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# Average Degree-Eccentricity Energy of Graphs

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**Abstract**

The concept of average degree-eccentricity matrix $ADE(G)$ of a connected graph $G$ is introduced. Some coefficients of the characteristic polynomial of $ADE(G)$ are obtained, as well as a bound for the eigenvalues of $ADE(G)$. We also introduce the average degree-eccentricity graph energy and establish bounds for it.

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## 1. Introduction

Throughout this paper, all graphs are assumed to be simple, finite and connected. Let $G = (V, E)$ be such a graph, with vertex set $\mathbf{V}$ and edge set $\mathbf{E}$. If $|\mathbf{V}| = p$ and $|\mathbf{E}| = q$, then $G$ is said to be a $(p, q)$-graph. The degree of a vertex $v$, denoted by $d(v)$, is the number of edges of $G$ incident with $v$. The distance $d(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of a shortest path connecting them. For a vertex $v$ of $G$, the eccentricity of $v$ is $e(v) = \max\{d(v, u), u \in \mathbf{V}(G)\}$. For additional graph-theoretical terminologies we refer to [8].

The adjacency matrix of $G$, $\mathbf{A}(G) = (a_{ij})$ is a $p \times p$ matrix, such that $a_{ij} = 1$ if $v_i v_j \in \mathbf{E}$ and $a_{ij} = 0$ otherwise. The energy of $G$, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{p} |x_i| \tag{1}$$

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where $x_1, x_2, \ldots, x_p$ are the eigenvalues of $\mathbf{A}(G)$. This concept was introduced almost 40 years ago [5] and has been extensively investigated [2,6,7,10]. Eventually, numerous other graph energies have been invented, based on eigenvalues of matrices different from the adjacency matrix; for more details see [1,6,7,9,11,13–17] and the references cited therein.

One of these graph energies is the *sum-eccentricity energy* [15,17], based on the eigenvalues of the *sum-eccentricity matrix* **SE**, whose elements are equal defined as

$$se_{ij} = \begin{cases} e(v_i) + e(v_j) & \text{if } v_i v_j \in \mathbf{E} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Another recently introduced graph energy is the *first Zagreb energy* [9], based on the eigenvalues of the *first Zagreb matrix* **ZG**, whose elements are defined as

$$zg_{ij} = \begin{cases} d(v_i) + d(v_j) & \text{if } v_i v_j \in \mathbf{E} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In this article, we introduce the concept of *average degree-eccentricity matrix* **ADE**.

**Definition 1.1.** Let $G = (V, E)$ be a simple connected graph with $p$ vertices $v_1, v_2, \ldots, v_p$ and let $d_i$ and $e(v_i)$ be, respectively, the degree and eccentricity of $v_i$, $i = 1, 2, \ldots, p$. Then the average degree-eccentricity matrix $\mathbf{ADE} = \mathbf{ADE}(G)$ of $G$ is the $p \times p$ matrix whose elements are given by

$$m_{ij} = \begin{cases} \frac{1}{4}[d(v_i) + d(v_j) + e(v_i) + e(v_j)] & \text{if } v_i v_j \in \mathbf{E} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Bearing in mind Equations (2) and (3), we see that **ADE** is conceived as a linear combination of the sum-eccentricity and Zagreb matrices, i.e.,

$$\mathbf{ADE} = \frac{1}{4}\big[\mathbf{SE} + \mathbf{ZG}\big] .$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ of $\mathbf{ADE}(G)$ form the average degree-eccentricity spectrum or the **ADE**-spectrum of $G$. As usual, the **ADE**-spectrum of $G$ with $n_i$-fold degenerate eigenvalues $\lambda_i$ is written as

$$S_p(G) = \{(\lambda_1)^{n_1}, (\lambda_2)^{n_2}, \ldots, (\lambda_p)^{n_p}\} .$$

**ADE** is a real symmetric matrix. Therefore, its eigenvalues are real numbers, and $\sum\limits_{i=1}^{p} \lambda_i = 0$.

The following result will be useful in the proof of our results.

**Theorem 1.2.** [3] (*Gershgorin's Theorem*) *Every eigenvalue* $\lambda$ *of a* $p \times p$ *matrix* $M = (m_{ij})$ *satisfies:*

$$|\lambda - m_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{p} |m_{ij}|.$$

**Corollary 1.3.** [4] (*Hadamard's Inequality*) *If the entries of a* $p \times p$ *matrix* $M$ *are bounded by* $B$*, then* $|\det(M)| \leq B^p \, p^{p/2}$.

## 2. Average Degree-Eccentricity Energy

**Definition 2.1.** The average degree-eccentricity energy $E_{ade}(G)$ of a graph $G$ is

$$E_{ade}(G) = \sum_{i=1}^{p} |\lambda_i| \,. \tag{5}$$

Evidently, the average degree-eccentricity energy is defined in analogy to the ordinary graph energy, Equation (1).

**Example 2.2.** For a graph $G_1$ in Figure 1,

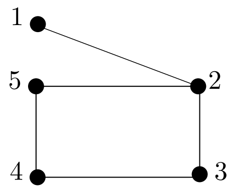


Figure 1: $G_1$.

the average degree-eccentricity matrix of $G_1$ is

$$\mathbf{ADE}(G_1) = \begin{bmatrix} 0 & \frac{9}{4} & 0 & 0 & 0 \\ \frac{9}{4} & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\ 0 & \frac{9}{4} & 0 & \frac{9}{4} & 0 \\ 0 & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\ 0 & \frac{9}{4} & 0 & \frac{9}{4} & 0 \end{bmatrix}$$

The characteristic polynomial of $\mathbf{ADE}(G_1)$ is, $P(G_1, \lambda) = |\lambda I_p - \mathbf{ADE}(G_1)| = \lambda^5 - \frac{405}{16}\lambda^3 + \frac{6561}{128}\lambda$ and the average degree-eccentricity eigenvalues of $G_1$ are $\lambda_1 \approx 4.8, \lambda_2 \approx 1.5, \lambda_3 = 0, \lambda_4 \approx -1.5, \lambda_5 \approx -4.8$. Then the average degree-eccentricity energy of $G_1$ is $E_{ade}(G_1) = 4.8 + 1.5 + 1.5 + 4.8 = 12.6$.

We now calculate the coefficient $c_i$ of $\lambda^{p-i}(i = 0, 1, 2, p)$ in the characteristic polynomial of the average degree-eccentricity matrix $\mathbf{ADE}(G)$. Clearly $c_0 = 1$, $c_1 =$trace$(\mathbf{ADE}(G)) = 0$. Now

$$c_2 = \sum_{1\leq i<j\leq p} \begin{vmatrix} 0 & m_{ij} \\ m_{ji} & 0 \end{vmatrix} = \sum_{1\leq i<j\leq p} -m_{ij}^2 \,.$$

In view of Equation (4) we get

$$c_2 = -\sum_{v_i v_j \in \mathbf{E}} \left[ \frac{d(v_i) + e(v_i) + d(v_j) + e(v_j)}{4} \right]^2 .$$

For $c_3$ we have

$$c_3 = (-1)^3 \sum_{1\leq i<j<r\leq n} \begin{vmatrix} m_{ii} & m_{ij} & m_{ir} \\ m_{ji} & m_{jj} & m_{jr} \\ m_{ri} & m_{rj} & m_{rr} \end{vmatrix} .$$

The number of non-zero terms in the above sum is equal to the number of triangles in $G$. Therefore, $c_3 = 0$ if $G$ has no triangle.

Finally, $c_p = \det(\mathbf{ADE}(G))$.

**Lemma 2.3.** *Let $G$ be a connected $(p, q)$-graph and $uv \in \mathbf{E}$. Then*

$$\frac{1}{4}[d(u) + d(v) + e(u) + e(v)] \leq \frac{p}{2} \,. \tag{6}$$

*Equality in (6) holds for all $uv \in \mathbf{E}$ only if $G \cong K_p$.*

*Proof.* Without loss of generality, we may assume that $e(u) \leq e(v)$. So, we have

$$\begin{aligned} d(u) + d(v) + e(u) + e(v) &\leq d(u) + d(v) + 2e(v) \\ &\leq d(u) + d(v) + 2[p - (d(u) + d(v)) + 1] \\ &= 2p - (d(u) + d(v)) + 2 \leq 2p \,. \end{aligned}$$

If $G \cong K_p$, then for any $uv \in \mathbf{E}$ we have $d(u) = d(v) = p-1$ and $e(u) = e(v) = 1$, implying that the left–hand side of (6) is equal to $p/2$. For all other (connected) graphs, for some $uv \in \mathbf{E}$ the inequality in (6) will be strict. □

**Lemma 2.4.** *Let $G$ be a connected $(p, q)$-graph. Then*

$$trace\, \mathbf{ADE}^2(G) \leq trace\, \mathbf{ADE}^2(K_p) = \frac{(p-1)p^3}{4} \,. \tag{7}$$

*Equality in (7) holds if and only if $G \cong K_p$.*

*Proof.* Since

$$\mathbf{ADE}(K_p)_{ij} = \begin{cases} \dfrac{p}{2} & \text{if } v_i v_j \in \mathbf{E} \\ 0 & \text{otherwise} \end{cases}$$

we get that for $i \neq j$,

$$\mathbf{ADE}^2(K_p)_{ij} = (p-2)\left(\frac{p}{2}\right)^2$$

whereas for $i = j$,

$$\mathbf{ADE}^2(K_p)_{ii} = (p-1)\left(\frac{p}{2}\right)^2$$

implying that

$$trace\,\mathbf{ADE}^2(K_p) = p \times (p-1)\left(\frac{p}{2}\right)^2 = \frac{(p-1)p^3}{4}\,.$$

Bearing in mind Lemma 2.3 and formula (4), we immediately see that $\mathbf{ADE}(G)_{ij} \leq \mathbf{ADE}(K_p)_{ij}$, and that if $G \ncong K_p$, then the inequality is strict for at least some of $ij$. Consequently, inequality (7) holds. □

**Theorem 2.5.** *For any $(p,q)$-graph, with average degree-eccentricity eigenvalue $\lambda_j$,*

$$|\lambda_j| \leq \frac{p(p-1)}{2}\,. \tag{8}$$

*Proof.* By Lemma 2.4, the trace of $\mathbf{ADE}^2(K_p)$ is equal to $\frac{(p-1)p^3}{4}$. Then for any $(p,q)$-graph $G$ with average degree-eccentricity eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$, we have $\sum\limits_{i=1}^{p} |\lambda_i|^2 \leq \frac{(p-1)p^3}{4}$. By the Cauchy–Schwarz inequality,

$$\left(\sum_{\substack{i=1\\ i\neq j}}^{p} \lambda_i\right)^2 = (p-1)\sum_{\substack{i=1\\ i\neq j}}^{p} \lambda_i^2\,.$$

Since $\sum\limits_{i=1}^{p} \lambda_i^2 = -2c_2$ and $\sum\limits_{i=1}^{p} \lambda_i = 0$, we get

$$\lambda_j^2 \leq (p-1)\left[\frac{(p-1)p^3}{4} - \lambda_j^2\right]$$

which implies (8). □

**Proposition 2.6.** *Let $G$ be a graph of order $p$, and average degree-eccentricity eigenvalue $\lambda_i$. Then*

$$\prod_{i=1}^{p} |\lambda_i| \leq \left(\frac{p}{2}\right)^p p^{p/2}\,.$$

*Proof.* By Corollary 1.3 and by the definition of **ADE**, setting $B = p/2$. □

**Theorem 2.7.** *Let $G$ be a $(p,q)$-graph. Then*

$$E_{ade}(G) \leq \frac{(p-1)p^2}{2}\,.$$

*Proof.* By Gershgorin's Theorem and Lemma 2.3, we have

$$\begin{aligned} E_{ade}(G) &= \sum_{i=1}^{p} |\lambda_i| = \sum_{i=1}^{p} |\lambda_i - 0| \leq \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j\neq i}}^{p} m_{ij} \\ &\leq \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j\neq i}}^{p} \frac{p}{2} = \frac{(p-1)p^2}{2}\,. \end{aligned}$$

□

**Theorem 2.8.** *Let $G$ be a connected $(p,q)$-graph. Then*

$$E_{ade}(G) \geq \sqrt{2(q|\det(\mathbf{ADE})|^{2/p} - c_2)}\,.$$

*Proof.*

$$E_{ade}(G)^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 = \sum_{i=1}^{p} \lambda_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i||\lambda_j| = -2c_2 + \sum_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i||\lambda_j|\,.$$

From relation between the arithmetic and geometric means, we get

$$\begin{aligned} \sum_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i||\lambda_j| &\geq p(p-1)\left(\prod_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i||\lambda_j|\right)^{\frac{1}{p(p-1)}} = p(p-1)\left(\prod_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}} \\ &= p(p-1)\left(\prod_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i|\right)^{2/p} \geq 2q\left(\prod_{\substack{i=1 \\ i\neq j}}^{p} |\lambda_i|\right)^{2/p} = 2q|\det(\mathbf{ADE})|^{2/p}\,. \end{aligned}$$

Then

$$E_{ade}(G)^2 \geq 2q|\det(\mathbf{ADE})|^{2/p} - 2c_2 = 2\big[q|\det(\mathbf{ADE})|^{2/p} - c_2\big]$$

and finally,

$$E_{ade}(G) \geq \sqrt{2\big[q|\det(\mathbf{ADE}|)^{2/p} - c_2\big]}\,.$$

□

Note that Theorem 2.8 and its proof are just a replica of the classical McClelland inequality for ordinary graph energy [12].

**Corollary 2.9.** *Let $G$ be a connected $(p,q)$-graph. Then*

$$\sqrt{2(q|\det(\mathbf{ADE})|^{2/p} - c_2)} \leq E_{ade}(G) \leq \frac{(p-1)p^2}{2}\,.$$

# 3. Average Degree-Energy of Some Classes of Graphs

In this section, we compute the average degree-eccentricity energies of some well-known graphs.

**Example 3.1.** Let $G$ be a complete graph $K_p$. Then $S_p(\mathbf{ADE}(K_p)) = \{(\frac{p}{2})^{p-1}, ((p-1)(\frac{p}{2}))^1\}$ and $E_{ade}(K_p) = 2(p-1)(\frac{p}{2})$.

*Proof.* Let $G$ be the complete graph $K_p$. Then

$$\begin{aligned}
|\lambda I - \mathbf{ADE}(K_p)| &= \begin{vmatrix} \lambda & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ -\frac{p}{2} & \lambda & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{p}{2} & -\frac{p}{2} & -\frac{p}{2} & \cdots & \lambda \end{vmatrix} \\
&= \left(\lambda + \frac{p}{2}\right)^{p-1} \begin{vmatrix} \lambda & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\ -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} \\
&= \left(\lambda + \frac{p}{2}\right)^{p-1} \left[\lambda - (p-1)\frac{p}{2}\right].
\end{aligned}$$

Then the average degree-eccentricity energy of the complete graph is

$$E_{ade}(K_p) = 2(p-1)\,\frac{p}{2}\,.$$

□

**Example 3.2.** Let $G$ be a complete bipartite graph $K_{m,n}$, $m,n \geq 2$. Then

$$S_p(\mathbf{ADE}(K_{m,n})) = \left\{ \left(\frac{p+4}{4}\sqrt{mn}\right)^1, (0)^{p-2}, \left(-\frac{(p+4)}{4}\sqrt{mn}\right)^1 \right\} \tag{9}$$

and

$$E_{ade}(K_{m,n}) = \frac{p+4}{2}\sqrt{mn}\,. \tag{10}$$

*Proof.* Let the vertex set of $K_{m,n}$ be $V = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$. Then, $p = m + n$, $q = mn$, and

$$
\begin{aligned}
|\lambda I - \mathbf{ADE}(K_{m,n})| &= \begin{vmatrix}
\lambda & 0 & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\
0 & \lambda & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \lambda & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\
-\frac{p+4}{4} & \cdots & -\frac{p+4}{4} & 0 & \lambda & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{p+4}{4} & \cdots & -\frac{p+4}{4} & 0 & \cdots & 0 & \lambda
\end{vmatrix} \\
&= \lambda^p - \left(\frac{p+4}{4}\right)^2 (mn)\lambda^{p-2} = \lambda^{p-2}\left[\lambda^2 - \left(\frac{p+4}{4}\right)^2 (mn)\right].
\end{aligned}
$$

Then, $\lambda^{p-2}\Big[\lambda^2 - (\frac{p+4}{4})^2(mn)\Big] = 0$ implies $\lambda^{p-2} = 0$, or $\lambda^2 = (\frac{p+4}{4})^2(mn)$, resulting in (9) and (10). □

**Example 3.3.** For the star graph $K_{1,p-1}$,

$$S_p(\mathbf{ADE}(K_{1,p-1})) = \left\{ \left(\frac{p+3}{4}\sqrt{p-1}\right)^1, (0)^{p-2}, \left(-\frac{(p+3)}{4}\sqrt{p-1}\right)^1 \right\}$$

and

$$E_{ade}(K_{1,p-1}) = \frac{p+3}{2}\sqrt{p-1}\,.$$

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**Confilcts of Interest.** The authors declare that they have no conflicts of interest.

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