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Fixed Point Theorems for *kg*-Contractive Mappings in a Complete Strong Fuzzy Metric Space

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Abstract

In this paper, we introduce a new class of contractive mappings in fuzzy metric spaces and we present some fixed point results for this class of maps.

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1. Introduction

The concept of a fuzzy metric space was introduced by Kramosil and Micálek [7]. Afterwards, George and Veeramani [1] modified the concept of fuzzy metric space due to [7]. Later on, Gregori and Sapene [4] introduced fuzzy contraction mappings and proved a fixed point theorem in fuzzy metric space in the sense of George and Veeramani. In particular, Miheţ enlarged the class of fuzzy contractive mappings of Gregori and Sapene [4] in a complete non-Archimedean(strong) fuzzy metric space and proved a fuzzy Banach contraction theorem using a strong condition for completeness, now called the completeness in the sense of Grabiec, or G-completeness. Motivated by the "Generalized weak contractions" introduced by Singh et al. [13] in metric spaces. In this paper, we introduce $kg-$contractive

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maps in fuzzy metric spaces and we prove the existence of fixed points in complete strong fuzzy metric spaces.

2. Preliminaries

We begin with some basic definitions and results which will be used in the main part of our paper.

Definition 2.1. [12] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions :

(T1) $*$ is associative and commutative;

(T2) $*$ is continuous;

(T3) $a * 1 = a$ for all $a \in [0,1]$;

(T4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Remark 1. A t-norm $*$ is called positive if $a * b > 0$ for all $a, b \in (0,1)$.

Some examples of continuous t-norms are Lukasievicz t-norm, i.e. $a *_L b = \max\{a + b - 1, 0\}$, product t-norm, i.e. $a * b = ab$, and minimum t-norm, i.e. $a *_M b = \min\{a, b\}$, for $a, b \in [0,1]$.

The concept of fuzzy metric space is defined by George and Veeramani [1] as follows.

Definition 2.2. [1] Let X be a nonempty set and $*$ be a continuous t-norm. Assume that, for each $x, y, z \in X$ and $t, s > 0$, a fuzzy set $M : X \times X \times (0,\infty) \to [0,1]$ satisfies the following conditions:

(M1) $M(x, y, t) > 0$,

(M2) $M(x, y, t) = 1$ if and only if $x = y$,

(M3) $M(x, y, t) = M(y, x, t)$,

(M4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(M5) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Then, we call M a fuzzy metric on X, and we call the 3-tuple $(X, M, *)$ a fuzzy metric space.

Definition 2.3. [5] Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is said to be strong (non- Archimedean) if, for each $x, y, z \in X$ and each $t > 0$, it satisfies

$$(M4^{'}) : M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

Remark 2. The axiom $(M4^{'})$ cannot replace the axiom $(M4)$ in the definition of fuzzy metric since in that case M could not be a fuzzy metric on X (see Example 8 in [11]).

Note that it is possible to define a strong fuzzy metric by replacing $(M4)$ by $(M4^{'})$ and demanding in $(M5)$ that the function $M(x, y, \cdot)$ be an increasing continuous function on t, for each $x, y \in X$ (in fact, in such a case we have $M(x, z, t+s) \geq M(x, y, t+s) * M(y, z, t+s) \geq M(x, y, t) * M(y, z, s)$).

Remark 3. Every fuzzy metric space is not strong fuzzy metric space.

The following example shows that there are non -strong fuzzy metric spaces.

Example 2.4. [6] Let $X = \{x, y, z\}$, $* = \cdot$ (usual product) and $M : X \times X \times (0, \infty) \to [0, 1]$ defined for each $t > 0$ as $M(x, x, t) = M(y, y, t) = M(z, z, t) = 1$, $M(x, z, t) = M(z, x, t) = M(y, z, t) = M(z, y, t) = \frac{t}{t+1}$, $M(x, y, t) = M(y, x, t) = \frac{t^2}{(t+2)^2}$. Then, $(X, M, *)$ is a non-strong fuzzy metric space.

Lemma 2.5. *[2] Let $(X, M, *)$ be a fuzzy metric space. For all $x, y \in X$, $M(x, y, \cdot)$ is non-decreasing function on $(0, \infty)$.*

Remark 4. We observe that $0 < M(x, y, t) < 1$ provided $x \neq y$, for all $t > 0$ (see [8]). Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X, called the topology induced by the fuzzy metric M. This topology is metrizable (see [3]).

Definition 2.6. [1] Let $(X, M, *)$ be a fuzzy metric space.

1. A sequence$\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$.
2. A sequence$\{x_n\}$ in X is called a Cauchy sequence if, for each $0 < \epsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$, for each $n, m \geq n_0$.
3. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
4. A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Remark 5. In a fuzzy metric space, the limit of a convergent sequence is unique.

Definition 2.7. [9] Let $(X, M, *)$ be a fuzzy metric space. Then, the mapping M is said to be continuous on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

when $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times (0, \infty)$ which converges to a point $(x, y, t) \in X \times X \times (0, \infty)$, *i.e.,*

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 2.8. *[10] If $(X, M, *)$ is a fuzzy metric space, then M is a continuous function on $X \times X \times (0, \infty)$.*

Definition 2.9. [4] A fuzzy contractive mapping on a fuzzy metric space in the sense of George and Veeramani $(X, M, *)$ is a self-mapping f of X with the property

$$\frac{1}{M(fx, fy, t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1) \quad \text{for all } x, y \in X, \text{ for all } t > 0. \tag{1}$$

First, we define the following :

Definition 2.10. Let $\psi : (0, 1] \to [1, \infty)$ be a function which satisfies the following conditions:

1. ψ is continuous and non-increasing;

2. $\psi(x) = 1$ if and only if $x = 1$.

We denote by Ψ the class of all functions which satisfies the above conditions.

Note that $\Psi \neq \emptyset$, in fact the map $\psi : (0, 1] \to [1, \infty)$ defined by $\psi(t) = \frac{1}{t}$ is a member of Ψ.

Definition 2.11. Let $\phi : (0, 1] \to (0, 1]$ be a function which satisfies the following conditions:

1. ϕ is upper semi continuous;

2. $\phi(s) = 1$ if and only if $s = 1$.

We denote by Φ the class of all functions which satisfies the above conditions.

Note that $\Phi \neq \emptyset$, in fact the map $\phi : (0, 1] \to (0, 1]$ defined by $\phi(t) = \sqrt{t}$ is a member of Φ.

Now, we introduce a kg-contractive mapping in a fuzzy metric space.

Definition 2.12. Let $(X, M, *)$ be a fuzzy metric space. We say that a mapping $T : X \to X$ is a kg-contractive mapping if there exists $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\begin{gathered} M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t) \\ \text{implies} \\ \psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N(x, y, t)), \end{gathered} \tag{2}$$

for all x, y in X and $t > 0$, where

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\}.$$

Example 2.13. Let $X = [0, \infty)$ and $M(x, y, t) = (\frac{t}{t+1})^{d(x,y)}$, where $d(x, y) = |x - y|$, and $*$ be the product continuous t-norm. Here, $(X, M, *)$ is a complete fuzzy metric space. Let $T : X \to X$ be a map defined by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{if } x \leq 1 \\ 0, & \text{if } x > 1 \end{cases}$$

we can easily see that T is a $kg-$contractive map.

In Section 3, we prove the existence of fixed points of kg-contractive mappings in a complete strong fuzzy metric space.

3. Main Results

The following proposition is important to prove our main result.

Proposition 3.1. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $T : X \to X$ be a kg-contractive mapping. Fix $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$, and the sequence $\{M(x_n, x_{n+1}, t)\}$ is increasing in $[0, 1]$ for all $t > 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since the mapping T is a kg-contractive map, there exists $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N(x, y, t))), \tag{3}$$

for all x, y in X and $t > 0$.

Suppose that sequence $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \geq 1$, there are positive integers $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ such that

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon. \tag{4}$$

We assume that $m(k)$ is the least integer exceeding $n(k)$ and satisfies the above inequality, that is equivalent to

$$M(x_{n(k)}, x_{m(k)-1}, t_0) > 1 - \epsilon \text{ and } M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon.$$

Now we have

$$\begin{aligned} 1 - \epsilon \geq M(x_{n(k)}, x_{m(k)}, t_0) &\geq M(x_{n(k)}, x_{m(k)-1}, t_0) * M(x_{m(k)-1}, x_{m(k)}, t_0) \\ &> (1 - \epsilon) * M(x_{m(k)-1}, x_{m(k)}, t_0). \end{aligned}$$

Then $\lim_{k\to\infty}(1 - \epsilon) * M(x_{m(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon$. Hence,

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, t_0)$$

exists and $\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, t_0) = 1 - \epsilon$.

First we prove that

(i) $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1-\epsilon$;

(i) $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1-\epsilon$;

(iii) $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1-\epsilon$.

Since

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, t_0) &\geq M(x_{m(k)}, x_{m(k)-1}, t_0) \\ &\quad * M(x_{m(k)-1}, x_{n(k)-1}, t_0) \\ &\quad * M(x_{n(k)-1}, x_{n(k)}, t_0) \end{aligned} \tag{5}$$

and

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}, t_0) &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) \\ &\quad * M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad * M(x_{n(k)}, x_{n(k)-1}, t_0), \end{aligned} \tag{6}$$

by taking the limit superior in (5) and the limit inferior in (6), we get

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) \tag{7}$$

and

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) \geq 1-\epsilon. \tag{8}$$

Since the limit superior is always greater than or equal to the limit inferior, from (7) and (8), we obtain

$$\begin{aligned} \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) &= 1-\epsilon, \\ \liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) &= 1-\epsilon. \end{aligned}$$

Thus, $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0)$ exists and equal to $1-\epsilon$. Thus (i) holds. We now prove (ii). By condition $(M4^{'})$ of the strong fuzzy metric space, we have

$$M(x_{m(k)-1}, x_{n(k)}, t_0) \geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)}, x_{n(k)}, t_0), \tag{9}$$

and

$$M(x_{m(k)}, x_{n(k)}, t_0) \geq M(x_{m(k)}, x_{m(k)-1}, t_0) * M(x_{m(k)-1}, x_{n(k)}, t_0). \tag{10}$$

Taking the limit inferior in (9) and the limit superior in (10) as $k \to \infty$, we have

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq 1-\epsilon,$$

and

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0).$$

This implies

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq \liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq 1-\epsilon.$$

Thus,

$$\limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = \liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1 - \epsilon.$$

Hence, $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0)$ exists and $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1 - \epsilon$, which proves (ii). We now prove (iii). By the condition $(M4^{'})$ in a strong fuzzy metric space, we have

$$M(x_{n(k)-1}, x_{m(k)}, t_0) \geq M(x_{n(k)-1}, x_{n(k)}, t_0) * M(x_{n(k)}, x_{m(k)}, t_0), \tag{11}$$

and

$$M(x_{n(k)}, x_{m(k)}, t_0) \geq M(x_{n(k)}, x_{n(k)-1}, t_0) * M(x_{n(k)-1}, x_{m(k)}, t_0). \tag{12}$$

Taking the limit inferior in (11) and the limit superior in (12) as $k \to \infty$, we obtain

$$\liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq 1 - \epsilon,$$

and

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0).$$

This implies

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq \liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq 1 - \epsilon.$$

Thus,

$$\limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = \liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon.$$

Hence $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0)$ exists and $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon$, so (iii) holds.

Now, since $m(k) + 1 > m(k) \geq n(k) + 1 > n(k)$ and $\{M(x_n, x_{n+1}, t)\}$ is increasing for any t, it follows that $M(x_{m(k)}, x_{m(k)+1}, t_0) \geq M(x_{n(k)}, x_{n(k)+1}, t_0)$. On the other hand, from the property of the fuzzy metric space we have

$$M(x_{m(k)}, x_{n(k)}, t_0) * M(x_{n(k)}, x_{n(k)+1}, t_0) \leq M(x_{n(k)}, x_{n(k)+1}, t_0).$$

Thus,

$$M(x_{m(k)}, x_{n(k)}, t_0) * M(x_{n(k)}, x_{n(k)+1}, t_0) \leq M(x_{m(k)}, x_{m(k)+1}, t_0).$$

From (3) we have

$$\begin{aligned} \psi(M(x_{m(k)+1}, x_{n(k)+1}, t_0)) &= \psi(M(Tx_{m(k)}, Tx_{n(k)}, t_0)) \qquad (13) \\ &\leq \psi(N(x_{n(k)}, x_{m(k)}, t_0))\phi(N(x_{m(k)}, x_{n(k)}, t_0)), \end{aligned}$$

where

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}, t_0) &= \min\{M(x_{m(k)}, x_{n(k)}, t_0), M(x_{m(k)}, Tx_{m(k)}, t_0), \\ &\quad M(x_{n(k)}, Tx_{n(k)}, t_0), \\ &\quad \max\{M(x_{m(k)}, Tx_{n(k)}, t_0), M(x_{n(k)}, Tx_{m(k)}, t_0)\}\}. \end{aligned}$$

Equivalently

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}, t_0) &= \min\{M(x_{m(k)}, x_{n(k)}, t_0), M(x_{m(k)}, x_{m(k)+1}, t_0), \\ &\quad M(x_{n(k)}, x_{n(k)+1}, t_0), \\ &\quad \max\{M(x_{m(k)}, x_{n(k)+1}, t_0), M(x_{n(k)}, x_{m(k)+1}, t_0)\}\}. \end{aligned}$$

As $k \to \infty$,

$$\lim_{k\to\infty} N(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon. \tag{14}$$

Since ψ is continuous and ϕ is upper semi continuous, by taking the limit superior as $k \to \infty$ in (13), it follows that

$$\psi(1-\epsilon) \leq \psi(1-\epsilon)\phi(1-\epsilon).$$

So that, $\phi(1-\epsilon) = 1$. Hence from the property of ϕ, we have $\epsilon = 0$, which contradicts that $0 < \epsilon < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. $\square$

Theorem 3.2. *Let $(X, M, *)$ be a strong fuzzy metric space and $T : X \to X$ be kg-contractive mapping. Then, T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary element of X. We define a sequence $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is the fixed point of T.

Assume that $x_n \neq x_{n+1},$ for all $n = 0, 1, 2, 3, \ldots$.

Since T is kg-contractive mapping, there exists $(\phi, \psi) \in \Psi \times \Phi$ such that

$$\begin{gathered} M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t) \\ \text{implies that} \\ \psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t)) \end{gathered}$$

for all x, y in X and $t > 0$. Thus, for $x_{n-1} \neq x_n$, we have

$$M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t) \leq M(x_{n-1}, x_n, t).$$

Therefore,

$$\psi(M(Tx_{n-1}, Tx_n, t)) \leq \psi(N(x_{n-1}, x_n, t))\phi(N(x_{n-1}, x_n, t)), \tag{15}$$

where

$$\begin{aligned}
N(x_{n-1}, x_n, t) &= \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), \\
&\quad \max\{M(x_{n-1}, Tx_n, t), M(x_n, Tx_n, t)\}\} \\
&= \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), \\
&\quad \max\{M(x_{n-1}, x_{n+1}, t), M(x_n, x_{n+1}, t)\}\} \\
&= \min\{M(x_{n-1}, x_n, t), M(x_{n+1}, x_n, t)\}.
\end{aligned}$$

If $\min\{M(x_{n-1}, x_n, t), M(x_{n+1}, x_n, t)\} = M(x_{n+1}, x_n, t)$ for some n and $t > 0$, then from (15), we have

$$\psi(M(x_{n+1}, x_n, t)) \leq \psi(M(x_{n+1}, x_n, t))\phi(M(x_{n+1}, x_n, t)). \tag{16}$$

This implies that $\phi(M(x_{n+1}, x_n, t)) = 1$. By the property of ϕ, we have $M(x_{n+1}, x_n, t) = 1$. Thus $x_n = x_{n+1}$. This contradicts $x_n \neq x_{n+1}$ for all n. Therefore $\min\{M(x_{n-1}, x_n, t), M(x_{n+1}, x_n, t)\} = M(x_{n-1}, x_n, t)$ for all n and $t > 0$. Thus from (15) we have

$$\psi(M(x_{n+1}, x_n, t)) \leq \psi(M(x_n, x_{n-1}, t))\phi(M(x_n, x_{n-1}, t)) < \psi(M(x_n, x_{n-1}, t)).$$

Since ψ is non-increasing $M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, t)$ for each n and $t > 0$. Therefore, for every $t > 0$, $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence of real numbers in $(0, 1]$. Since every bounded and monotone sequence is convergent, $\{M(x_n, x_{n+1}, t)\}$ converges to some number in $(0, 1]$.

Let $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = l_t$. We Show that $l_t = 1$, for all $t > 0$.

Let $t > 0$, by taking the limit superior as $k \to \infty$ in the inequality (16), the continuity of ψ and the upper semi continuity ϕ show that $\phi(l_t) \geq 1$. Hence, by the property of ϕ, $\phi(l_t) = 1$ implies $l_t = 1$. Now by Proposition 3.1, the sequence $\{x_n\}$ is Cauchy.

Since X is a complete strong fuzzy metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. We show that z is a fixed point of T.

(i) Suppose there exist a positive integer k and $t_k > 0$ such that

$$M(x_n, z, t_k) * M(z, Tz, t_k) > M(x_n, x_{n+1}, t_k) \tag{17}$$

for all $n \geq k$. Taking limit as $n \to \infty$ in (17) we get $M(z, Tz, t_k) \geq 1$, which implies that $Tz = z$.

(ii) Suppose for each positive integer k and $t > 0$, there exists $n_k \geq k$ such that

$$M(x_{n_k}, z, t) * M(z, Tz, t) \leq M(x_{n_k}, x_{n_k+1}, t).$$

Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$M(x_{n_k}, z, t) * M(z, Tz, t) \leq M(x_{n_k}, x_{n_k+1}, t).$$

Hence, from (2) we have

$$\psi(M(Tz, x_{n_k+1}, t)) \leq \psi(N(z, x_{n_k}, t))\phi(N(z, x_{n_k}, t)), \tag{18}$$

where

$$\begin{aligned} N(z, x_{n_k}, t) \quad &= \quad \min\{M(x_{n_k}, z, t), M(z, Tz, t), M(x_{n_k}, x_{n_k+1}, t), \\ &\qquad \max\{M(z, x_{n_k+1}, t), M(x_{n_k}, Tz, t)\}\}. \end{aligned}$$

Since $x_{n_k} \to z$ as $k \to \infty$ (being a subsequence of $\{x_n\}$) and $\lim_{k\to\infty} M(x_{n_k}, x_{n_k+1}, t) = 1$, it follows that $\lim_{k\to\infty} N(z, x_{n_k}, t) = M(z, Tz, t)$. Since ψ is continuous and ϕ upper semi continuous, by taking the limit superior in (18) we get

$$\psi(M(Tz, z, t)) \leq \psi(M(Tz, z, t))\phi(M(Tz, z, t)).$$

So that $\phi(M(Tz, z, t)) = 1$, thus $M(Tz, z, t) = 1$, and hence $Tz = z$. Therefore z is a fixed point of T.

Now, we show the uniqueness of fixed points of T. Let u and v be two fixed points of T. Thus, $Tu = u$ and $Tv = v$. Since T is a kg-contractive map for $u, v \in X$, and $t > 0$ we have

$$M(u, v, t) * M(u, Tu, t) \leq M(v, Tv, t), \tag{19}$$

which implies that

$$\psi(M(u, v, t)) = \psi(M(Tu, Tv, t)) \leq \psi(N(u, v, t))\phi(N(u, v, t)),$$

where

$$\begin{aligned} N(u, v, t) \quad &= \quad \min\{M(u, v, t), M(u, Tu, t), M(v, Tv, t), \\ &\qquad\qquad \max\{M(u, Tv, t), M(v, Tu, t)\}\} \\ &= \quad \min\{M(u, v, t), 1, 1, M(u, v, t)\} = M(u, v, t). \end{aligned} \tag{20}$$

From (19) and (20), we observe that

$$\psi(M(u, v, t)) \leq \psi(M(u, v, t))\phi((M(u, v, t)).$$

This implies $\phi((M(u, v, t)) = 1$. Thus, $M(u, v, t) = 1$, which implies $u = v$. Therefore the fixed point of T is unique. $\square$

Corollary 3.3. *Let $(X, M, *)$ be a strong fuzzy metric space and T be a self map of X. If there exists $(\psi, \phi) \in \Psi \times \Phi$ such that*

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N(x, y, t))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\}.$$

Then, T has a unique fixed point.

If we take $\psi(t) = \frac{1}{t}$ and $\phi(s) = \sqrt{s}$ in Theorem 3.2, we get the following corollary.

Corollary 3.4. *Let $(X, M, *)$ be a strong fuzzy metric space and T be a self map of X. For $x, y \in X$ and $t > 0$ if $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$ implies that $M(Tx, Ty, t) \geq \sqrt{N(x, y, t)}$, where*

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\}.$$

Then, T has a unique fixed point.

Corollary 3.5. *Let $(X, M, *)$ be a strong fuzzy metric space and T be a self map of X. For $x, y \in X$ and $t > 0$ if $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$ implies that $M(Tx, Ty, t) \geq \sqrt{M(x, y, t)}$, then T has a unique fixed point.*

Proof. Let $x, y \in X$ $t > 0$. Suppose $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$. Then $M(Tx, Ty, t) \geq \sqrt{M(x, y, t)}$. Since $N(x, y, t) \leq M(x, y, t)$, where

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\},$$

we have $M(Tx, Ty, t) \geq \sqrt{N(x, y, t)}$. Therefore, by Corollary 3.4, T has a unique fixed point. □

By taking $\psi(t) = \frac{1}{t}$ and $\phi(t) = k(1-t)+t$, for $0 < k < 1$ we draw the following corollary.

Corollary 3.6. *Let $(X, M, *)$ be a strong fuzzy metric space and T be a self map of X. For $x, y \in X$ and $t > 0$ if there exists $k \in (0, 1)$ such that $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$ implies $(\frac{1}{M(Tx,Ty,t)} - 1) \leq k(\frac{1}{N(x,y,t)} - 1)$, where*

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\},$$

then T has a unique fixed point.

Corollary 3.7. *Let $(X, M, *)$ be a strong fuzzy metric space and T be a self map of X. For $x, y \in X$ and $t > 0$ if there exists $k \in (0, 1)$ such that $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$ implies $(\frac{1}{M(Tx,Ty,t)} - 1) \leq k(\frac{1}{M(x,y,t)} - 1)$. Then T has a unique fixed point.*

Proof. Let $x, y \in X$, $t > 0$, and $M(x, y, t) * M(y, Ty, t) \leq M(x, Tx, t)$. Then

$$(\frac{1}{M(Tx, Ty, t)} - 1) \leq k(\frac{1}{M(x, y, t)} - 1).$$

Since $N(x, y, t) \leq M(x, y, t)$ where

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \max\{M(x, Ty, t), M(y, Tx, t)\}\}.$$

But $k(\frac{1}{M(x,y,t)} - 1) \leq k(\frac{1}{N(x,y,t)} - 1)$, thus $(\frac{1}{M(Tx,Ty,t)} - 1) \leq k(\frac{1}{N(x,y,t)} - 1)$. By Corollary 3.6, T has a unique fixed point. □

4. Examples

In this section, we provide some examples in support of the main results of Section 3. The following example is in support of Theorem 3.2.

Example 4.1. Let $X = \{(1,1),(1,4),(4,1)\}$ and $M(x,y,t) = (\frac{t}{t+1})^{d(x,y)}$, where $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, and $x = (x_1, y_1), y = (x_2, y_2)$, $*$ be the product continuous t-norm. Here $(X, M, *)$ is a complete strong fuzzy metric space. Let $T : X \to X$ be a map defined by

$$T(x_1, y_1) = \begin{cases} (1,1), & \text{if } x_1 \leq y_1 \\ (1,4), & \text{if } x_1 > y_1 \end{cases}.$$

Then, T is a kg-contractive map for $\psi(t) = \frac{1}{t}$ and $\phi(t) = \sqrt{t}$. Clearly, $(\psi, \phi) \in \Psi \times \Phi$. Now we wish to show

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x,y,t))\phi(N(x,y,t)). \tag{21}$$

(i) Let $(x,y) = ((1,1),(1,1)), T(1,1) = (1,1)$ and let $a = \frac{t}{t+1}$.
Here $M(x,y,t) = 1$, $M(Tx,Ty,t) = 1$, $M(x,Tx,t) = 1$, $M(y,Ty,t) = 1$, $M(x,Ty,t) = 1$ and $M(y,Tx,t) = 1$. It follows that $M(x,y,t)*M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = 1$. Then $\psi(M(Tx,Ty,t)) = 1$, $\psi(N(x,y,t)) = 1$ and $\phi(N(x,y,t)) = 1$, which implies that

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x,y,t))\phi(N(x,y,t)).$$

(ii) Let $(x,y) = ((1,4),(1,4)), T(1,4) = (1,1)$ and let $a = \frac{t}{t+1}$.
Here $M(x,y,t) = 1$, $M(Tx,Ty,t) = 1$, $M(x,Tx,t) = a^3$, $M(y,Ty,t) = a^3$, $M(x,Ty,t) = a^3$ and $M(y,Tx,t) = a^3$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^3$. Then $\psi(M(Tx,Ty,t)) = 1$, $\psi(N(x,y,t)) = a^{-3}$ and $\phi(N(x,y,t)) = a^{\frac{3}{2}}$. Since $0 > -3 + \frac{3}{2} = \frac{-3}{2}$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(iii) Let $(x,y) = ((4,1),(4,1)), T(4,1) = (1,4)$ and let $a = \frac{t}{t+1}$.
Here $M(x,y,t) = 1$, $M(Tx,Ty,t) = 1$, $M(x,Tx,t) = a^6$, $M(y,Ty,t) = a^6$, $M(x,Ty,t) = a^6$ and $M(y,Tx,t) = a^6$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^6$. Then $\psi(M(Tx,Ty,t)) = 1$, $\psi(N(x,y,t)) = a^{-6}$ and $\phi(N(x,y,t)) = a^3$. Since $0 > -3$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(iv) Let $(x,y) = ((1,1),(1,4)), T(1,1) = (1,1), T(1,4) = (1,1)$ and let $a = \frac{t}{t+1}$.
Here $M(x,y,t) = a^3$, $M(Tx,Ty,t) = 1$, $M(x,Tx,t) = 1$, $M(y,Ty,t) = a^3$, $M(x,Ty,t) = 1$ and $M(y,Tx,t) = a^3$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^3$. Then $\psi(M(Tx,Ty,t)) = 1$, $\psi(N(x,y,t)) = a^{-3}$ and $\phi(N(x,y,t)) = a^{\frac{3}{2}}$. Since $0 > \frac{-3}{2}$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(v) Let $(x,y) = ((1,1),(4,1)), T(1,1) = (1,1), T(4,1) = (1,4)$ and let $a = \frac{t}{t+1}$. Here $M(x,y,t) = a^3$, $M(Tx,Ty,t) = a^3$, $M(x,Tx,t) = 1$, $M(y,Ty,t) = a^6$, $M(x,Ty,t) = a^3$ and $M(y,Tx,t) = a^3$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^6$. Then $\psi(M(Tx,Ty,t)) = a^{-3}$, $\psi(N(x,y,t)) = a^{-6}$ and $\phi(N(x,y,t)) = a^3$. Since $-3 \geq -6+3$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(vi) Let $(x,y) = ((1,4),(4,1)), T(1,4) = (1,1), T(4,1) = (1,4)$ and let $a = \frac{t}{t+1}$. Here $M(x,y,t) = a^6$, $M(Tx,Ty,t) = a^3$, $M(x,Tx,t) = a^3$, $M(y,Ty,t) = a^6$, $M(x,Ty,t) = 1$ and $M(y,Tx,t) = a^3$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^6$. Then $\psi(M(Tx,Ty,t)) = a^{-3}$, $\psi(N(x,y,t)) = a^{-6}$ and $\phi(N(x,y,t)) = a^3$. Since $-3 \geq -6+3$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(vii) Let $(x,y) = ((4,1),(1,4)), T(4,1) = (1,4), T(1,4) = (1,1)$ and let $a = \frac{t}{t+1}$. Here $M(x,y,t) = a^6$, $M(Tx,Ty,t) = a^3$, $M(x,Tx,t) = a^6$, $M(y,Ty,t) = a^3$, $M(x,Ty,t) = a^3$ and $M(y,Tx,t) = 1$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^6$. Then $\psi(M(Tx,Ty,t)) = a^{-3}$, $\psi(N(x,y,t)) = a^{-6}$ and $\phi(N(x,y,t)) = a^3$. Since $-3 \geq -6+3$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

(viii) Let $(x,y) = ((4,1),(1,1)), T(4,1) = (1,4), T(1,1) = (1,1)$ and let $a = \frac{t}{t+1}$. Here $M(x,y,t) = a^3$, $M(Tx,Ty,t) = a^3$, $M(x,Tx,t) = a^6$, $M(y,Ty,t) = 1$, $M(x,Ty,t) = a^3$ and $M(y,Tx,t) = a^3$. Now in this case $M(x,y,t) * M(y,Ty,t) > M(x,Tx,t)$, but $N(x,y,t) = a^6$, $\psi(M(Tx,Ty,t)) = a^{-3}$, $\psi(N(x,y,t)) = a^{-6}$ and $\phi(N(x,y,t)) = a^3$ imply that $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$. Thus the relation (2) is true.

(ix) Let $(x,y) = ((1,4),(1,1)), T(1,4) = (1,1), T(1,1) = (1,1)$ and let $a = \frac{t}{t+1}$. Here $M(x,y,t) = a^3$, $M(Tx,Ty,t) = 1$, $M(x,Tx,t) = a^3$, $M(y,Ty,t) = 1$, $M(x,Ty,t) = a^3$ and $M(y,Tx,t) = 1$. It follows that $M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t)$ and $N(x,y,t) = a^3$. Then $\psi(M(Tx,Ty,t)) = 1$, $\psi(N(x,y,t)) = a^{-3}$ and $\phi(N(x,y,t)) = a^{\frac{3}{2}}$. Since $-0 \geq -3+\frac{3}{2}$, we have $\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t))$.

From (Case i-Case ix) we observe that there exists $(\psi,\phi) \in \Psi \times \Phi$ such that

$$\begin{gathered} M(x,y,t) * M(y,Ty,t) \leq M(x,Tx,t) \\ \text{implies} \\ \psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N(x,y,t)), \end{gathered} \tag{22}$$

for all x,y in X and $t > 0$, where

$$N(x,y,t) = \min\{M(x,y,t), M(x,Tx,t), M(y,Ty,t), \max\{M(x,Ty,t), M(y,Tx,t)\}\}.$$

Therefore, T is a kg-contractive mapping and thus by Theorem 3.2, T has a unique fixed point. In fact, $(1, 1)$ is a fixed point of T.

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