.3.

Theorem 3.5. [10] *Suppose that a_i and b_i , $1\le i\le n$, are positive real numbers. Then*

$$\left|n\sum_{i=1}^{n}a_i\,b_i-\sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i\right| \leq \alpha(n)(A-a)(B-b)$$

where a, b, A, and B are real constants, such that for each i, $1\le i\le n$, $a\le a_i\le A$, and $b\le b_i\le B$. In addition, $\alpha(n)=n\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)$.

Theorem 3.6. [2] *Let a_i and b_i , $1\le i\le n$, be non-negative real numbers. Then*

$$\sum_{i=1}^{n}b_i^2+rR\sum_{i=1}^{n}a_i^2 \leq (r+R)\left(\sum_{i=1}^{n}a_i\,b_i\right)$$

where r and R are real constants, such that for each i, $1\le i\le n$, $r\,a_i\le b_i\le R\,a_i$ holds.

Theorem 3.7. *Let G be an (n,m)-graph and let $\xi_{max} = \max_{1\le i\le n} |\xi_i|$ and $\xi_{min} = \min_{1\le i\le n} |\xi_i|$. Then*

$$E_{SL^+}(G) \geq \sqrt{nM - \frac{n^2}{4}(\xi_{max} - \xi_{min})^2}\,. \tag{7}$$

Proof. Applying Theorem 3.4 for $a_i = 1$ and $b_i = |\xi_i|$ we get

$$\begin{aligned} \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i|\right)^2 &\leq \frac{n^2}{4}(\xi_{max} - \xi_{min})^2 \\ nM - (E_{SL^+}(G))^2 &\leq \frac{n^2}{4}(\xi_{max} - \xi_{min})^2 \end{aligned}$$

from which inequality (7) is straightforward. □

Theorem 3.8. *Let G, ξ_{max}, and ξ_{min} be same as in Theorem 3.7. Then*

$$E_{SL^+}(G) \geq \frac{2\sqrt{nM\,\xi_{max}\,\xi_{min}}}{\xi_{max} + \xi_{min}}\,. \tag{8}$$

Proof. Applying Theorem 3.3 for $a_i = 1$ and $b_i = |\xi_i|$ we get

$$\begin{aligned} \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 &\leq \frac{1}{4}\left(\sqrt{\frac{\xi_{max}}{\xi_{min}}} + \sqrt{\frac{\xi_{min}}{\xi_{max}}}\right)^2 \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \\ nM &\leq \frac{1}{4}\left(\frac{(\xi_{max} + \xi_{min})^2}{\xi_{max}\,\xi_{min}}\right) \left[E_{SL^+}(G)\right]^2 \end{aligned}$$

and inequality (8) follows. □

Theorem 3.9. *Let G, ξ_{max}, and ξ_{min} be same as in Theorem 3.7. Then*

$$E_{SL^+}(G) \geq \sqrt{nM - \alpha(n)\,(\xi_{max} - \xi_{min})^2} \tag{9}$$

where $\alpha(n)$ is the parameter defined in Theorem 3.5.

Proof. Applying Theorem 3.5 for $a_i = |\xi_i| = b_i$, $a = \xi_{min} = b$, and $A = \xi_{max} = B$, we get

$$\begin{aligned} \left| n \sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \right| &\leq \alpha(n)\,(\xi_{max} - \xi_{min})^2 \\ nM - (E_{SL^+}(G))^2 &\leq \alpha(n)\,(\xi_{max} - \xi_{min})^2 \end{aligned}$$

from which inequality (9) directly follows. □

Since $\alpha(n) \leq \frac{n^2}{4}$, by Theorem 3.9 we get:

Corollary 3.10.

$$E_{SL^+}(G) \geq \sqrt{nM - \frac{n^2}{4}\left(\xi_{max} - \xi_{min}\right)^2}\,.$$

Theorem 3.11. *Let G, ξ_{max}, and ξ_{min} be same as in Theorem 3.7. Then*

$$E_{SL^+}(G) \geq \frac{M + n\,\xi_{max}\,\xi_{min}}{(\xi_{max} + \xi_{min})}\,. \tag{10}$$

Proof. Applying Theorem 3.6 for $b_i = |\xi_i|$, $a_i = 1$, $r = \xi_{min}$, and $R = \xi_{max}$ we get

$$\begin{aligned} \sum_{i=1}^{n} |\xi_i|^2 + \xi_{max}\,\xi_{min} \sum_{i=1}^{n} 1 &\leq (\xi_{max} + \xi_{min}) \sum_{i=1}^{n} |\xi_i| \\ M + n\,\xi_{max}\,\xi_{min} &\leq (\xi_{max} + \xi_{min})\, E_{SL^+}(G)\,. \end{aligned}$$

Inequality (10) follows. □

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