Mathematics Interdisciplinary Research **3** (2018) 31 − 43

On Powers of Some Graph Operations

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Abstract

Let $G * H$ be the product $*$ of G and H. In this paper we determine the r^{th} power of the graph $G * H$ in terms of G^r, H^r and $G^r * H^r$, when $*$ is the join, Cartesian, symmetric difference, disjunctive, composition, skew and corona product. Then we solve the equation $(G * H)^r = G^r * H^r$. We also compute the Wiener index and Wiener polarity index of the skew product.

Keywords: Graph product, power graphs, graph indices.

2010 Mathematics Subject Classification: 05C12, 05C76.

How to cite this article
M. A. Seoud, H. M. Hafez, On powers of some graph operations, *Math. Interdisc. Res.* **3** (2018) 31-43.

1. Introduction

The r^{th} power of a graph G, denoted by G^r, is the graph with vertex set $V(G)$ where two vertices are adjacent if they are within distance r in G, i.e., the length of the shortest path joining them is at most r. The maximum distance between any pair of vertices in a graph G is called the diameter of G and denoted by $diam(G)$. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The join (sum) $G + H$ has $V(G) \cup V(H)$ as its vertex set and its edge set consists of $E(G) \cup E(H)$ and all edges joining $V(G)$ with $V(H)$. The cartesian product $G \times H$ has its vertex set $V(G) \times V(H)$ and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[x_1 = x_2$ and $y_1y_2 \in E(H)]$ or $[y_1 = y_2$ and $x_1x_2 \in E(G)]$. The symmetric difference $G \oplus H$ has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $x_1x_2 \in E(G)$ or $y_1y_2 \in E(H)$ but not both. The disjunctive product $G \vee H$ has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $x_1x_2 \in E(G)$ or $y_1y_2 \in E(H)$ or both. The corona product $G \circ H$ is the graph obtained by taking one copy of G (which has n_1

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Academic Editor: Yongtang Shi
Received 11 May 2017, Accepted 14 January 2018
DOI: 10.22052/mir.2018.85618.1062

vertices) and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The composition $G[H]$ has its vertex set $V(G) \times V(H)$ and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[x_1x_2 \in E(G)]$ or $[x_1 = x_2$ and $y_1y_2 \in E(H)]$. The conjunction product $G \wedge H$ has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. The strong product $G \otimes H$ has $V(G) \times V(H)$ as its vertex set and its edge set $E(G \otimes H) = E(G \times H) \cup E(G \wedge H)$. The skew product $G \bigtriangleup H$, defined in [10], has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[x_1 = x_2$ and $y_1y_2 \in E(H)]$ or $[x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)]$. The converse skew product $G \bigtriangledown H$ has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[y_1 = y_2$ and $x_1x_2 \in E(G)]$ or $[x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)]$. The complete graph with n vertices is denoted by K_n. For the graphs $G(V(G), E(G))$ and $H(V(H), E(H))$, we denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ by $G \cup H$. For a graph $G(V(G), E(G))$ and any set of edges $F \subset E(G)$, we assume $G - F$ to be the graph obtained from G by deleting the edges in F. We denote the distance between u and v in G by $d_G(u, v)$. By $G = H$ we mean that G is isomorphic to H, sometimes written $G \cong H$. A graph is null if it has no edges. Denote the degree of a vertex u in a graph G by $deg_G(u)$. For more details see [1–3]. In [7–9], Seoud proved necessity and sufficiency conditions for:

1. $G^2 + H^2 = (G + H)^2$;
2. $G^2 \times H^2 = (G \times H)^2$;
3. $G^2 \circ H^2 = (G \circ H)^2$;
4. $G^2[H^2] = (G[H])^2$;
5. $G^2 \vee H^2 = (G \vee H)^2$;
6. $G^2 \oplus H^2 = (G \oplus H)^2$.

Here, we determine the graph $(G * H)^r$ in terms of G^r, H^r and $G^r * H^r$ for $r \geq 2$, where the operation $G * H$ represents a product of G and H. According to the definitions of graph products one can prove the following lemma, for more details say [1, 12].

Lemma 1.1. *Let G, H be two graphs and $(u_1, v_1), (u_2, v_2)$ be two distinct vertices in $V(G) \times V(H)$, where $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$, then:*

1. $d_{G\times H}((u_1, v_1), (u_2, v_2)) = d_G(u_1, u_2) + d_H(v_1, v_2)$.
2. $$d_{G+H}(u, v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \text{ and } v \in V(H)) \\ 2 & \text{otherwise.} \end{cases}$$

3. $d_{G[H]}((u_1,v_1),(u_2,v_2)) = \begin{cases} d_G(u_1,u_2) & u_1 \neq u_2 \\ d_H(v_1,v_2) & u_1 = u_2, deg_G(u_1) = 0, \\ 1 & u_1 = u_2\,, deg_G(u_1) \neq 0\, and\, v_1v_2 \in E(H) \\ 2 & u_1 = u_2\,, deg_G(u_1) \neq 0\, and\, v_1v_2 \notin E(H). \end{cases}$

4. $d_{G\vee H}((u_1,v_1),(u_2,v_2)) = \begin{cases} 0 & u_1 = u_2 \quad v_1 = v_2 \\ 1 & u_1u_2 \in E(G) \quad and \quad v_1v_2 \in E(H) \\ 2 & otherwise. \end{cases}$

5. $d_{G\oplus H}((u_1,v_1),(u_2,v_2)) = \begin{cases} 0 & u_1 = u_2 \quad v_1 = v_2 \\ 1 & u_1u_2 \in E(G)\ and\ v_1v_2 \in E(H)\ but\ not\ both \\ 2 & otherwise. \end{cases}$

6. $d_{G\otimes H}((u_1,v_1),(u_2,v_2)) = Max\{d_G(u_1,u_2), d_H(v_1,v_2)\}$.

2. Main Results

Lemma 2.1. *For any two graphs G and H*

1. $G^r[H^r] \subset (G[H])^r$ *($\subset$ means spanning subgraph of).*
2. $G^r + H^r \subset (G+H)^r$.
3. $G^r \times H^r \subset (G \times H)^r$.
4. $G^r \circ H^r \subset (G \circ H)^r$.
5. $G^r \vee H^r \subset (G \vee H)^r$.
6. $G^r \oplus H^r \subset (G \oplus H)^r$.

Proof. 1. $G^r[H^r]$ and $(G[H])^r$ have the same set of vertices, which is $V(G) \times V(H)$ and $(u_1,u_2),(v_1,v_2)$ are adjacent in $G^r[H^r]$ means that $\big[u_1 = u_2$ and v_1 is adjacent to v_2 in $H^r\big]$ or $\big[u_1$ is adjacent to u_2 in $G^r\big]$. That is equivalent to $\big[u_1 = u_2 \,\text{and}\, d_H(v_1,v_2) \leq r\big]$ or $\big[d_G(u_1,u_2) \leq r\big]$.
First, if $u_1 = u_2$ and $d_H(v_1,v_2) \leq r$, then there exists in H the path $v_1x_1x_2...x_{r-1}v_2$ between v_1 and v_2 of length at most r. In $G[H]$ we have the path (u_1,v_1) (u_2,x_1) $(u_2,x_2)\cdots$ $(u_2,x_{r-1})(u_2,v_2)$ of length at most r. Then (u_1,u_2) and (v_1,v_2) are adjacent in $(G[H])^r$.
Second, if $d_G(u_1,u_2) \leq r$, then there exists in G the path $u_1y_1y_2...y_{r-1}u_2$ between u_1 and u_2 of length at most r. In $G[H]$ we have the path $(u_1,v_1)(y_1,v_2)$ $(y_2,v_2)\cdots(y_{r-1},v_2)(u_2,v_2)$ of length at most r. Then (u_1,v_1) and (u_2,v_2) are adjacent in $(G[H])^r$.

2. Let u, v be adjacent in $G^r + H^r$. Note that possible cases for u, v are either $\big[u \in V(G)$ and v$\in V(H)\big]$ or $\big[v \in V(G)$ and $u \in V(H)\big]$ or $\big[u, v \in V(G)\big]$ or $\big[u, v \in V(H)\big]$. Now we discuss the possible cases as follows:

Case 1: If $[u \in V(G)\, and\, v \in V(H)]$, then the result follows trivially.
Case 2: If $[u, v \in V(G)]$, then there exists a vertex w in $V(H)$ such that $uw, vw \in E(G+H)$. It follows that $uv \in E((G+H)^2)$ and $(G+H)^2 \subset (G+H)^r$ implies that $uv \in E((G+H)^r)$.
Case 3: $u, v \in V(H)$ analogous to case 2.
3, 4 and 5 are not difficult to be proved. □

Theorem 2.2. $(G+H)^r = K_{|V(G)|+|V(H)|}$.

Proof. According to Lemma 1.1, we have $diam(G+H) = 2$. Hence $(G+H)^r$, for any $r \geq 2$, is complete. It is clear now that both sides in the above equation represent a complete graph on $|V(G)| + |V(H)|$ vertices. □

Now, we are ready to discuss the equation $(G+H)^r = G^r + H^r$.

Corollary 2.3.

$$(G+H)^r = G^r + H^r \tag{1}$$

if and only if G *and* H *are connected and* $diam(G) \leq r$ *and* $diam(H) \leq r$.

Proof. If G is not connected, then there are two vertices u and v that are not adjacent in G and so in G^r. Hence there are two vertices u and v not adjacent in $G^r + H^r$ contradicts that $G^r + H^r = (G+H)^r = K_{|V(G)|+|V(H)|}$. Therefore G and H must be connected. Again, if G is connected with $diam(G) > r$, then there exist two vertices u and v which are not adjacent in G^r, and hence non-adjacent in $G^r + H^r$ a contradiction to $G^r + H^r = (G+H)^r$. So we must have $diam(G) \leq r$ and $diam(H) \leq r$. Conversely, if G and H are connected and $diam(G) \leq r$ and $diam(H) \leq r$, then $G^r + H^r = K_{|V(G)|} + K_{|V(H)|} = K_{|V(G)|+|V(H)|} = (G+H)^r$. □

Theorem 2.4. $(G \times H)^r = G^r \times H^r \cup_{i=1}^{r-1} G^i \wedge H^{r-1-i} - \cup_{i=1}^{r-2} E(G^i \wedge H^{r-2-i})$.

Proof. Let (u_1, v_1) be adjacent to (u_2, v_2) in $(G \times H)^r$, then $d((u_1, v_1), (u_2, v_2)) \leq r$ in $G \times H$. Therefore we must have one of the following cases:

(1) $u_1 = u_2$ and $d_H(v_1, v_2) = d_{G\times H}((u_1, v_1), (u_2, v_2)) \leq r$, or

(2) $v_1 = v_2$ and $d_H(v_1, v_2) = d_{G\times H}((u_1, v_1), (u_2, v_2)) \leq r$, or

(3) $u_1 u_2 \in E(G)$ and $d_H(v_1, v_2) = d_{G\times H}((u_1, v_1), (u_2, v_2)) - 1 \leq r - 1$, or

(5) $d_G(u_1, u_2) = 2$ and $d_H(v_1, v_2) = d_{G\times H}((u_1, v_1), (u_2, v_2)) - 2 \leq r - 2$, or

$\vdots$

(r+1) $d_G(u_1, u_2) = \lfloor \frac{r}{2} \rfloor$ and $d_H(v_1, v_2) = d_{G\times H}((u_1, v_1), (u_2, v_2)) - \lfloor \frac{r}{2} \rfloor \leq \lceil \frac{r}{2} \rceil$.

In other words, $(u_1, v_1)(u_2, v_2) \in E(G^r \times H^r) \cup_{i=1}^{r-1} E(G^i \wedge H^{r-1-i})$. Since $E(G \wedge H^{r-3})$ is computed one time in $E(G \wedge H^{r-2})$ and another time in $E(G^2 \wedge H^{r-3})$ in the union $\cup_{i=1}^{r-1} E(G^i \wedge H^{r-1-i})$, we must subtract it once. In general any term in the union $\cup_{i=1}^{r-2} E(G^i \wedge H^{r-2-i})$ is computed twice in $\cup_{i=1}^{r-1} E(G^i \wedge H^{r-1-i})$, so we must subtract it once. □

The following result is a consequence of Theorem 2.4.

Corollary 2.5. $(G \times H)^r = G^r \times H^r$ *if and only if* G *is null or* H *is null.*

Proof. Let $(G \times H)^r = G^r \times H^r$ and G and H be non-null, then there exist u_1, u_2 adjacent in G and v_1, v_2 adjacent in H. Hence (u_1, v_1) is adjacent to (u_2, v_1) in $G \times H$ and (u_2, v_1) is adjacent to (u_2, v_2) in $G \times H$, consequently (u_1, v_1) is adjacent to (u_2, v_2) in $(G \times H)^r$, but they are not adjacent in $G^r \times H^r$, a contradiction to $(G \times H)^r = G^r \times H^r$. Hence $E(G)$ or $E(H)$ must be empty. Conversely, we note that $E(G^i \wedge H^{r-1-i})$ is empty if G is null or H is null and therefore $(G \times H)^r = G^r \times H^r$. □

Theorem 2.6. $(G \otimes H)^r = G^r \otimes H^r$.

Proof. Let (u_1, v_1) be adjacent to (u_2, v_2) in $(G \otimes H)^r$. Therefore $d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$ and one of the following must happen:

1. $d_G(u_1, u_2) = d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$ and $v_1 = v_2$, or
2. $d_H(v_1, v_2) = d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$ and $u_1 = u_2$, or
3. $d_G(u_1, u_2) = d_H(v_1, v_2) = d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$, or
4. $1 \leq d_G(u_1, u_2) < d_H(v_1, v_2) = d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$, or
5. $1 \leq d_H(v_1, v_2) < d_G(u_1, u_2) = d_{G\otimes H}((u_1, v_1), (u_2, v_2)) \leq r$.

Which would imply that $(u_1, v_1)(u_2, v_2)$ belongs to $G^r \otimes H^r$. □

Theorem 2.7. *If* G *and* H *don't contain isolated vertices, then* $(G \oplus H)^r = K_{|V(G)\times V(H)|}$.

Proof. For any two vertices (u_1, v_1) and (u_2, v_2) in $(G \oplus H)$, $d_{G\oplus H}((u_1, v_1), (u_2, v_2)) \leq 2$, by Lemma 1.1, and $(G \oplus H)^r$ is complete graph on $|V(G)| \times |V(H)|$ vertices. □

Corollary 2.8. $(G \oplus H)^r = G^r \oplus H^r$ *has the only solutions:*

(i) $G = K_1$, H *is any graph;*

(ii) $H = K_1$, G *is any graph;*

(iii) Trivial solution, $G = nK_1, H = nK_1$.

Proof. Let $(G \oplus H)^r = G^r \oplus H^r$ and G and H be non-null, then there exist $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$ such that u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. It follows that (u_1, v_1) is not adjacent to (u_2, v_2) in $G^r \oplus H^r$, a contradiction to $G^r \oplus H^r = (G \oplus H)^r = K_{|V(G)\times V(H)|}$. Hence $E(G)$ or $E(H)$ must be empty. Similarly: if G is null with more than one vertex and H is connected, then there exist u_1, u_2 in $V(G)$ and v_1, v_2 in $V(H)$ such that v_1 is adjacent to v_2

in H. It follows that (u_1, v_1) is not adjacent to (u_2, v_1) in $G^r \oplus H^r$, a contradiction again. Conversely, if $G = K_1$ or $H = K_1$, then $(G \oplus H)^r = G^r[K_1] \cong G^r \cong G^r \oplus H^r$ or $(G \oplus H)^r = K_1[K_{|V(H)|}] \cong K_{|V(H)|} = G^r \oplus H^r$, respectively. □

Theorem 2.9. *If G and H don't contain isolated vertices, then* $(G \vee H)^r = K_{|V(G)\times V(H)|}$.

Proof. By Lemma 1.1 $d_{G\vee H}((u_1, v_1), (u_2, v_2)) \leq 2$ and the result follows immediately. □

Corollary 2.10. $(G \vee H)^r = G^r \vee H^r$ *has the only solutions:*

(i) G and H are connected with $diam\,(G) \leq r \,\&\, diam(H) \leq r$;

(ii) $G = K_1$, H is any graph;

(iii) $H = K_1$, G is any graph;

(iv) Trivial solution, $G = nK_1, H = nK_1$.

Proof. Firstly, assume that $G^r \vee H^r = (G \vee H)^r$.

(a) If H is connected with $diam(H) > r$, and G is any graph. Let G (without any loss of generality) be connected. It follows that there exist paths $u_1u_2u_3......u_ru_{r+1}$ in $H : d_H(u_i, u_{i+1}) = 1, 1 \leq i \leq r+1$, and v_1v_2 in G. Now (v_1, u_1) is not adjacent to (v_1, u_{r+1}) in $G^r \vee H^r$ by definition, a contradiction to $G^r \vee H^r = (G \vee H)^r = K_{|V(G)\times V(H)|}$.

(b) If G is connected with $diam(G) > r$, and H is any graph, the proof proceeds as case (a).

(c) If G is disconnected with more than one component and H is any graph, then H or G must have at least two vertices that are adjacent (since otherwise one gets case iv). Let u_1, v_1 be any two vertices in $V(G)$ that lie in distinct components of G and the vertices u_2, v_2 in $V(H)$ that are adjacent (such vertices must exist in either H or G). Then (u_1, u_2) is not adjacent to (v_1, u_2) in $G^r \vee H^r$. A contradiction again to $G^r \vee H^r = (G \vee H)^r = K_{|V(G)\times V(H)|}$.

Conversely,

(i) If G and H are connected with $diam\,(G) \leq r \,\&\, diam(H) \leq r$, then $G^r \vee H^r = (G \vee H)^r = K_{|V(G)\times V(H)|}$.

(ii) If $G = K_1$ and H is any graph, then $G \vee H \cong H$, $(G \vee H)^r \cong H^r \cong G^r \vee H^r$.

Case (iii) is similar to (ii). □

Theorem 2.11. *If G is connected, then* $(G[H])^r = G^r[K_{|V(H)|}]$.

Proof. According to Lemma 1.1, $d_{G[H]}((u_1, v_1), (u_2, v_2)) \leq r$ is equivalent to: $[d_G(u_1, u_2) \leq r$ when $u_1 \neq u_2]$ or $[u_1 = u_2 \quad and \quad d_H(v_1, v_2) \leq 2]$. Which implies that $(u_1, v_1)(u_2, v_2) \in G^r[K_{|V(H)|}]$ and the proof is complete. □

Theorem 2.12. *$(G[H])^r = G^r[H^r]$ has the only solutions:*

(i) $G = nK_1$, H is any graph;

(ii) H is connected with $diam(H) \leq r$ and G is any graph.

Proof. Assume that $(G[H])^r = G^r[H^r]$ and G is not null, we prove that H is connected with $diam(H) \leq r$, indeed: if H is disconnected or connected with diameter $\geq r+1$, then there exist two vertices v_1, v_2 in H which are not adjacent in H^r. Since G is not null, then there exist u_1, u_2 in $G: d_G(u_1, u_2) = 1$. The vertices (u_1, v_2) and (u_1, v_1) are not adjacent in $G^r[H^r]$, but (u_1, v_2) and (u_1, v_1) are adjacent in $(G[H])^r$, and this is a contradiction. Assume that $(G[H])^r = G^r[H^r]$ and G is null, we prove that H may be any graph, indeed: for any graph H, we have $(G[H])^r \cong |V(G)|H^r \cong G^r[H^r]$. Conversely, assume G, H be two graphs satisfying the conditions in the theorem, then we have:

(i) If $G = nK_1$ and H is any graph, then $G[H] \cong H$ and $(G[H])^r \cong nH^r \cong G^r[H^r]$.

(ii) H is connected with $diam(H) \leq r$ and G is any graph. It is sufficient to prove that $(G[H])^r \subseteq G^r[H^r]$. For this, if G is connected, then $(G[H])^r = G^r[K_{|V(H)|}] = G^r[H^r]$. Note that $H^r = K_{|V(H)|}$ because $diam(H) \leq r$. Now, let $u = (u_1, v_1)$ be adjacent to $v = (u_2, v_2)$ in $(G[H])^r$, it follows that $d(u, v) \leq r$ in $(G[H])^r$. Then there exists the path $P = (u_1, v_1)(u_3, v_3) \cdots (u_m, v_m)(u_2, v_2)$ of length at most r in $G[H]$. We have the following cases:
Case 1: if $u_1 = u_3 = u_4 = \cdots = u_m = u_2$, then we have $d(v_1, v_2) \leq r$ in H. Since H is connected with diameter $\leq r$, it follows that $u_1 = u_2$ and v_1 is adjacent to v_2 in H^r. Then u, v are adjacent in $G^r[H^r]$.
Case 2: if for some $i = 3, 4, \cdots, m-1, u_i \neq u_{i+1}$, then the existence of the path P implies that u_i and u_{i+1} must be adjacent in G, of course this is the case for u_1, u_3 and u_m, u_2. Hence if $u_1 \neq u_3 \neq \cdots \neq u_m \neq u_2$, then u_1 is adjacent to u_3, u_3 is adjacent to $u_4, \cdots$, and u_m is adjacent to u_2, i.e. $d(u_1, u_2) \leq r$ in G, it follows that u_1 is adjacent to u_2 in G^r. Hence u and v are adjacent in $G^r[H^r]$.

□

Theorem 2.13. *$(G \circ H)^r = G^r \circ H^{diam(H)} \cup_{u \in V(G)} E(N(u) + H_u)$, where $N(u) = \{v \in V(G) : d_G(u, v) \leq r-1\} \cup \{v \in V(H_x) : d_G(u, x) \leq r-2\}$ and H_u is the copy of H corresponding to the vertex u.*

Proof. The copy of H corresponding to vertex $x \in V(G)$ is denoted by H_x. Let u, v be adjacent in $E(G \circ H)^r$. We must have either $u, v \in V(G)$ or $u, v \in V(H_x)$

or $u \in V(G)$ and $v \in V(H_x)$ or $u \in V(H_x)$ and $v \in V(H_y)$, where $x \neq y$. One of the following cases must happen:

(1) If $u, v \in V(G)$, then $d_G(u,v) \leq r$.

(2) For any two vertices $u, v \in V(H_x)$, $d_{G\circ H}(u,v) = 2$.

(3) If $u \in V(G)$ and $v \in V(H_x), x \neq u$, then $d_{G\circ H}(u,v) \leq r-1$.

(4) If $u \in V(H_x)$ and $v \in V(H_y)$, where $x \neq y$, then $d_{G\circ H}(u,v) \leq r-2$.

In all cases, we get $uv \in E(G^r \circ H^{diam(H)}) \cup_{u\in V(G)} E(N(u)+H_u)$ and the proof is complete. □

Corollary 2.14. *$(G\circ H)^r = G^r \circ H^r$ if and only if G is null and each component of H has diameter $\leq r$.*

Proof. Let $(G\circ H)^r = G^r \circ H^r$, then $diam(H) \leq r$ and $\cup_{u\in V(G)} E(N(u)+H_u)$ is empty, where $N(u) = \{v \in V(G) : d_G(u,v) \leq r-1\} \cup \{v \in V(H_x) : d_G(u,x) \leq r-2\}$ and H_u is the copy of H corresponding to the vertex u. Note that $\cup_{u\in V(G)} E(N(u)+H_u)$ is empty if and only if $E(G)$ is empty. □

3. Conjunction and Skew Product

We follow notions in [5]. Recall a walk between two vertices u, v is a sequence of vertices $ux_1x_2x_3...x_nv$ in which any two consecutive vertices are adjacent. While a path is a walk in which all vertices are distinct and not repeated. The length of a walk is the number of edges in it. Let $d_G(u,v)$ be the distance between u, v in G, define $d'_G(u,v)$ to be the length of the shortest $u-v$ walk satisfying $d_G(u,v) + d'_G(u,v)$ is odd and ∞ otherwise. We call $(u_1,v_1) \sim (u_2,v_2)$ whenever $d_G(u_1,u_2) + d_H(v_1,v_2)$ is even, otherwise we call $(u_1,v_1) \nsim (u_2,v_2)$. Distance between (u_1,v_1) and (u_2,v_2), in the conjunction product $G \wedge H$, is defined in [5] as follows:

Lemma 3.1. *[5]*

(a)If $(u_1,v_1) \sim (u_2,v_2)$, then $d_{G\wedge H}((u_1,v_1),(u_2,v_2)) = Max\{d_G(u_1,u_2), d_H(v_1,v_2)\}$.

*(b) If $(u_1,v_1) \nsim (u_2,v_2)$, then $d_{G\wedge H}((u_1,v_1),(u_2,v_2))$
$= Min\{Max\{d_G(u_1,u_2), d'_H(v_1,v_2)\}, Max\{d_H(v_1,v_2), d'_G(u_1,u_2)\}\}$.*

Lemma 3.2. *If each edge in $E(G)$ is contained in a triangle and each edge in $E(H)$ is contained in a triangle, then $(G\wedge H)^r = G^r \otimes H^r$.*

Proof. Since each edge belongs to a triangle in G, then $d'_G(u_1,u_2) = d_G(u_1,u_2)+1$ except for $d_G(u_1,u_2) = 0$, where $d'_G(u_1,u_2) = 3$, the same for H. If $(u_1,v_1) \sim (u_2,v_2)$, then $d_{G\wedge H}((u_1,v_1),(u_2,v_2)) = Max\{d_G(u_1,u_2), d_H(v_1,v_2)\} = d_{G\otimes H}((u_1,v_1),(u_2,v_2))$. If $(u_1,v_1) \nsim (u_2,v_2)$, then $d_{G\wedge H}((u_1,v_1),(u_2,v_2)) = Min\{Max\{d_G(u_1,u_2), d_H($

$v_1, v_2)+1\big\}, Max\big\{d_H(v_1, v_2), d_G(u_1, u_2)+1\big\}\big\}$. Assume without any loss of generality that $d_G(u_1, u_2) > d_H(v_1, v_2)$ such that $(d_G(u_1, u_2), d_H(v_1, v_2)) \neq (1, 0)$. Then $d_G(u_1, u_2) \geq d_H(v_1, v_2) + 1$ and $d_{G\wedge H}((u_1, v_1), (u_2, v_2)) = d_G(u_1, u_2) = Max\{d_G(u_1, u_2),\, d_H(v_1, v_2)\}$. If $(d_G(u_1, u_2)\ , d_H(v_1, v_2)) = (1, 0)$, then $d_{G\wedge H}((u_1, v_1), (u_2, v_2)) = 2$ and $(u_1, u_2), (v_1, v_2)$ are adjacent in $(G \wedge H)^r$ for all $r \geq 2$. □

Lemma 3.3. *$d_{G\triangle H}((u_1, v_1), (u_2, v_2)) = Max\{d_G(u_1, u_2), d_H(v_1, v_2)\}$ when $(u_1, u_2) \sim (v_1, v_2)$ or $(u_1, u_2) \nsim (v_1, v_2)$ and $d_G(u_1, u_2) < d_H(v_1, v_2)$, otherwise $d_{G\triangle H}((u_1, v_1), (u_2, v_2)) = 1 + Max\{d_G(u_1, u_2), d_H(v_1, v_2)\} = 1 + d_G(u_1, u_2)$.*

Proof. Assume that $(u_1, u_2) \sim (v_1, v_2)$.Since $G\wedge H \subset G \triangle H$, then $d_{G\triangle H}((u_1, v_1), (u_2, v_2)) \leq d_{G\wedge H}((u_1, v_1), (u_2, v_2)) = Max\big\{d_G(u_1, u_2), d_H(v_1, v_2)\big\}$. But $G \triangle H \subset G \otimes H$, hence $d_{G\triangle H}(\ (u_1, v_1), (u_2, v_2)) \geq d_{G\otimes H}((u_1, v_1), (u_2, v_2)) = Max\big\{d_G(u_1, u_2), d_H(v_1, v_2)\big\}$. If $(u_1,\ u_2) \nsim (v_1, v_2)$ and $d_G(u_1, u_2) < d_H(v_1, v_2)$, then there exists a vertex (u_2, x) such that $(u_2, x) \sim (u_1, v_1)$, where $x \in V(H)$ satisfying $d_H(v_1, x) = Min\big\{d_H(v_1, y) : y \in V(H)\, and\, (\ u_2, y) \sim (u_1, v_1)\big\}$. Hence $d_{G\triangle H}((u_1, v_1), (u_2, x)) = d_H(v_1, x)$ and $d_{G\triangle H}((u_1, v_1), (u_2, v_2)) = d_H(v_1, x) + d_H(x, v_2) = d_H(v_1, v_2) = Max\{d_G(u_1, u_2), d_H(v_1, v_2)\}$. By the same technique, it is not difficult to prove the other case. □

Let $G^{[r]}$ be the graph with the same vertex set as G and two vertices $u, v \in V(G^{[r]})$ are adjacent whenever $d_G(u, v) = r$.

Lemma 3.4. *$(G \triangle H)^r = G^{r-1} \wedge H^r \cup G^{[r]} \wedge H^{[k]} \cup G^{2\lfloor \frac{r}{2} \rfloor} \times H^r$, where $k \leq r$ and $k + r$ is even.*

Proof. For any edge $(u_1, v_1)(u_2, v_2) \in E(G^{r-1} \wedge H^r) \cup E(G^{[r]} \wedge H^{[k]}) \cup E(G^{2\lfloor \frac{r}{2} \rfloor} \times H^r)$, where $k \leq r$ and $k+r$ is even, by Lemma 3.3 $d_{G\triangle H}((u_1, v_1), (u_2, v_2)) \leq r$ and $(u_1, v_1)(u_2, v_2) \in E((G \triangle H)^r)$. Conversely, let $(u_1, v_1)(u_2, v_2) \in E((G \triangle H)^r)$, then $d_{G\triangle H}((u_1, v_1), (u_2, v_2)) \leq r$ and we have the following cases:

(1) When $u_1 = u_2$, then $1 \leq d_H(v_1, v_2) \leq r$.

(2) When $v_1 = v_2$, then $1 \leq d_G(u_1, u_2) \leq r - 1$.

(3) When $1 \leq d_G(u_1, u_2) \leq 2\lfloor \frac{r}{2} \rfloor$, then $1 \leq d_H(v_1, v_2) \leq r$.

(4) When $d_G(u_1, u_2) = r$, then $d_H(v_1, v_2) = k$, where $r \geq k \in Z^+$ and $r + k$ is even.

In other words $(u_1, v_1)(u_2, v_2) \in E(G^{2\lfloor \frac{r}{2} \rfloor} \times H^r)$ or $(u_1, v_1)(u_2, v_2) \in E(G^{r-1} \wedge H^r)$ or $(u_1, v_1)(u_2, v_2) \in \cup E(G^{[r]} \wedge H^{[k]})$, where $k \leq r$ and $k + r$ is even. □

Following the same technique in Corollary 2.5, it is not difficult to prove the following corollary.

Corollary 3.5. *$(G \triangle H)^r = G^r \triangle H^r$ iff G is null or H is null.*

From the definitions of skew product and converse skew product, we conclude that:

Theorem 3.6. *$(G \bigtriangledown H)^r = H^{r-1} \wedge G^r \cup H^{[r]} \wedge G^{[k]} \cup H^{2\lfloor \frac{r}{2} \rfloor} \times G^r$, where $k \leq r$ and $k + r$ is even.*

Corollary 3.7. *$(G \bigtriangledown H)^r = G^r \bigtriangledown H^r$ iff G is null or H is null.*

4. Wiener Index and Wiener Polarity Index of the Skew Product of Graphs

In this section, we compute the Wiener index and the Wiener polarity index of $G \bigtriangleup H$ and $G \bigtriangledown H$. The Wiener index [11] defined as the sum of distances between all vertex pairs in a connected graph. While the Wiener polarity index [11] is defined to be the number of unordered pairs of vertices $\{u, v\}$ of $V(G)$ such that $d_G(u, v) = 3$. Let $W_k(G)$ be the number of unordered pairs of vertices $\{u, v\}$ of $V(G)$ such that $d_G(u, v) = k$. Then the Wiener polarity index of a graph G, denoted by $W_P(G) = W_3(G)$ and the Wiener index $W(G) = \sum_{k \geq 1} kW_k(G)$. The Wiener index of cartesian product, join, composition and corna is computed in [12]. In [6], the Wiener index of strong product is computed. The Wiener polarity index of cartesian product, composition, strong product is computed in [4].

Lemma 4.1. *Let G and H be two connected graphs, then $W_P(G \bigtriangleup H) = W_p(H)\big[|V(G)| + 2|E(G)| + 2W_2(G) + 2W_p(G)\big] + 2|E(H)|\big[W_p(G) + W_2(G)\big]$.*

Proof. According to Lemma 3.3, two vertices $(u_1, v_1), (u_2, v_2)$ are at distance 3 in $G \bigtriangleup H$ whenever: $(d_G(u_1, u_2), d_H(v_1, v_2))) = (0, 3), (1, 3), (2, 3), (3, 3), (3, 1), (2, 1)$. We distinguish between the following cases:
Case 1: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (0, 3)$, then $W_P(G \bigtriangleup H) = W_p(H)|V(G)|$.
Case 2: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (1, 3)$, then $W_P(G \bigtriangleup H) = 2W_p(H)|E(G)|$.
Case 3: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (2, 3)$, then $W_P(G \bigtriangleup H) = 2W_p(H)W_2(G)$.
Case 4: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (3, 3)$, then $W_P(G \bigtriangleup H) = 2W_p(H)W_p(G)$.
Case 5: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (3, 1)$, then $W_P(G \bigtriangleup H) = 2W_p(G)|E(H)|$.
Case 6: when $(d_G(u_1, u_2), d_H(v_1, v_2)) = (2, 1)$, then $W_P(G \bigtriangleup H) = 2W_2(G)|E(H)|$.
Therefore $W_P(G \bigtriangleup H) = W_P(H)\big[|V(G)| + 2|E(G)| + 2W_2(G) + 2W_P(G)\big] + 2|E(H)|\big[W_P(G) + W_2(G)\big]$. □

Lemma 4.2. *Let G and H be two connected graphs, then $W_P(G \bigtriangledown H) = W_p(G)\big[|V(H)| + 2|E(H)| + 2W_2(H) + 2W_p(H)\big] + 2|E(G)|\big[W_p(H) + W_2(H)\big]$.*

Theorem 4.3. *Let G and H be connected graphs, then $W(G \bigtriangleup H) = n_1^2 W(H) + n_2[W(G) + \sum_{i=1}^{i=\lfloor \frac{D}{2} \rfloor} W_{2i-1}(G)] + 2\sum_{i=2}^{i=D} W_i(G)[\sum_{j=2}^{j=i}[iW_{j-1}(H) + (i-1)W_{j-2}(H)]$, where $|V(G)| = n_1, |V(H)| = n_2$ and $D = diam(G \bigtriangleup H) = Max\{diam(G) + 1, diam(H)\}$.*

Proof. Let G and H be connected graphs with $|V(G)| = n_1, |V(H)| = n_2$, then $G \triangle H$ is connected. We note that when k is even, then

$$W_k(G\triangle H) = W_k(H)\big[n_1+2\sum_{i=1}^{i=k} W_i(G)\big]+(W_k(G)+W_{k-1}(G))\big[n_2+2\sum_{i=1}^{i=\frac{k-2}{2}} W_{2i}(H)\big],$$

for $k > 2$ and $W_2(G \triangle H) = W_2(H)\big[n_1 + 2\sum_{i=1}^{i=2} W_i(G)\big] + n_2\big[W_2(G) + W_1(G)\big]$. When k is odd, $W_k(G\triangle H) = W_k(H)\big[n_1+2\sum_{i=1}^{i=k} W_i(G)\big]+2\big[W_k(G)+W_{k-1}(G)\big] \sum_{i=1}^{i=\frac{k-1}{2}} W_{2i-1}(H)$, for $k \geq 3$ and $W_1(G \triangle H) = W_1(H)\big[n_1 + 2W_1(G)\big]$. Let $D = Max\{diam(G) + 1, diam(H)\}$. Assume without loss of generality that D is even, then

$$\begin{aligned}
W =& W_1(H)\big[n_1 + 2W_1(G)\big]+\\
& 2\Big[W_2(H)\big[n_1 + 2\sum_{i=1}^{i=2} W_i(G)\big] + n_2\big[W_2(G) + W_1(G)\big]\Big]+\\
& 3\Big[W_3(H)\big[n_1 + 2\sum_{i=1}^{i=3} W_i(G)\big] + 2\big[W_3(G) + W_2(G)\big]W_1(H)\Big]+\\
& 4\Big[W_4(H)\big[n_1 + 2\sum_{i=1}^{i=4} W_i(G)\big] + (W_4(G) + W_3(G))\big[n_2 + 2W_2(H)\big]\Big]+\\
& \vdots\\
& D\Big[W_D(H)\big[n_1 + 2\sum_{i=1}^{i=D} W_i(G)\big] + (W_D(G)+\\
& W_{D-1}(G))\big[n_2 + 2\sum_{i=3}^{i=\frac{D-2}{2}} W_{2i}(H)\big]\Big].
\end{aligned}$$

Which would imply that

$$\begin{aligned}
W =& W(H)\big[n_1 + 2\sum_{i=1}^{i=D} W_i(G)\big] + 2n_2\big[\sum_{i=1}^{i=D}\lceil\frac{i}{2}\rceil W_i(G)\big]+\\
& 2\sum_{i=2}^{i=D} W_i(G)\Big[\sum_{j=2}^{j=i}\big[iW_{j-1}(H) + (i-1)W_{j-2}(H)\big]\Big].\\
=& n_1^2W(H) + 2n_2\big[\sum_{i=1}^{i=D}\lceil\frac{i}{2}\rceil W_i(G)\big]+
\end{aligned}$$

$$2\sum_{i=2}^{i=D} W_i(G)\Big[\sum_{j=2}^{j=i}\big[iW_{j-1}(H)+(i-1)W_{j-2}(H)\big]\Big].$$

$$=n_1^2 W(H)+n_2\big[W(G)+\sum_{i=1}^{i=\lfloor\frac{D}{2}\rfloor} W_{2i-1}(G)\big]+$$

$$2\sum_{i=2}^{i=D} W_i(G)\Big[\sum_{j=2}^{j=i}\big[iW_{j-1}(H)+(i-1)W_{j-2}(H)\big]\Big].$$

□

Theorem 4.4. *Let G and H be connected graphs, then*

$$W(G\bigtriangledown H)=n_2^2 W(G)+n_1\Big[W(H)+\sum_{i=1}^{i=\lfloor\frac{D}{2}\rfloor} W_{2i-1}(H)\Big]+$$

$$2\sum_{i=2}^{i=D} W_i(H)\Big[\sum_{j=2}^{j=i}\big[iW_{j-1}(G)+(i-1)W_{j-2}(G)\big]\Big],$$

where $|V(G)|=n_1, |V(H)|=n_2$ *and* $D=diam(G\bigtriangledown H)=Max\{diam(G)(H)+1\}$.

Confilcts of Interest. The authors declare that they have no conflicts of interest.

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