Average Degree-Eccentricity Energy of Graphs

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Abstract

The concept of average degree-eccentricity matrix $ADE(G)$ of a connected graph $G$ is introduced. Some coefficients of the characteristic polynomial of $ADE(G)$ are obtained, as well as a bound for the eigenvalues of $ADE(G)$. We also introduce the average degree-eccentricity graph energy and establish bounds for it.

Keywords: Average degree-eccentricity matrix, average degree-eccentricity eigenvalue, average degree-eccentricity energy.

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1. Introduction

Throughout this paper, all graphs are assumed to be simple, finite and connected. Let $G = (V, E)$ be such a graph, with vertex set $V$ and edge set $E$. If $|V| = p$ and $|E| = q$, then $G$ is said to be a $(p,q)$-graph. The degree of a vertex $v$, denoted by $d(v)$, is the number of edges of $G$ incident with $v$. The distance $d(u,v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of a shortest path connecting them. For a vertex $v$ of $G$, the eccentricity of $v$ is $e(v) = \max\{d(v,u), u \in V(G)\}$. For additional graph-theoretical terminologies we refer to [8].

The adjacency matrix of $G$, $A(G) = (a_{ij})$ is a $p \times p$ matrix, such that $a_{ij} = 1$ if $v_i v_j \in E$ and $a_{ij} = 0$ otherwise. The energy of $G$, denoted by $E(G)$, is defined as

$$E(G) = \sum_{|x_i|}^{P}$$
where \(x_1, x_2, \ldots, x_p\) are the eigenvalues of \(A(G)\). This concept was introduced almost 40 years ago [5] and has been extensively investigated [2, 6, 7, 10]. Eventually, numerous other graph energies have been invented, based on eigenvalues of matrices different from the adjacency matrix; for more details see [1, 6, 7, 9, 11, 13–17] and the references cited therein.

One of these graph energies is the sum-eccentricity energy [15, 17], based on the eigenvalues of the sum-eccentricity matrix \(SE\), whose elements are defined as

\[
se_{ij} = \begin{cases} 
e(v_i) + e(v_j) & \text{if } v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}
\]  

(2)

Another recently introduced graph energy is the first Zagreb energy [9], based on the eigenvalues of the first Zagreb matrix \(ZG\), whose elements are defined as

\[
zg_{ij} = \begin{cases} d(v_i) + d(v_j) & \text{if } v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}
\]  

(3)

In this article, we introduce the concept of average degree-eccentricity matrix \(ADE\).

**Definition 1.1.** Let \(G = (V, E)\) be a simple connected graph with \(p\) vertices \(v_1, v_2, \ldots, v_p\) and let \(d_i\) and \(e(v_i)\) be, respectively, the degree and eccentricity of \(v_i\), \(i = 1, 2, \ldots, p\). Then the average degree-eccentricity matrix \(ADE = ADE(G)\) of \(G\) is the \(p \times p\) matrix whose elements are given by

\[
m_{ij} = \begin{cases} \frac{1}{4}[d(v_i) + d(v_j) + e(v_i) + e(v_j)] & \text{if } v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}
\]  

(4)

Bearing in mind Equations (2) and (3), we see that \(ADE\) is conceived as a linear combination of the sum-eccentricity and Zagreb matrices, i.e.,

\[
ADE = \frac{1}{4}[SE + ZG].
\]

The eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_p\) of \(ADE(G)\) form the average degree-eccentricity spectrum or the \(ADE\)-spectrum of \(G\). As usual, the \(ADE\)-spectrum of \(G\) with \(n_i\)-fold degenerate eigenvalues \(\lambda_i\) is written as

\[
S_p(G) = \{(\lambda_1)^{n_1}, (\lambda_2)^{n_2}, \ldots, (\lambda_p)^{n_p}\}.
\]

\(ADE\) is a real symmetric matrix. Therefore, its eigenvalues are real numbers, and \(\sum_{i=1}^{p} \lambda_i = 0\).

The following result will be useful in the proof of our results.
Theorem 1.2. [3] (Gershgorin’s Theorem) Every eigenvalue $\lambda$ of a $p \times p$ matrix $M = (m_{ij})$ satisfies:

$$|\lambda - m_{ii}| \leq \sum_{j=1 \atop j \neq i}^{p} |m_{ij}|.$$  

Corollary 1.3. [4] (Hadamard’s Inequality) If the entries of a $p \times p$ matrix $M$ are bounded by $B$, then $|\det(M)| \leq B^p p^{p/2}$.

2. Average Degree-Eccentricity Energy

Definition 2.1. The average degree-eccentricity energy $E_{ade}(G)$ of a graph $G$ is

$$E_{ade}(G) = \sum_{i=1}^{p} |\lambda_i|.$$  \hspace{1cm} (5)

Evidently, the average degree-eccentricity energy is defined in analogy to the ordinary graph energy, Equation (1).

Example 2.2. For a graph $G_1$ in Figure 1,

![Figure 1: G1.](image)

the average degree-eccentricity matrix of $G_1$ is

$$\text{ADE}(G_1) = \begin{bmatrix}
0 & \frac{9}{4} & 0 & 0 & 0 \\
\frac{9}{4} & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\
0 & \frac{9}{4} & 0 & \frac{9}{4} & 0 \\
0 & 0 & \frac{9}{4} & 0 & \frac{9}{4} \\
0 & \frac{9}{4} & 0 & \frac{9}{4} & 0 
\end{bmatrix}$$

The characteristic polynomial of $\text{ADE}(G_1)$ is, $P(G_1, \lambda) = |\lambda I_p - \text{ADE}(G_1)| = \lambda^5 - \frac{405}{16} \lambda^3 + \frac{5624}{128} \lambda$ and the average degree-eccentricity eigenvalues of $G_1$ are $\lambda_1 \approx 4.8, \lambda_2 \approx 1.5, \lambda_3 = 0, \lambda_4 \approx -1.5, \lambda_5 \approx -4.8$. Then the average degree-eccentricity energy of $G_1$ is $E_{ade}(G_1) = 4.8 + 1.5 + 1.5 + 4.8 = 12.6$. 
We now calculate the coefficient \(c_i\) of \(\lambda^{p-i}(i = 0, 1, 2, p)\) in the characteristic polynomial of the average degree-eccentricity matrix \(\text{ADE}(G)\). Clearly \(c_0 = 1\), \(c_1 = \text{trace}(\text{ADE}(G)) = 0\). Now

\[
c_2 = \sum_{1 \leq i < j \leq p} \begin{vmatrix} 0 & m_{ij} \\ m_{ji} & 0 \end{vmatrix} = \sum_{1 \leq i < j \leq p} -m_{ij}^2.
\]

In view of Equation (4) we get

\[
c_2 = -\sum_{v_i, v_j \in E} \left[ \frac{d(v_i) + e(v_i) + d(v_j) + e(v_j)}{4} \right]^2.
\]

For \(c_3\) we have

\[
c_3 = (-1)^3 \sum_{1 \leq i < j < r \leq n} \begin{vmatrix} m_{ii} & m_{ij} & m_{ir} \\ m_{ji} & m_{jj} & m_{jr} \\ m_{ri} & m_{rj} & m_{rr} \end{vmatrix}.
\]

The number of non-zero terms in the above sum is equal to the number of triangles in \(G\). Therefore, \(c_3 = 0\) if \(G\) has no triangle.

Finally, \(c_p = \det(\text{ADE}(G))\).

**Lemma 2.3.** Let \(G\) be a connected \((p, q)\)-graph and \(uv \in E\). Then

\[
\frac{1}{4}[d(u) + d(v) + e(u) + e(v)] \leq \frac{p}{2}.
\]

Equality in (6) holds for all \(uv \in E\) only if \(G \cong K_p\).

**Proof.** Without loss of generality, we may assume that \(e(u) \leq e(v)\). So, we have

\[
d(u) + d(v) + e(u) + e(v) \leq d(u) + d(v) + 2e(v) \\
\leq d(u) + d(v) + 2[p - (d(u) + d(v)) + 1] \\
= 2p - (d(u) + d(v)) + 2 \leq 2p.
\]

If \(G \cong K_p\), then for any \(uv \in E\) we have \(d(u) = d(v) = p - 1\) and \(e(u) = e(v) = 1\), implying that the left–hand side of (6) is equal to \(p/2\). For all other (connected) graphs, for some \(uv \in E\) the inequality in (6) will be strict. \(\square\)

**Lemma 2.4.** Let \(G\) be a connected \((p, q)\)-graph. Then

\[
\text{trace} \text{ADE}^2(G) \leq \text{trace} \text{ADE}^2(K_p) = \frac{(p - 1)p^3}{4}.
\]

Equality in (7) holds if and only if \(G \cong K_p\).
Proof. Since
\[ \text{ADE}(K_p)_{ij} = \begin{cases} \frac{p}{2} & \text{if } v_iv_j \in E \\ 0 & \text{otherwise} \end{cases} \]
we get that for \( i \neq j \),
\[ \text{ADE}^2(K_p)_{ij} = (p - 2) \left( \frac{p}{2} \right)^2 \]
whereas for \( i = j \),
\[ \text{ADE}^2(K_p)_{ii} = (p - 1) \left( \frac{p}{2} \right)^2 \]
implying that
\[ \text{trace } \text{ADE}^2(K_p) = p \times (p - 1) \left( \frac{p}{2} \right)^2 = \frac{(p - 1)p^3}{4}. \]

Bearing in mind Lemma 2.3 and formula (4), we immediately see that \( \text{ADE}(G)_{ij} \leq \text{ADE}(K_p)_{ij} \), and that if \( G \not\cong K_p \), then the inequality is strict for at least some of \( ij \). Consequently, inequality (7) holds.

**Theorem 2.5.** For any \((p,q)\)-graph, with average degree-eccentricity eigenvalue \( \lambda_j \),
\[ |\lambda_j| \leq \frac{p(p - 1)}{2} . \]

**Proof.** By Lemma 2.4, the trace of \( \text{ADE}^2(K_p) \) is equal to \( \frac{(p - 1)p^3}{4} \). Then for any \((p,q)\)-graph \( G \) with average degree-eccentricity eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_p \), we have
\[ \sum_{i=1}^{p} |\lambda_i|^2 \leq \frac{(p - 1)p^3}{4}. \]

By the Cauchy–Schwarz inequality,
\[ \left( \sum_{i=1}^{p} \lambda_i \right)^2 = (p - 1) \sum_{i=1}^{p} \lambda_i^2 . \]

Since \( \sum_{i=1}^{p} \lambda_i^2 = -2c_2 \) and \( \sum_{i=1}^{p} \lambda_i = 0 \), we get
\[ \lambda_j^2 \leq (p - 1) \left[ \frac{(p - 1)p^3}{4} - \lambda_j^2 \right] \]
which implies (8).

**Proposition 2.6.** Let \( G \) be a graph of order \( p \), and average degree-eccentricity eigenvalue \( \lambda_i \). Then
\[ \prod_{i=1}^{p} |\lambda_i| \leq \left( \frac{p}{2} \right)^p p^{p/2}. \]
Proof. By Corollary 1.3 and by the definition of ADE, setting $B = p/2$.

**Theorem 2.7.** Let $G$ be a $(p,q)$-graph. Then

$$E_{ade}(G) \leq \frac{(p - 1)p^2}{2}.$$ 

Proof. By Gershgorin’s Theorem and Lemma 2.3, we have

$$E_{ade}(G) = \sum_{i=1}^{p} |\lambda_i| = \sum_{i=1}^{p} |\lambda_i - 0| \leq \sum_{i=1}^{p} \sum_{j \neq i} m_{ij}$$

$$\leq \sum_{i=1}^{p} \sum_{j \neq i} \frac{p}{2} = \frac{(p - 1)p^2}{2}.$$

**Theorem 2.8.** Let $G$ be a connected $(p,q)$-graph. Then

$$E_{ade}(G) \geq \sqrt{2(q|\det(ADE)|^{2/p} - c_2)}.$$

Proof.

$$E_{ade}(G)^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 = \sum_{i=1}^{p} \lambda_i^2 + \sum_{i=1}^{p} |\lambda_i||\lambda_j| = -2c_2 + \sum_{i=1}^{p} |\lambda_i||\lambda_j|.$$ 

From relation between the arithmetic and geometric means, we get

$$\sum_{i=1}^{p} |\lambda_i||\lambda_j| \geq p(p - 1) \left(\prod_{i=1}^{p} |\lambda_i||\lambda_j|\right)^{\frac{1}{p(p - 1)}} = p(p - 1) \left(\prod_{i=1}^{p} |\lambda_i|^2(p-1)\right)^{\frac{1}{2p}}$$

$$= p(p - 1) \left(\prod_{i=1}^{p} |\lambda_i|\right)^{2/p} \geq 2q \left(\prod_{i=1}^{p} |\lambda_i|\right)^{2/p} = 2q|\det(ADE)|^{2/p}.$$

Then

$$E_{ade}(G)^2 \geq 2q|\det(ADE)|^{2/p} - 2c_2 = 2[q|\det(ADE)|^{2/p} - c_2]$$

and finally,

$$E_{ade}(G) \geq \sqrt{2[q|\det(ADE)|^{2/p} - c_2]}.$$
Note that Theorem 2.8 and its proof are just a replica of the classical McClelland inequality for ordinary graph energy [12].

**Corollary 2.9.** Let $G$ be a connected $(p,q)$-graph. Then
\[
\sqrt{2(q} \det(\text{ADE})^{(2/p - c_2}) \leq E_{ade}(G) \leq \frac{(p - 1)p^2}{2}.
\]

3. Average Degree-Energy of Some Classes of Graphs

In this section, we compute the average degree-eccentricity energies of some well-known graphs.

**Example 3.1.** Let $G$ be a complete graph $K_p$. Then $S_p(\text{ADE}(K_p)) = \{(\frac{p}{2})^{p-1}, ((p - 1)(\frac{p}{2}))^1\}$ and $E_{ade}(K_p) = 2(p - 1)(\frac{p}{2})$.

**Proof.** Let $G$ be the complete graph $K_p$. Then
\[
|\lambda I - \text{ADE}(K_p)| = \begin{vmatrix}
\lambda & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\
-\frac{p}{2} & \lambda & -\frac{p}{2} & \cdots & -\frac{p}{2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{p}{2} & -\frac{p}{2} & -\frac{p}{2} & \cdots & \lambda \\
\end{vmatrix}
= \begin{vmatrix}
\lambda + \frac{p}{2} & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\
-1 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & 0 & 0 & \cdots & 1 \\
\end{vmatrix}
= \begin{vmatrix}
\lambda + \frac{p}{2} & -\frac{p}{2} & -\frac{p}{2} & \cdots & -\frac{p}{2} \\
-1 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & 0 & 0 & \cdots & 1 \\
\end{vmatrix}.
\]

Then the average degree-eccentricity energy of the complete graph is
\[
E_{ade}(K_p) = 2(p - 1)\frac{p}{2}.
\]

**Example 3.2.** Let $G$ be a complete bipartite graph $K_{m,n}$, $m,n \geq 2$. Then
\[
S_p(\text{ADE}(K_{m,n})) = \left\{ \left( \frac{p + 4}{4} \sqrt{mn} \right)^1, (0)^{p-2}, \left( -\frac{p + 4}{4} \sqrt{mn} \right)^1 \right\} \quad \text{(9)}
\]
and
\[
E_{ade}(K_{m,n}) = \frac{p + 4}{2} \sqrt{mn}. \quad \text{(10)}
\]
Proof. Let the vertex set of $K_{m,n}$ be $V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$. Then,

\[ p = m + n, \quad q = mn, \quad \text{and} \]

\[
|\lambda - \text{ADE}(K_{m,n})| = \begin{vmatrix}
\lambda & 0 & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\
0 & \lambda & \cdots & 0 & -\frac{p+4}{4} & \cdots & -\frac{p+4}{4} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{p+4}{4} & \cdots & -\frac{p+4}{4} & 0 & \cdots & 0 & \lambda \\
\end{vmatrix} = \lambda^p - \left(\frac{p+4}{4}\right)^2 (mn) \lambda^{p-2} = \lambda^{p-2} \left[\lambda^2 - \left(\frac{p+4}{4}\right)^2 (mn)\right].
\]

Then, $\lambda^{p-2} \left[\lambda^2 - \left(\frac{p+4}{4}\right)^2 (mn)\right] = 0$ implies $\lambda^{p-2} = 0$ or $\lambda^2 = \left(\frac{p+4}{4}\right)^2 (mn)$, resulting in (9) and (10).

Example 3.3. For the star graph $K_{1,p-1}$,

\[ S_p(\text{ADE}(K_{1,p-1})) = \left\{\left(\frac{p+3}{4}\sqrt{p-1}\right)^{p-1}, (0)^{p-2}, -\left(\frac{p+3}{4}\sqrt{p-1}\right)^{p-1}\right\} \]

and

\[ E_{\text{ade}}(K_{1,p-1}) = \frac{p+3}{2\sqrt{p-1}}. \]

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