

Eigenvalues and Energy of the Cayley Graph of Some Groups with respect to a Normal Subset

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Abstract

Set $X = \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Z_n, T_{4n}, SD_{8n}, Sz(q), G_2(q), V_{8n}\}$, where $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ are Mathieu groups and $Z_n, T_{4n}, SD_{8n}, Sz(q), G_2(q)$ and V_{8n} denote the cyclic, dicyclic, semi-dihedral, Suzuki, Ree and a group of order $8n$ presented by

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = e, aba = b^{-1}, ab^{-1}a = b \rangle,$$

respectively. In this paper, we compute all eigenvalues of $Cay(G, T)$, where $G \in X$ and T is minimal, second minimal, maximal or second maximal normal subset of $G \setminus \{e\}$ with respect to its size. In the case that S is a minimal normal subset of $G \setminus \{e\}$, the summation of the absolute value of eigenvalues, energy of the Cayley graph, are evaluated.

Keywords: Simple group, Cayley graph, eigenvalue, energy.

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1. Introduction

Suppose $\Gamma = (V, E)$ is a simple graph, where $V = V(\Gamma)$ is the set of vertices and $E = E(\Gamma)$ is the set of edges of Γ . If G is a finite group, S is a subset of G such that $S = S^{-1}$ and $S \subseteq G \setminus \{e\}$ then the Cayley graph $\Gamma = Cay(G, S)$ is defined by $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$ [4]. It is clear that a Cayley graph $\Gamma = Cay(G, S)$ is connected if and only if $G = \langle S \rangle$.

A subset S of a finite group G is called normal, if $g^{-1}Sg = S$, for all $g \in G$. It is called symmetric, if $S^{-1} = S$. A normal, symmetric and generating subset of G with this property that $e \notin S$ is said to be an NS of G . A minimal or second minimal NS is denoted by MNS or $SMNS$, respectively.

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The semi-dihedral group SD_{8n} , dicyclic group T_{4n} and the group V_{8n} have the following presentations, respectively:

$$\begin{aligned} SD_{8n} &= \langle a, b \mid a^{4n} = b^2 = e, bab = a^{2n-1} \rangle, \\ T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\ V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e, aba = b^{-1}, ab^{-1}a = b \rangle. \end{aligned}$$

It is easy to see the dicyclic group T_{4n} has order $4n$ and the cyclic subgroup $\langle a \rangle$ of T_{4n} has index 2 [13]. The groups SD_{8n} and V_{8n} have order $8n$ and their character tables computed in [11, 6], respectively.

Set $X = \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Z_n, T_{4n}, SD_{8n}, Sz(q), G_2(q), V_{8n}\}$, where $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ are denoted Mathieu groups, $Sz(q)$ is Suzuki group of order $(q-1)(q^2+1)$ and $G_2(q)$ is Ree group. The energy $E(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix [10]. The aim of this paper is to compute the eigenvalues of $Cay(G, S)$, where S is an MNS or $SMNS$ and G is isomorphic to a group in X . In the case that S is a minimal normal subset of $G \setminus \{e\}$, the summation of the absolute value of eigenvalues, energy of the Cayley graph, are evaluated.

A subset $A \subseteq G$ is called rational if, for every character $\chi \in Irr(G)$, $\chi(A) = \sum_{x \in A} \chi(x)$ is an integer. Alpering in some recent papers [1, 2], proved some very nice results on integrality of Cayley graphs. Among these results, two are very important as follows:

- if the Cayley graph of G on a set S is an integral Cayley graph then S is a rational set.
- for abelian groups a set S is rational if and only if the Cayley graph on S is an integral Cayley graph.

In this paper we study the integral Cayley graphs on simple group. Our calculations are made by computer algebra system GAP [17]. Our notation is standard and can be taken from [5, 12, 13].

2. Main Results

If Γ is a graph then $Spec(\Gamma)$ denotes the multi-set of all Γ -eigenvalues. Two graphs Γ_1 and Γ_2 are said to be co-spectral, if they have the same spectrum. Diaconis and Shahshahani [7], was the first mathematician considered the problem of computing eigenvalues of Cayley graphs into account. They used the character table of the group under consideration to calculate the eigenvalues of $Cay(G, S)$, where G is a finite group and S is an NS of G . We refer to [14, 19] for more information on this topic. The aim of this section is to compute all eigenvalues of the Cayley graph of groups in the set X with respect to a minimal, second minimal, maximal or second maximal normal subset of G .

To compute the eigenvalues of a Cayley graph $Cay(G, S)$ with respect to a normal subset S of group $G \setminus \{e\}$, we apply a result of Zieschang [19, Theorem 1]. For the sake of completeness, we mention here this result.

Proposition 2.1. *Let $Cay(G, T)$ denote the Cayley graph of a finite group G with respect to a normal subset T of $G \setminus \{e\}$. Let further $\{\chi_1, \dots, \chi_s\}$ be the set of all irreducible complex characters of G and define $\lambda_j = \frac{1}{\chi_j(e)} \sum_{t \in T} \chi_j(t)$, where $1 \leq j \leq s$. Then $\{\lambda_1, \dots, \lambda_s\}$ is the set of all values of the spectrum of $Cay(G, T)$. Moreover, if m_j is the multiplicity of λ_j , then*

$$m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^s (\chi_k(e))^2.$$

Suppose $Irr(G)$ denotes the set of all irreducible characters of G and $Cl(G)$ is the set of conjugacy classes of G . We also assume that $\mathcal{A}(G)$ is the set of all normal subsets S of $G \setminus \{e\}$. Define $S' = G \setminus (S \cup \{e\})$. One can see that, S is a minimal element of $\mathcal{A}(G)$ if and only if S' is a second maximal element of $\mathcal{A}(G)$. Notice that the first maximal is $G \setminus \{e\}$ and the Cayley graph $Cay(G, G \setminus \{e\})$ is $(|G| - 1)$ -regular. This implies that $Cay(G, G \setminus \{e\})$ is complete and $Spec(Cay(G, G \setminus \{e\})) = \{-1(|G| - 1 \text{ times}), |G| - 1\}$.

Proposition 2.2. *Suppose G is a finite group with exactly n conjugacy classes and S, S' are normal subsets of $G \setminus \{e\}$ such that $S' = G \setminus S \cup \{e\}$. Moreover, we assume that $\Gamma = Cay(G, S)$, $\Gamma' = Cay(G, S')$. Set $Spec(\Gamma) = \{\lambda_1, \dots, \lambda_n\}$ and $Spec(\Gamma') = \{\beta_1, \dots, \beta_n\}$. Then $\beta_i = -\lambda_i - 1$, $1 \leq i \leq n$.*

Proof. It is clear that S' is a symmetric and generating subset of G . So, by Proposition 2.1,

$$\begin{aligned} \beta_i &= \frac{1}{\chi_i(e)} \sum_{s' \in S'} \chi_i(s') \\ &= \frac{1}{\chi_i(e)} \left(\sum_{s' \in G} \chi_i(s') - \sum_{s \in S \cup \{e\}} \chi_i(s) \right) \\ &= \frac{1}{\chi_i(e)} \sum_{s' \in G} \chi_i(s') - \frac{1}{\chi_i(e)} \sum_{s \in S} \chi_i(s) - \frac{1}{\chi_i(e)} \chi_i(e) \\ &= -\lambda_i - 1, \end{aligned}$$

proving the result. \square

Proposition 2.3. *Suppose $G = \{g_1, g_2, \dots, g_n\}$ and F is a subset of $\{1, 2, \dots, n\}$ such that for all $j \in F$, the order of g_j is a power of a prime p_j . Define $S = \bigcup_{j \in F} g_j^G$ and $\Gamma = Cay(G, S)$. We also assume that for each $\chi \in Irr(G)$, $\chi(g_j)$ is*

an integer. Then $\text{Spec}(\Gamma) = \{\lambda_1, \dots, \lambda_n\}$, where $\lambda_i = \sum_{j \in F} |g_j^G| \left(1 + \frac{k_j p_j}{\chi_i(e)}\right)$, for some integer k_j .

Proof. Suppose χ_l , $1 \leq l \leq |\text{Irr}(G)|$, is an arbitrary irreducible character of G . Then by [13, Corollary 22.27], $\chi_l(g_j) \equiv \chi_l(e) \pmod{p_j}$. Thus, $\chi_l(g_j) - \chi_l(e) = p_j k_j$, for some integer k_j . This implies that $\frac{\chi_l(g_j)}{\chi_l(e)} = 1 + \frac{p_j k_j}{\chi_l(e)}$ and so,

$$\lambda_l = \sum_{s \in S} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j \in F} \sum_{s \in g_j^G} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j \in F} \sum_{s \in g_j^G} \left(1 + \frac{p_j k_j}{\chi_l(e)}\right) = \sum_{j \in F} |g_j^G| \left(1 + \frac{p_j k_j}{\chi_l(e)}\right).$$

□

Corollary 2.4. *Suppose G is a finite group, S is a normal subset of G such that all elements of S are involutions and $\Gamma = \text{Cay}(G, S)$. If F is a representative set for G -conjugacy classes of S then the following statements hold:*

1. if G is abelian then all Γ -eigenvalues are odd or all of them are even,
2. if G is a simple group then $\lambda_\chi = \sum_{g \in F} |g^G| \left(1 + \frac{4k_g}{\chi(e)}\right)$.

Proof. To prove (1), it is enough to apply Proposition 2.3 and the fact that the irreducible characters of G are linear. We now assume that G is simple, g is an involution in G and $\chi \in \text{Irr}(G)$. Then $\chi(g) \equiv \chi(e) \pmod{4}$ or G has a normal subgroup of index 2. Since G is simple, there exists k_g such that $\frac{\chi(g)}{\chi(1)} = 1 + \frac{4k_g}{\chi(1)}$. By Proposition 2.1, $\lambda_\chi = \sum_{g \in S} \frac{\chi(g)}{\chi(e)} = \sum_{x \in F} \sum_{g \in x^G} \left(1 + \frac{4k_g}{\chi(e)}\right) = \sum_{g \in F} |g^G| \left(1 + \frac{4k_g}{\chi(e)}\right)$. □

Suppose G is a group, A is the set of all character values of G , $Q(A)$ denotes the extension of Q by A and Λ is the Galois group of this extension. It is well-known that there exists ε such that $Q(A) \subseteq Q(\varepsilon)$, where ε is a primitive n -th root of unity. Thus, if $\alpha \in \Lambda$ then there exists a unique positive integer r such that $(r, n) = 1$ and $\alpha(\varepsilon) = \varepsilon^r$. So, it is well define to use the notation $\alpha = \sigma_r$. The group Λ acts on the set of irreducible characters and conjugacy classes of G by $\chi^\alpha(g) = \alpha(\chi(g))$ and $(x^G)^{\sigma_r} = (x^r)^G$, respectively.

Proposition 2.5. *Suppose G is a finite group and S, T are two Λ -conjugate conjugacy classes of G . Then $\Gamma_1 = \text{Cay}(G, S)$ and $\Gamma_2 = \text{Cay}(G, T)$ are co-spectral.*

Proof. Let G be a finite group, χ be an irreducible character of G and S, T are two Λ -conjugate conjugacy classes of G . If $\{\chi_1, \dots, \chi_r\}$ is the orbit of χ under action of the Galois group, then $\{\chi_1(S), \dots, \chi_r(S)\} = \{\chi_1(T), \dots, \chi_r(T)\}$. Suppose $\text{Spec}(\Gamma_1) = \{\lambda_1, \dots, \lambda_r\}$ and $\text{Spec}(\Gamma_2) = \{\mu_1, \dots, \mu_r\}$. If $\chi_i(S) = \chi_i(T)$, $\chi_i \in \text{Irr}(G)$, then $\lambda_i = \frac{1}{\chi_i(e)} |T| \chi_i(T) = \frac{1}{\chi_i(e)} |S| \chi_i(S) = \mu_i$. In other case, if $\{\chi_{n_1}, \dots, \chi_{n_s}\}$ is the orbit of χ_i under the action of Galois group then the values of

$\chi_{n_1}(S), \dots, \chi_{n_r}(S)$ can be permuted to find $\chi_{n_1}(T), \dots, \chi_{n_r}(T)$. This shows that there exists positive integer k such that $\chi_i(e) = \chi_k(e)$. Thus, $\lambda_i = \frac{1}{\chi_i(e)}|T|\chi_i(T) = \frac{1}{\chi_k(e)}|S|\chi_k(S) = \mu_k$. Therefore, $\Gamma_1 = Cay(G, S)$ and $\Gamma_2 = Cay(G, T)$ are co-spectral. \square

The converse of Proposition 2.5 is not generally correct. To do this, we consider the following example:

Example 2.6. Consider the alternating group A_6 which has exactly two conjugacy classes S and T of elements of order 3. It is easy to see that these classes are not Galois conjugate, but the Cayley graphs $\Gamma_1 = Cay(G, S)$ and $\Gamma_2 = Cay(G, T)$ are co-spectral. In fact,

$$Spec(\Gamma_1) = \begin{pmatrix} -8 & -5 & 0 & 4 & 16 & 40 \\ 25 & 128 & 81 & 100 & 25 & 1 \end{pmatrix} = Spec(\Gamma_2).$$

Example 2.7. Consider the cyclic group $G = \langle a \rangle$ of order n and $S = \{a, a^{-1}\}$. Then S is an MNS with the following spectrum:

$$Spec(Cay(G, S)) = \{w^j + w^{-j} \mid 0 \leq j \leq n - 1\},$$

where $w = e^{\frac{2\pi i}{n}}$. The energy of this Cayley graph approximately is $\frac{4n}{\pi}$, see [9]. By a similar argument, one can prove $T = S \cup \{a^2, a^{-2}\}$ is an $SMNS$ for G , all eigenvalues of $Cay(G, T)$ are $4Cos(\frac{3\pi j}{n})Cos(\frac{\pi j}{n})$, $0 \leq j \leq n - 1$.

By method of subgroups of index 2 [13, p. 420], the group T_{4n} has exactly $n + 3$ conjugacy classes $\{e\}$, $\{a^n\}$, $\{a^r, a^{-r}\}$ ($1 \leq r \leq n - 1$), $\{a^{2j}b \mid 0 \leq j \leq n - 1\}$ and $\{a^{2j+1}b \mid 0 \leq j \leq n - 1\}$. It is easy to see that T_{4n} has exactly four linear characters and $n - 1$ non-linear irreducible characters recorded in Table 1.

Example 2.8. An MNS of T_{4n} can be computed as $S = (ab)^{T_{4n}} \cup b^{T_{4n}}$ and $S = a^{T_{4n}} \cup b^{T_{4n}}$, when n is odd and even, respectively. The simple eigenvalues of Cayley graph $Cay(T_{4n}, S)$ are $\{n+2, -n-2, n-2, -n+2\}$ and $\{2n, -2n\}$, for even and odd n , respectively. Moreover, if n is even then this graph has eigenvalues $2Cos(\frac{\pi j}{n})$ with multiplicity four, where $1 \leq j \leq n - 1$. If n is odd, 0 is an eigenvalue of $Cay(T_{4n}, S)$ with multiplicity $4n - 2$. To see this, it is easy to see that S is an $MNSG$ of T_{4n} . By Proposition 2.1, if n is even then $\lambda_{\chi_i} = \frac{1}{\chi_i(e)}(2\chi_i(a) + n\chi_i(b))$. If n is odd, then $\lambda_{\chi_i} = \frac{n}{\chi_i(e)}(\chi_i(b) + \chi_i(ab))$. So, by Table 1, one can easily compute the eigenvalues of $Cay(T_{4n}, S)$. If n is odd then $E(Cay(T_{4n}, S)) = 4n$ and if n is even then the energy of this Cayley graph is computed as follows:

$$\begin{aligned} E(Cay(T_{4n}, S)) &= 4n + 8 \sum_{j=1}^{n-1} |Cos(\frac{\pi j}{n})| \\ &\approx 4n + \frac{8n}{\pi} \int_0^\pi |Cos(x)|dx - 8 = 4n + \frac{16n}{\pi} - 8. \end{aligned}$$

Therefore we have

$$E(\text{Cay}(T_{4n}, S)) \approx \begin{cases} 4n & \text{if } n \text{ is odd} \\ 4n + \frac{16n}{\pi} - 8 & \text{if } n \text{ is even.} \end{cases}$$

By a similar argument, one can see that $T = S \cup \{a^n\}$ is an *SMNS* for T_{4n} . If n is odd then the simple eigenvalues of $\text{Cay}(T_{4n}, T)$ are $1 \pm 2n$. Other eigenvalues can be computed as -1 with multiplicity $2n$ and 1 with multiplicity $2n - 2$. If n is even then the simple eigenvalues of $\text{Cay}(T_{4n}, T)$ are $3 \pm n$ and $-1 \pm n$. Other eigenvalues can be computed as $2\text{Cos}(\frac{\pi j}{n}) + (-1)^j$, $1 \leq j \leq n - 1$, with multiplicity 4.

Example 2.9. The character table of V_{8n} computed in [13, 421], when n is odd. Darafsheh and Poursalavati [6] generalized this group in the case that n is even and computed its character table. This group has exactly $2n + 3$ conjugacy classes, if n is odd, and $2n + 6$ conjugacy classes, for even n . If n is odd then the conjugacy classes are: $\{1\}$, $\{b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}$ ($0 \leq r \leq n - 1$), $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ ($1 \leq s \leq \frac{n-1}{2}$), $\{a^j b^k \mid j \text{ is even} \ \& \ k = 1, 3\}$, $\{a^j b^k \mid j \text{ is odd} \ \& \ k = 1, 3\}$. The conjugacy classes, for even n , are: $\{1\}$, $\{b^2\}$, $\{a^n\}$, $\{a^n b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}$ ($0 \leq r \leq n - 1$), $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ ($1 \leq s \leq \frac{n}{2} - 1$), $\{a^{2k}b^{(-1)^k} \mid 0 \leq k \leq n - 1\}$, $\{a^{2k}b^{(-1)^{k+1}} \mid 0 \leq k \leq n - 1\}$, $\{a^{2k+1}b^{(-1)^k} \mid 0 \leq k \leq n - 1\}$, $\{a^{2k+1}b^{(-1)^{k+1}} \mid 0 \leq k \leq n - 1\}$. An *MNS* of V_{8n} can be computed at $S = a^{V_{8n}} \cup (a^{-1})^{V_{8n}} \cup b^{V_{8n}}$ and $S = a^{V_{8n}} \cup b^{V_{8n}} \cup (a^{-1})^{V_{8n}} \cup (b^{-1})^{V_{8n}}$, when n is odd and even, respectively. The simple eigenvalues of Cayley graphs $\text{Cay}(V_{8n}, S)$ are $\{\pm(2n + 4), \pm(2n - 4)\}$. Moreover these graphs have eigenvalues 0, with multiplicities $4n$ and $4\text{Cos}(\frac{\pi j}{n})$, $1 \leq j \leq n - 1$, with multiplicities four. By our calculations, the energy of V_{8n} can be evaluated as follows:

$$E(\text{Cay}(V_{8n}, S)) \approx \begin{cases} 16 & n = 1 \\ 8n + \frac{32n}{\pi} - 16 & \text{otherwise.} \end{cases}$$

A second minimal is $T = S \cup \{b^2\}$. The simple eigenvalues of $\text{Cay}(V_{8n}, T)$ are $5 \pm 2n$ and $-3 \pm 2n$. If n is even then other eigenvalues are -1 with multiplicity 4, 1 with multiplicity $4(n - 1)$ and $4\text{Cos}(\frac{\pi j}{n}) + 1$, $1 \leq j \leq n - 1$, each of which with multiplicity four. If n is odd then other eigenvalues are -1 with multiplicity $4n$ and $4\text{Cos}(\frac{\pi j}{n}) + 1$, $1 \leq j \leq n - 1$, each of which with multiplicity four

The group SD_{8n} is presented by $SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1 \ \& \ bab = a^{2n-1} \rangle$. In the following result the energy of the Cayley graph of this group with respect to its unique *MNS* is approximately computed.

Proposition 2.10. *Suppose S is an *MNS* of SD_{8n} . The energy of $E(\text{Cay}(SD_{8n}), S)$ can be evaluated as follows:*

$$E(\text{Cay}(SD_{8n}), S) \approx \begin{cases} \frac{32n}{\pi} + 4n & n = 1, 3 \\ 8n + \frac{32n}{\pi} - 16 & \text{otherwise.} \end{cases}$$

Proof. All $8n$ elements of SD_{8n} may be given by $\{1, a, \dots, a^{4n-1}, b, ba, \dots, ba^{4n-1}\}$. Following Hormozi and Rodtes [11], we define $C^{even} = C_1 \cup C_2^{even} \cup C_3^{even}$ and $C^{odd} = C_1 \cup C_2^{odd} \cup C_3^{odd}$, where $C_1 = \{0, 2, 4, \dots, 2n\}$, $C_2^{even} = \{1, 3, 5, \dots, n-1\}$, $C_3^{even} = \{2n+1, 2n+3, 2n+5, \dots, 3n-1\}$, $C_2^{odd} = \{1, 3, 5, \dots, n\}$, $C_3^{odd} = \{2n+1, 2n+3, 2n+5, \dots, 3n\}$, $C_{even}^\dagger = C_1 \setminus \{0, 2n\}$ and $C_{odd}^\dagger = C_2^{even} \cup C_3^{even}$. Moreover, we assume that $C_\star^{even} = C^{even} \setminus \{0, 2n\}$ and $C_\star^{odd} = C^{odd} \setminus \{0, n, 2n, 3n\}$. Then by [11, Proposition 2.2], the conjugacy classes of SD_{8n} , $n \geq 2$, can be computed as follows:

- If n is even, there are $2n + 3$ conjugacy classes as follows:
 - 2 classes of size one being $[1] = \{1\}$ and $[a^{2n}] = \{a^{2n}\}$,
 - $2n - 1$ conjugacy classes of size two being $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_\star^{even}$,
 - 2 classes of size $2n$ being $[b] = \{ba^{2t} \mid 0 \leq t \leq 2n - 1\}$ and $[ba] = \{ba^{2t+1} \mid 0 \leq t \leq 2n - 1\}$.
- If n is odd, then there are $2n + 6$ conjugacy classes as follows:
 - 4 classes of size one being $[1] = \{1\}$, $[a^n] = \{a^n\}$, $[a^{2n}] = \{a^{2n}\}$ and $[a^{3n}] = \{a^{3n}\}$,
 - $2n - 2$ classes of size two being $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_\star^{odd}$,
 - 4 classes of size n being $[b] = \{ba^{4t} \mid 0 \leq t \leq n - 1\}$, $[ba] = \{ba^{4t+1} \mid 0 \leq t \leq n - 1\}$, $[ba^2] = \{ba^{4t+2} \mid 0 \leq t \leq n - 1\}$ and $[ba^3] = \{ba^{4t+3} \mid 0 \leq t \leq n - 1\}$.

An *MNS* of SD_{8n} can be computed as $S = (a)^{SD_{8n}} \cup (a^{-1})^{SD_{8n}} \cup b^{SD_{8n}}$. This graph has simple eigenvalues $\pm(4 + 2n)$ and $\pm(4 - 2n)$. If n is even then other eigenvalues are $4\text{Cos}(\frac{h\pi}{2n})$, where $h \in C_{even}^\dagger$, each of which with multiplicity four and 0 with multiplicity $4n$. If n is odd, other eigenvalues are $4\text{Cos}(\frac{h\pi}{2n})$, where $h \in C_{even}^\dagger$, each of which with multiplicity four, $\pm n$ and 0 with multiplicities two and $4n - 8$ respectively. By [19, Theorem 1], if n is even then $\lambda_{\chi_i} = \frac{2}{\chi_i(e)}(\chi_i(a) + \chi_i(a^{-1}) + n\chi_i(b))$. If n is odd, then $\lambda_{\chi_i} = \frac{1}{\chi_i(e)}(2\chi_i(a) + 2\chi_i(a^{-1}) + n\chi_i(b))$. By these calculations and a similar argument as Example 2.8, the energy of SD_{8n} can be computed as follows:

$$E(\text{Cay}(SD_{8n}), S) \approx \begin{cases} \frac{32n}{\pi} + 4n & n = 1, 3 \\ 8n + \frac{32n}{\pi} - 16 & \text{otherwise.} \end{cases}$$

which proves the result. □

Example 2.11. In this example, the eigenvalues of $\text{Cay}(G, S)$ are computed, where G is a Mathieu group and S is an *MNS* or *SMNS* for G . Since G is a simple group, $\langle A \rangle = G$, if A is a conjugacy class of G . So, if a real conjugacy class S of G has the minimum size between all conjugacy classes of G then S is an

MNS for G . It is easy to see that for each Mathieu group, the conjugacy class $2A$ is real and it has the minimum size between all real conjugacy classes of G . The conjugacy class $2B$ in M_{12} and M_{24} is also real and has the second size between all real conjugacy classes of these groups and their union. The conjugacy class $3A$ in M_{11} , M_{22} and M_{23} is again real and has the second size between all real conjugacy classes of these groups and their union. These conjugacy classes are $SMNS$ of G and all eigenvalues with respect to these classes are recorded in Tables 2–6.

For the sake of completeness, we present here some details on the Suzuki group. Following Suzuki [16], a group G is called a ZT -group if G acts on a set Ω in such a way that, (1) G is a doubly transitive group on $1 + N$ symbols, (2) the identity is the only element which leaves three distinct symbols invariant, (3) G contains no normal subgroup of order $1 + N$, and (4) N is even. Suzuki [16] proved that for each prime power $q = 2^{2s+1}$, there is a unique ZT -group $Sz(q)$ of order $q^2(q-1)(q^2+1)$ which is called later the Suzuki group. This group is simple, when $q > 2$. Suppose that a is a symbol on which G acts and $H = G_a$. By [16], it follows from the conditions (1) and (2) that H is a Frobenius group on $\Omega \setminus \{a\}$. Apply a well-known result of Frobenius to deduce that H contains a regular normal subgroup Q of order N such that every non-identity element of Q leaves only the symbol a invariant. Suppose $b \in \Omega \setminus \{a\}$ and $K = H_b$. Suppose $x \in N_G(K)$ is an involution. Then it is well-known that the Suzuki group are containing two elements y and z such that y is an involution and $xyx = z^{-1}xz$, and three cyclic subgroups A_0 , A_1 and A_2 of orders $q-1$, $q+r+1$ and $q-r+1$, respectively. By [16], The conjugacy classes of $Sz(q)$ can be computed as follows: $\{e\}$, $y^{Sz(q)}$, $z^{Sz(q)}$, $(z^{-1})^{Sz(q)}$, $b_0^{Sz(q)}$, $b_1^{Sz(q)}$ and $b_2^{Sz(q)}$ of lengths 1, $(q-1)(q^2+1)$, $\frac{1}{2}q(q-1)(q^2+1)$, $\frac{1}{2}q(q-1)(q^2+1)$, $q^2(q-1)(q+r+1)$, $q^2(q+r+1)(q-r+1)$ and $q^2(q-1)(q-r+1)$, respectively. Here, b_0, b_1 and b_2 are non-identity elements of A_i , $i = 0, 1, 2$, respectively. Note that there are $\frac{q-r}{4}$, $\frac{q}{2} - 1$ and $\frac{q+r}{4}$ conjugacy classes of types $b_0^{Sz(q)}$, $b_1^{Sz(q)}$ and $b_2^{Sz(q)}$, respectively.

One can also find the character table of this group in [16]. Hence, by applying above information on Suzuki groups and Proposition 2.1 we have the following proposition:

Proposition 2.12. *Consider the Suzuki group $Sz(q)$ with $q = 2^{2s+1}$, $r = 2^{s+1}$ and $s \geq 1$. The conjugacy class $S = y^{Sz(q)}$ and the normal subset $T = z^{Sz(q)} \cup (z^{-1})^{Sz(q)}$ are the MNS and $SMNS$ of $Sz(q)$, respectively. Moreover, $|S| = (q-1)(q^2+1)$, $|T| = q(q-1)(q^2+1)$ and the simple eigenvalues of $Cay(Sz(q), S)$ and $Cay(Sz(q), T)$ are $|S|$ and $|T|$, respectively. The Cayley graph $Cay(Sz(q), S)$ has eigenvalues 0 , $-(q^2+1)$, $(q-1)$, $\frac{(1+q^2)(r-1)}{q-r+1}$ and $\frac{-(1+q^2)(r+1)}{q+r+1}$ with multiplicities q^4 , $\frac{(q-1)^2(r^2)}{2}$, $\frac{q-2}{2}(q^2+1)^2$, $\frac{q+r}{4}((q-r+1)(q-1))^2$ and $\frac{q-r}{4}((q+r+1)(q-1))^2$, respectively. The energy of $Cay(Sz(q), S)$ is as follows:*

$$\begin{aligned}
E(\text{Cay}(Sz(q), S)) &= -\frac{\sqrt{2q}}{2} + \frac{\sqrt{2q^3}}{2} - \sqrt{2q^7} + \frac{\sqrt{2q^5}}{2} + \frac{3\sqrt{2q^9}}{2} - \frac{3\sqrt{2q^{11}}}{2} \\
&+ \frac{\sqrt{2q^{13}}}{2} - q^5 + \frac{\sqrt{q^2}}{2} - q^3 + q^4 + \frac{\sqrt{q^6}}{2}.
\end{aligned}$$

The Cayley graph $\text{Cay}(Sz(q), S)$ has eigenvalues 0 , $q(q-1)$, $-\frac{q(q^2+1)}{q-r+1}$ and $-\frac{q(q^2+1)}{q+r+1}$ with multiplicities $\frac{1}{4}(4q^4+r^2(q-1)^2)$, $\frac{1}{2}(q-2)(q^2+1)^2$, $\frac{1}{4}(q-r+1)^2(q-1)^2(q+r)$ and $\frac{1}{4}(q+r+1)^2(q-1)^2(q-r)$, respectively.

We end our paper by computing eigenvalues of a group of type Ree of characteristic q . We refer to [15, 16, 18] for our notations and known results concerning this important class of simple groups. A finite group G has Ree type if G satisfies the following conditions:

- The Sylow 2-subgroups of G are elementary abelian of order 8.
- The group G has no normal subgroup of index 2.
- There is an involution J in G such that the centralizer $C_G(J) \cong \langle J \rangle \times LF(2, q)$, where $L = LF(2, q)$ denotes the linear fractional group on $GF(q)$.
- If $\langle R \rangle$ denotes a cyclic subgroup of order $\frac{q+e}{2}$ in L , then for any subgroup $1 \neq \langle R_0 \rangle$, we have $N_G(\langle R_0 \rangle) \leq C_G(J)$.
- Let J' be an involution of L and S an element of L of order $\frac{q-e}{4}$ which centralizes J' . Then an element of G of order 3 which normalizes $\langle J, J' \rangle$ does not centralize S . We call q the characteristic of G .

Note that the (I) implies that $q \equiv 4 + e \pmod{8}$ where $e = \pm 1$. In the end of this paper, we consider the simple Ree group $G_2(q)$ of characteristic q and order $q^3(q-1)(q^3+1)$, where $q = 3^{2k+1}$ and $k \geq 1$. This group has exactly $q+8$ conjugacy classes [18]. Suppose $m = 3^k$. Then we have:

Proposition 2.13. *The conjugacy class $S = J^{G_2(q)}$ is the unique MNS of $G_2(q)$ with size $\frac{q(q^2-q+1)}{q^2+1}$. The energy of $\text{Cay}(G_2(q), S)$ can be computed by the following formula:*

$$\begin{aligned}
E(\text{Cay}(G_2(q), S)) &= \frac{1}{4(1+3^{4k+2})} (19 \cdot 3^{22k} + 4e3^{8k} + 3^{4k+3} + 3^{2k+1} + (2e-11)3^{21k} \\
&+ 2e3^{2k+1} + 3^{25k} + 280 \cdot 3^{18k} + 3^{13k} - 4 \cdot 3^{10k}) \\
&+ \frac{4}{(3^{2k+1} + 3^{k+1} + 1)(1 + 3^{4k+2})} (3^{22k} + 2 \cdot 3^{19k} + 20 \cdot 3^{16k} + 3^{8k} \\
&- 11 \cdot 3^{10k} + 14 \cdot 3^{13k}).
\end{aligned}$$

Proof. Since $G_2(q)$ has exactly one conjugacy class of involutions, $S = S^{-1}$, and since $G_2(q)$ is simple, $G_2(q) = \langle S \rangle$. On the other hand, the lengths of non trivial conjugacy classes of $G_2(q)$ are as follows:

$$\begin{aligned} & q^3(q^2 - q + 1), \frac{q^3(q-1)(q^3+1)}{q^2+1}, \frac{q^3(q-1)(q^3+1)}{q^2-3^{\frac{q+2}{2}}}, \\ & \frac{q^3(q-1)(q^3+1)}{q^2+3^{\frac{q+2}{2}}}, \frac{(q-1)(q^3+1)}{q^2}, \frac{q(q-1)(q^3+1)}{2}, \\ & \frac{(q-1)(q^3+1)}{2q}, \frac{q(q-1)(q^3+1)}{3}, \frac{q(q^2-q+1)}{q^2+1}. \end{aligned}$$

By above information on conjugacy lengths of $G_2(q)$ and some tedious calculations, one can see that $S = J^{G_2(q)}$ is the unique MNS of $G_2(q)$. The simple eigenvalues of $\text{Cay}(G_2(q), S)$ is $|S|$. This graph has eigenvalues

$$\begin{aligned} & 0, -\frac{q}{q^2+1}, \frac{3q}{q^2+1}, \frac{q}{q^2+1}, \\ & \frac{q(q^2-q+1)}{m(q^2+1)(q+3m+1)}, \frac{q(q^2-q+1)}{m(q^2+1)(q-3m+1)} \end{aligned}$$

with multiplicities, $3^{14k+6} - 3^{12k+5} - 5 \cdot 3^{10k+4} - 7 \cdot 3^{6k+2} + 2 \cdot 3^{8k+3} - 3^{4k+1} - 3^{2k+1}$, $\frac{1}{8} \cdot 3^{14k+8} - \frac{11}{8} \cdot 3^{12k+6} + \frac{e}{4} \cdot 3^{12k+6} + \frac{5}{2} \cdot 3^{10k+5} - \frac{11}{4} \cdot 3^{8k+4} + \frac{7}{4} \cdot 3^{6k+3} - \frac{1}{2} \cdot 3^{4k+2} - \frac{1}{8} \cdot 3^{2k+1} + \frac{1}{8} + \frac{e}{2} \cdot 3^{6k+3} + \frac{e}{4}$, $\frac{1}{8} \cdot 3^{14k+6} - \frac{7}{8} \cdot 3^{12k+5} + \frac{5}{2} \cdot 3^{10k+4} - \frac{17}{4} \cdot 3^{8k+3} + \frac{19}{4} \cdot 3^{6k+2} - \frac{7}{2} \cdot 3^{4k+1} + \frac{13}{8} \cdot 3^{2k} - \frac{1}{8}$, $\frac{1}{4} \cdot 3^{14k+7} - \frac{1}{2} \cdot 3^{12k+6} + \frac{e}{4} \cdot 3^{12k+6} + \frac{1}{2} \cdot 3^{8k+4} - 3^{6k+3} + \frac{e}{2} \cdot 3^{6k+3} + \frac{1}{4} \cdot 3^{2k+1} - \frac{1}{2} + \frac{e}{4}$, $2 \cdot 3^{10k+4} + 4 \cdot 3^{9k+4} + 2 \cdot 3^{8k+4} - 4 \cdot 3^{7k+3} - 16 \cdot 3^{6k+2} - 4 \cdot 3^{5k+2} + 2 \cdot 3^{4k+2} + 4 \cdot 3^{3k+1} + 2 \cdot 3^{2k}$, $2 \cdot 3^{10k+4} - 4 \cdot 3^{9k+4} + 2 \cdot 3^{8k+4} + 4 \cdot 3^{7k+3} - 16 \cdot 3^{6k+2} + 4 \cdot 3^{5k+2} + 2 \cdot 3^{4k+2} - 4 \cdot 3^{3k+1} - 2 \cdot 3^{2k}$, respectively. Therefore, the energy of $\text{Cay}(G_2(q), S)$ is as follows:

$$\begin{aligned} E(\text{Cay}(G_2(q), S)) &= \frac{1}{4(1+3^{4k+2})} (19 \cdot 3^{22k} + 4e3^{8k} + 3^{4k+3} + 3^{2k+1} + (2e-11)3^{21k} \\ &+ 2e3^{2k+1} + 3^{25k} + 280 \cdot 3^{18k} + 3^{13k} - 4 \cdot 3^{10k}) \\ &+ \frac{4}{(3^{2k+1} + 3^{k+1} + 1)(1+3^{4k+2})} (3^{22k} + 2 \cdot 3^{19k} + 20 \cdot 3^{16k} + 3^{8k} \\ &- 11 \cdot 3^{10k} + 14 \cdot 3^{13k}). \end{aligned}$$

This completes the proof. \square

By the notation of [18] we have the following proposition.

Proposition 2.14. *The conjugacy class $T = X^{G_2(q)}$ is the unique SMNS of $G_2(q)$ with size $\frac{(q-1)(q^3+1)}{q^2}$. The simple eigenvalues of $\text{Cay}(G_2(q), T)$ is $|T|$ and the graph has the eigenvalues*

$$\begin{aligned} & 0, \frac{-(q-1)^2(q^3+1)}{q^2(q^2-q+1)}, \frac{(q-1)(q^3+1)}{q^2(q^2-q+1)}, \frac{-(q+m)(q^3+1)}{mq^2(q+3m+1)}, \frac{(q+m)(q^3+1)}{mq^2(q-3m+1)}, \\ & \frac{-(q^3+1)}{q^2(q+1)}, \frac{q-1}{q^2}, \frac{(2q-1)(q^3+1)}{q^2(q^2-q+1)}, \frac{-(q+1+3m)(q^3+1)}{(q+1)(q+1+3m)q^2}, \frac{-(q+1-3m)(q^3+1)}{(q+1)(q+1-3m)q^2}, \end{aligned}$$

with multiplicities, $q^6, 3^{8k+4} - 2 \cdot 3^{6k+3} + 3^{4k+3} - 2 \cdot 3^{2k+1} + 1, 3^{12k+6} - 2 \cdot 3^{10k+5} + 3^{8k+5} - 2 \cdot 3^{6k+3}, \frac{1}{2} \cdot 3^{10k+4} + \frac{1}{2} \cdot 3^{8k+4} + 3^{9k+4} - 4 \cdot 3^{6k+4} - 3^{7k+3} + \frac{1}{2} \cdot 3^{4k+2} - 3^{5k+2} + \frac{1}{2} \cdot 3^{2k} + 3^{3k+1} + 3^{3k+1}, \frac{1}{2} \cdot 3^{10k+4} + 3^{9k+4} + \frac{1}{2} \cdot 3^{8k+4} + 3^{7k+3} - 4 \cdot 3^{6k+2} + 3^{5k+2} + \frac{1}{2} \cdot 3^{4k+2} - 3^{3k+1} + \frac{1}{2} \cdot 3^{2k}, 2 \cdot 3^{10k+4} - 4 \cdot 3^{6k+2} + 2 \cdot 3^{2k}, \frac{1}{2} \cdot 3^{14k+7} - \frac{1}{2} \cdot (1 + \frac{\epsilon}{2}) \cdot 3^{12k+6} + 3^{8k+4} + (e - 2) \cdot 3^{6k+3} + \frac{1}{2} \cdot 3^{2k+1} - 1 + \frac{\epsilon}{2}, \frac{1}{2} (3^{14k+6} - 7 \cdot 3^{12k+5} + 20 \cdot 3^{10k+4} - 34 \cdot 3^{8k+3} + 38 \cdot 3^{6k+2} - 28 \cdot 3^{4k+1} + 13 \cdot 3^{2k} - 1), \frac{1}{2} \cdot 3^{14k+6} - \frac{7}{2} \cdot 3^{12k+5} + 10 \cdot 3^{10k+4} - 17 \cdot 3^{8k+3} + 19 \cdot 3^{6k+2} - 14 \cdot 3^{4k+1} - \frac{13}{2} \cdot 3^{2k} - \frac{1}{2}, \frac{1}{2} \cdot 3^{14k+6} + \frac{1}{2} \cdot 3^{13k+7} + \frac{11}{2} \cdot 3^{12k+5} + \frac{7}{2} \cdot 3^{11k+5} + \frac{5}{2} \cdot 3^{10k+4} - \frac{5}{2} \cdot 3^{9k+4} - 11 \cdot 3^{8k+3} - 7 \cdot 3^{7k+3} - \frac{13}{2} \cdot 3^{6k+2} + \frac{1}{2} \cdot 3^{5k+2} + \frac{11}{2} \cdot 3^{4k+1} + \frac{7}{2} \cdot 3^{3k+1} + \frac{7}{2} \cdot 3^{2k} + \frac{1}{2} \cdot 3^k, \frac{1}{2} \cdot 3^{14k+6} - \frac{1}{2} \cdot 3^{13k+6} - \frac{11}{2} \cdot 3^{12k+5} + \frac{1}{2} \cdot 3^{11k+6} - \frac{7}{2} \cdot 3^{10k+4} + \frac{1}{2} \cdot 3^{9k+5} + 3^{8k+3} - 3^{7k+4} + \frac{11}{2} \cdot 3^{6k+2} - \frac{1}{2} \cdot 3^{5k+3} - \frac{1}{2} \cdot 3^{4k+1} + \frac{1}{2} \cdot 3^{3k+2} - \frac{5}{2} \cdot 3^{2k} + \frac{1}{2} \cdot 3^k, respectively.$

Table 1: The Character Table of T_{4n} .

| g_i $ C_G(g_i) $ | e $4n$ | a^n $4n$ | a^r ($1 \leq r \leq n-1$) $2n$ | b 4 | ab 4 |
|----------------------------------|-----------|---------------|---------------------------------------|--------|---------|
| Non-Linear Characters | | | | | |
| ψ_j ($1 \leq j \leq n-1$) | 2 | $2(-1)^j$ | $w^{rj} + w^{-rj}$ | 0 | 0 |
| Linear Characters | | | | | |
| <i>n is odd</i> | | | | | |
| χ_1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | $(-1)^r$ | i | $-i$ |
| χ_3 | 1 | 1 | 1 | -1 | -1 |
| χ_4 | 1 | -1 | $(-1)^r$ | $-i$ | i |
| <i>n is even</i> | | | | | |
| λ_1 | 1 | 1 | 1 | 1 | 1 |
| λ_2 | 1 | 1 | 1 | -1 | -1 |
| λ_3 | 1 | 1 | $(-1)^r$ | 1 | -1 |
| λ_4 | 1 | 1 | $(-1)^r$ | -1 | 1 |

Table 2: The Eigenvalues of $Cay(M_{11}, 2A)$ and $Cay(M_{11}, 3A)$

| $2A - Eigenvalues$ | $Multiplicities$ | $3A - Eigenvalues$ | $Multiplicities$ |
|--------------------|------------------|--------------------|-------------------------|
| -33 | $2^3 \cdot 5^2$ | -55 | 2^9 |
| -11 | $3^4 \cdot 5^2$ | -10 | $2^4 \cdot 11^2$ |
| -3 | $5^2 \cdot 11^2$ | 0 | $3^4 \cdot 5^2$ |
| 0 | 2^9 | 8 | $5^2 \cdot 11^2$ |
| 15 | $2^4 \cdot 11^2$ | 44 | $2^2 \cdot 3 \cdot 5^2$ |
| 33 | $2^2 \cdot 5^2$ | 80 | 11^2 |
| 45 | 11^2 | 440 | 1 |
| 165 | 1 | - | - |

Table 3: The Eigenvalues of $Cay(M_{12}, 2A)$ and $Cay(M_{12}, 2B)$.

| $2A - \text{Eigenvalues}$ | Multiplicities | $2B - \text{Eigenvalues}$ | Multiplicities |
|---------------------------|----------------------------|---------------------------|---------------------------|
| -36 | $7 \cdot 11^3$ | -33 | $3^2 \cdot 5^2 \cdot 7^3$ |
| -9 | $2^8 \cdot 11^2$ | -9 | $2 \cdot 5^2 \cdot 11^2$ |
| -4 | $3^4 \cdot 11^2$ | 0 | $3^4 \cdot 5^2$ |
| 0 | $2^6 \cdot 3^2 \cdot 5^2$ | 15 | $3^2 \cdot 11^2 \cdot 13$ |
| 11 | $2^8 \cdot 3^4$ | 55 | $2^2 \cdot 3^6$ |
| 36 | $2^2 \cdot 3^2 \cdot 11^2$ | 63 | $5^2 \cdot 11^2$ |
| 44 | $3^4 \cdot 61$ | 135 | $2 \cdot 11^2$ |
| 99 | 2^9 | 495 | 1 |
| 396 | 1 | - | - |

Table 4: The Eigenvalues of $Cay(M_{22}, 2A)$ and $Cay(M_{22}, 3A)$.

| $2A - \text{Eigenvalues}$ | Multiplicities | $3A - \text{Eigenvalues}$ | Multiplicities |
|---------------------------|-------------------------------------|---------------------------|----------------------------|
| -77 | $2 \cdot 3^4 \cdot 5^2$ | -160 | $3^2 \cdot 7^2 \cdot 11^2$ |
| -33 | $2^7 \cdot 5^2 \cdot 7^2$ | -64 | $5^2 \cdot 7^2 \cdot 11^2$ |
| 3 | $5^2 \cdot 7^2 \cdot 11^2$ | 0 | $3^6 \cdot 19$ |
| 11 | $2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$ | 44 | $2^7 \cdot 5^2 \cdot 7^2$ |
| 35 | $3^4 \cdot 11^2$ | 80 | $2^2 \cdot 7^2 \cdot 11^2$ |
| 75 | $2^2 \cdot 7^2 \cdot 11^2$ | 224 | $5^2 \cdot 11^2$ |
| 147 | $5^2 \cdot 11^2$ | 1760 | $3^2 \cdot 7^2$ |
| 275 | $3^2 \cdot 7^2$ | 12320 | 1 |
| 1155 | 1 | - | - |

Table 5: The Eigenvalues of $Cay(M_{23}, 2A)$ and $Cay(M_{23}, 3A)$.

| $2A - Eigenvalues$ | $Multiplicities$ | $3A - Eigenvalues$ | $Multiplicities$ |
|--------------------|-------------------------------------|--------------------|--------------------------------------|
| -253 | $2 \cdot 3^4 \cdot 5^2$ | -736 | $2 \cdot 3^2 \cdot 7^2 \cdot 11^2$ |
| -69 | $2^4 \cdot 5^3 \cdot 11^2 \cdot 13$ | -253 | $2^{15} \cdot 7^2$ |
| 0 | $2^{15} \cdot 7^2$ | -28 | $2^6 \cdot 11^2 \cdot 23^2$ |
| 15 | $2^6 \cdot 11^2 \cdot 23^2$ | 0 | $3^4 \cdot 5^2 \cdot 1499$ |
| 99 | $3^4 \cdot 5^2 \cdot 23^2$ | 224 | $11^2 \cdot 23^2$ |
| 115 | $3^3 \cdot 7^2 \cdot 11^2$ | 368 | $2^3 \cdot 5^2 \cdot 7^2 \cdot 11^2$ |
| 195 | $11^2 \cdot 23^2$ | 1232 | $2^2 \cdot 5^2 \cdot 23^2$ |
| 363 | $2^2 \cdot 5^2 \cdot 23^2$ | 1472 | $3^2 \cdot 7^2 \cdot 11^2$ |
| 1035 | $2^2 \cdot 11^2$ | 10304 | $2^2 \cdot 11^2$ |
| 3795 | 1 | 56672 | 1 |

Table 6: The Eigenvalues of $Cay(M_{24}, 2A)$ and $Cay(M_{24}, 2B)$.

| $2A - Eigenvalues$ | $Multiplicities$ | $2B - Eigenvalues$ | $Multiplicities$ |
|--------------------|---|--------------------|---------------------------------------|
| -759 | $2 \cdot 3^4 \cdot 5^2$ | -1386 | $2 \cdot 23^2 \cdot 61$ |
| -231 | $2 \cdot 3^4 \cdot 5^2 \cdot 23^2$ | -1242 | $2 \cdot 3^2 \cdot 7^2 \cdot 11^2$ |
| -207 | $2^4 \cdot 5^3 \cdot 11^2 \cdot 13$ | -378 | $5^2 \cdot 11^2 \cdot 23^2$ |
| -135 | $7^2 \cdot 11^2 \cdot 23^2$ | -322 | $2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$ |
| -115 | $2^6 \cdot 3^4 \cdot 7^2 \cdot 11^2$ | -266 | $3^4 \cdot 11^2 \cdot 23^2$ |
| -55 | $2^4 \cdot 3^4 \cdot 7^2 \cdot 23^2$ | -154 | $2^5 \cdot 5^2 \cdot 23^2$ |
| -23 | $3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2$ | -138 | $3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2$ |
| 45 | $2^6 \cdot 11^2 \cdot 23^2$ | 0 | $2^{12} \cdot 5^2 \cdot 11^2$ |
| 105 | $2^3 \cdot 3^2 \cdot 11 \cdot 23 \cdot 29 \cdot 2783$ | 54 | $3^2 \cdot 7^2 \cdot 11^2 \cdot 23^2$ |
| 165 | $2^8 \cdot 3^4 \cdot 23^2$ | 138 | $2^6 \cdot 3^4 \cdot 7^2 \cdot 11^2$ |
| 207 | $2^{12} \cdot 5^2 \cdot 11^2$ | 154 | $2^8 \cdot 3^4 \cdot 23^2$ |
| 345 | $2 \cdot 3^2 \cdot 7^2 \cdot 11^2$ | 198 | $2^2 \cdot 5^2 \cdot 23^2$ |
| 297 | $3^4 \cdot 5^2 \cdot 23^2$ | 378 | $2^6 \cdot 11^2 \cdot 23^2$ |
| 585 | $11^2 \cdot 23^2$ | 414 | $2^3 \cdot 5^2 \cdot 7^2 \cdot 11^2$ |
| 297 | $3^4 \cdot 5^2 \cdot 23^2$ | 1078 | $2^4 \cdot 5^2 \cdot 23^2$ |
| 441 | $5^2 \cdot 11^2 \cdot 23^2$ | 1518 | $2^4 \cdot 3^4 \cdot 7^2$ |
| 585 | $11^2 \cdot 23^2$ | 3542 | $3^4 \cdot 5^2$ |
| 1265 | $2^4 \cdot 3^4 \cdot 7^2$ | - | - |
| 3465 | 23^2 | - | - |
| 11385 | 1 | - | - |

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