

Some Applications of Strong Product

Mostafa Tavakoli^{*}, Freydoon Rahbarnia and
Irandokht Rezaee Abdolhosseinzadeh

Abstract

Let G and H be two graphs. The strong product $G \boxtimes H$ of the graphs G and H is the graph with vertex set $V(G) \times V(H)$, and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever $(v_1 = v_2$ and u_1 is adjacent with $u_2)$ or $(u_1 = u_2$ and v_1 is adjacent with $v_2)$ or $(u_1$ is adjacent with u_2 and v_1 is adjacent with $v_2)$. In this paper, some applications of this product are presented. Finally, we pose one open problem related to this topic.

Keywords: Strong product, graph invariant, fence graph.

2010 Mathematics Subject Classification: Primary: 05C76, Secondary: 05C12, 05C07.

How to cite this article

M. Tavakoli, F. Rahbarnia and I. Rezaee Abdolhosseinzadeh, Some applications of strong product, *Math. Interdisc. Res.* **3** (2018) 55-65.

1. Introduction

A **topological index** is a real number related to a graph. It must be a structural invariant, i.e., it preserves by every graph automorphism. A topological index is a graph invariant applicable in chemistry. Suppose G is a graph with the vertex and edge sets of $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$, then the **distance** $d_G(x, y)$ (or $d(x, y)$ for short) between x and y is defined as the length of a minimum path connecting x and y . The **Wiener index** of G , $W(G)$, is defined as the summation of distances between all pairs of vertices in G . In other words, the Wiener index of a graph G is defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$ [21]. A topological index is called distance-based if it can be defined by the distance function $d(-, -)$. It is worthy to mention here that Wiener did not consider the

^{*}Corresponding author (Email: m_tavakoli@um.ac.ir)

Academic Editor: Gholam Hossein Fath-Tabar

Received 01 June 2016, Accepted 03 April 2018

DOI: 10.22052/mir.2018.55115.1033



distance function $d(-, -)$ in the seminal paper. Hosoya [12], presented a new simple formula for the Wiener index by using distance function. We encourage the readers to consult [6, 7] for more information on Wiener index.

The **hyper-Wiener index** of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. [16], generalized Randić's definition for all connected graphs, as a generalization of the Wiener index. It is defined as

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} (d(u,v) + d^2(u,v))$$

or

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u,v).$$

The mathematical properties and chemical meaning of this topological index are reported in [4, 5, 9, 15, 25].

As usual, the **degree** of a vertex u of G is denoted by $deg(u)$ and it is defined as the number of edges incident with u . The **Zagreb indices** have been introduced more than thirty years ago by Gutman and Trinajstić, [10]. They are defined as:

$$M_1(G) = \sum_{u \in V(G)} deg(u)^2,$$

$$M_2(G) = \sum_{uv \in E(G)} deg(u)deg(v).$$

We encourage the reader to consult [1, 10, 23] for historical background, computational techniques and mathematical properties of Zagreb indices.

The **eccentricity** $\varepsilon_G(u)$ is defined as the largest distance between u and other vertices of G . We will omit the subscript G when the graph is clear from the context. The **eccentric connectivity** index of a graph G is defined as $\xi^c(G) = \sum_{u \in V(G)} deg_G(u)\varepsilon_G(u)$ [19]. We encourage the reader to consult the papers [2, 3] for some applications and the papers [13, 17, 22, 24] for the mathematical properties of this topological index. For a given vertex $u \in V(G)$ we define its **distance sum** $D_G(u)$ as $D_G(u) = \sum_{v \in V(G)} d_G(u,v)$. The **eccentric distance sum** of G is summation of all quantity $D_G(u)\varepsilon_G(u)$ over all vertices of G [8]. In other words, $\xi^{SD}(G) = \sum_{u \in V(G)} D_G(u)\varepsilon_G(u)$. The concept of eccentricity also gives rise to a number of other topological invariants. For example, the **total eccentricity** $\zeta(G)$ of a graph G is defined as $\zeta(G) = \sum_{u \in V(G)} \varepsilon_G(u)$.

The **n -cube** Q_n ($n \geq 1$) is the graph whose vertex set is the set of all n -tuples of 0s and 1s, where two n -tuples are adjacent if they differ in precisely one coordinate. Q_n has 2^n vertices, $2^{n-1}n$ edges, and is a regular graph with n edges touching each vertex. A graph G is called **nontrivial** if $|V(G)| > 1$. Also, we denote the path graph, the complete and the cycle of order n by P_n , K_n and C_n , respectively.

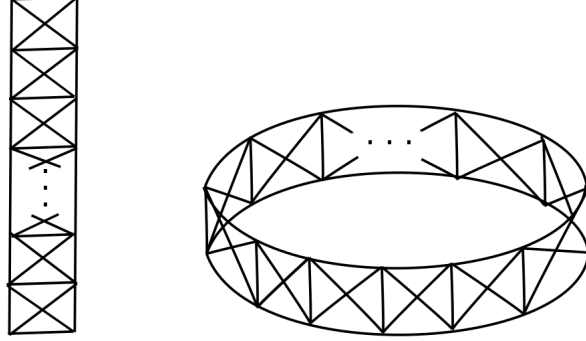


Figure 1: Open and closed fences.

The **Strong** product $G \boxtimes H$ of the graphs G and H has the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \boxtimes H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$, or $ab \in E(G)$ and $xy \in E(H)$. Occasionally one also encounters the names strong direct product or symmetric composition for the strong product [11]. As an example, see open and closed fences, $P_n \boxtimes K_2$, $C_n \boxtimes K_2$, Figure 1.

2. Results

For a connected graph G , the radius $r(G)$ and diameter $D(G)$ are, respectively, the minimum and maximum eccentricity among the vertices of G .

Lemma 2.1. [20] *Let G and H be two graphs. Then for every vertex (a, x) of $G \boxtimes H$, we have*

$$\varepsilon_{G \boxtimes H}((a, x)) = \max\{\varepsilon_G(a), \varepsilon_H(x)\}.$$

Theorem 2.2. [20] *Let G and H be nontrivial connected graphs. Then $G \boxtimes H$ is eulerian if and only if G and H are eulerian.*

By the above theorem, $C_n \boxtimes C_m$ and $K_{2n+1} \boxtimes C_m$ are eulerian.

Theorem 2.3. [20] *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} W(G \boxtimes H) &\geq (|V(G)| + 2|E(G)|)W(H) + (|V(H)| + 2|E(H)|)W(G) \\ &\quad + |V(G)||V(H)|(|V(G)||V(H)| - |V(G)| - |V(H)| + 1) \\ &\quad - 2|E(G)||V(H)|(|V(H)| - 1) - 2|E(H)|(|V(G)|^2|V(G)| - |E(G)|), \end{aligned}$$

with equality if and only if $\max\{D(G), D(H)\} \leq 2$, or $G \cong K_n$, or $H \cong K_n$.

We apply Theorem 2.3 to compute the Wiener index of $K_n \boxtimes C_m$ and $K_n \boxtimes P_m$. We have

$$\begin{aligned} W(K_n \boxtimes C_m) &= (|V(K_n)| + 2|E(K_n)|)W(C_m) + (|V(C_m)| + 2|E(C_m)|)W(K_n) \\ &\quad + |V(K_n)||V(C_m)|(|V(K_n)||V(C_m)| - |V(K_n)| - |V(C_m)| + 1) \\ &\quad - 2|E(K_n)||V(C_m)|(|V(C_m)| - 1) \\ &\quad - 2|E(C_m)|(|V(K_n)|^2 - |V(K_n)| - |E(K_n)|), \end{aligned}$$

on the other hand, by [18], $W(C_n) = \begin{cases} \frac{n^3}{8} & 2|n \\ \frac{n(n^2-1)}{8} & 2 \nmid n \end{cases}$ and $W(K_n) = \frac{n(n-1)}{2}$.

Using a tedious calculation, we have:

$$W(K_n \boxtimes C_m) = \begin{cases} \frac{1}{8}n^2m^3 + \frac{1}{2}n^2m - \frac{1}{2}nm & 2|m, \\ \frac{1}{8}n^2m^3 + \frac{3}{8}n^2m - \frac{1}{2}nm & 2 \nmid m. \end{cases} \quad (1)$$

Also, by [18], $W(P_n) = \frac{n(n^2-1)}{6}$, then

$$\begin{aligned} W(K_n \boxtimes P_m) &= (|V(K_n)| + 2|E(K_n)|)W(P_m) + (|V(P_m)| + 2|E(P_m)|)W(K_n) \\ &\quad + |V(K_n)||V(P_m)|(|V(K_n)||V(P_m)| - |V(K_n)| - |V(P_m)| + 1) \\ &\quad - 2|E(K_n)||V(P_m)|(|V(P_m)| - 1) \\ &\quad - 2|E(P_m)|(|V(K_n)|^2 - |V(K_n)| - |E(K_n)|) \\ &= \frac{1}{6}n^2m^3 + \frac{1}{3}n^2m - \frac{1}{2}nm. \end{aligned} \quad (2)$$

By replacing n with 2 in the relations (1) and (2), we obtain W of open and closed fences, as follow:

$$W(K_2 \boxtimes C_m) = \begin{cases} \frac{1}{2}m^3 + 2m - m & 2|m, \\ \frac{1}{2}m^3 + \frac{3}{2}m - m & 2 \nmid m, \end{cases}$$

$$W(K_2 \boxtimes P_m) = \frac{2}{3}m^3 + \frac{4}{3}m - m.$$

Theorem 2.4. [20] *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} W(G \boxtimes H) &\leq (|V(G)| + 2|E(G)|)W(H) + (|V(H)| + 2|E(H)|)W(G) \\ &+ D \left[\frac{|V(G)||V(H)|}{2} (|V(G)||V(H)| - |V(G)| - |V(H)| + 1) \right. \\ &- 2|E(H)| \binom{|V(G)|}{2} - 2|E(G)| \binom{|V(H)|}{2} \left. \right] \\ &+ 2|E(G)||E(H)|(D-1), \end{aligned}$$

where $D = \max\{D(G), D(H)\}$. Moreover, the upper bound is attained if and only if $D \leq 2$, or $G \cong K_n$, or $H \cong K_n$.

Theorem 2.5. [20] *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} WW(G \boxtimes H) &\geq (|V(G)| + 2|E(G)|)WW(H) + (|V(H)| + 2|E(H)|)WW(G) \\ &+ \frac{3}{2}|V(G)||V(H)|(|V(G)||V(H)| - |V(G)| - |V(H)| + 1) \\ &- 3|E(G)||V(H)|(|V(H)| - 1) \\ &- 3|E(H)|(|V(G)|^2 - |V(G)| - \frac{4}{3}|E(G)|), \end{aligned}$$

with equality if and only if $\max\{D(G), D(H)\} \leq 2$, or $G \cong K_n$, or $H \cong K_n$.

We apply Theorem 2.5 to compute the hyper-Wiener index of $K_n \boxtimes C_m$ and $K_n \boxtimes P_m$. We have:

$$\begin{aligned} WW(K_n \boxtimes C_m) &= (|V(K_n)| + 2|E(K_n)|)WW(C_m) \\ &+ (|V(C_m)| + 2|E(C_m)|)WW(K_n) \\ &+ \frac{3}{2}|V(K_n)||V(C_m)|(|V(K_n)||V(C_m)| - |V(K_n)| - |V(C_m)| + 1) \\ &- 3|E(K_n)||V(C_m)|(|V(C_m)| - 1) \\ &- 3|E(C_m)|(|V(K_n)|^2 - |V(K_n)| - \frac{4}{3}|E(K_n)|). \end{aligned}$$

On the other hand, by [14],

$$WW(C_n) = \begin{cases} \frac{n^2(n+1)(n+2)}{48} & 2|n, \\ \frac{n(n^2-1)(n+3)}{48} & 2 \nmid n. \end{cases}$$

Using a tedious calculation, we have:

$$WW(K_n \boxtimes C_m) = \begin{cases} \frac{1}{48}n^2m^2(m^2 + 3m + 2) + \frac{1}{2}mn(n-1) & 2|m, \\ \frac{1}{48}n^2m^2(m^2 + 3m - 1) + \frac{1}{2}mn(\frac{7}{8}n - 1) & 2 \nmid m. \end{cases} \quad (3)$$

Also, by [14], $WW(P_n) = \frac{1}{24}(n^4 + 2n^3 - n^2 - 2n)$, then

$$\begin{aligned}
WW(K_n \boxtimes P_m) &= (|V(K_n)| + 2|E(K_n)|)WW(P_m) \\
&\quad + (|V(P_m)| + 2|E(P_m)|)WW(K_n) \\
&\quad + \frac{3}{2}|V(K_n)||V(P_m)|(|V(K_n)||V(P_m)| - |V(K_n)| - |V(P_m)| + 1) \\
&\quad - 3|E(K_n)||V(P_m)|(|V(P_m)| - 1) \\
&\quad - 3|E(P_m)|(|V(K_n)|^2 - |V(K_n)| - \frac{4}{3}|E(K_n)|) \\
&= \frac{1}{24}m^2n^2(m^2 + 2m - 1) + \frac{1}{2}mn(\frac{5}{6}n - 1). \tag{4}
\end{aligned}$$

If $n = 2$ in the relations (3) and (4), we have the hyper-Wiener index of open and closed fences, as follow:

$$\begin{aligned}
WW(K_2 \boxtimes C_m) &= \begin{cases} \frac{1}{12}m^2(m^2 + 3m + 2) + m & 2|m, \\ \frac{1}{12}m^2(m^2 + 3m - 1) + \frac{3}{4}m & 2 \nmid m, \end{cases} \\
WW(K_2 \boxtimes P_m) &= \frac{1}{6}m^2(m^2 + 2m - 1) + \frac{2}{3}m.
\end{aligned}$$

Theorem 2.6. [20] *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned}
WW(G \boxtimes H) &\leq (|V(G)| + 2|E(G)|)WW(H) + (|V(H)| + 2|E(H)|)WW(G) \\
&\quad + \frac{1}{2}D(D+1) \left[\frac{|V(G)||V(H)|}{2} (|V(G)||V(H)| \right. \\
&\quad \left. - |V(G)| - |V(H)| + 1) \right. \\
&\quad \left. - 2|E(H)| \binom{|V(G)|}{2} - 2|E(G)| \binom{|V(H)|}{2} \right] \\
&\quad + |E(G)||E(H)|(D^2 + D - 2),
\end{aligned}$$

where $D = \max\{D(G), D(H)\}$. Moreover, the upper bound is attained if and only if $D \leq 2$, or $G \cong K_n$, or $H \cong K_n$.

Theorem 2.7. [20] *For graphs G and H , we have*

$$\begin{aligned}
M_1(G \boxtimes H) &= (|V(H)| + 4|E(H)|)M_1(G) + (|V(G)| + 4|E(G)|)M_1(H) \\
&\quad + M_1(G)M_1(H) + 8|E(G)||E(H)|.
\end{aligned}$$

By the previous theorem, we have

$$M_1(P_n \boxtimes C_m) = (|V(C_m)| + 4|E(C_m)|)M_1(P_n) + (|V(P_n)| + 4|E(P_n)|)M_1(C_m) \\ + M_1(P_n)M_1(C_m) + 8|E(P_n)||E(C_m)| = 64mn - 78m, \quad (5)$$

$$M_1(P_n \boxtimes P_m) = 64mn - 78n - 78m + 92, \quad (6)$$

$$M_1(C_n \boxtimes C_m) = 64nm.$$

If $n = 2$ in the relations (5) and (6), we have M_1 of open and closed fences, as follow:

$$M_1(P_2 \boxtimes C_m) = 128m - 78m = 50m,$$

$$M_1(P_2 \boxtimes P_m) = 50m - 64.$$

Consider Q_n on $n \geq 1$, then

$$M_1(Q_n) = \sum_{u \in V(Q_n)} \deg(u)^2 = n^2 \sum_{u \in V(Q_n)} 1 = n^2 2^n,$$

$$M_2(Q_n) = \sum_{uv \in E(Q_n)} \deg(u)\deg(v) = n^2 \sum_{uv \in E(Q_n)} 1 = n^3 2^{n-1}.$$

Therefore,

$$M_1(Q_n \boxtimes P_m) = (|V(P_m)| + 4|E(P_m)|)M_1(Q_n) + (|V(Q_n)| + 4|E(Q_n)|)M_1(P_m) \\ + M_1(Q_n)M_1(P_m) + 8|E(Q_n)||E(P_m)| \\ = 2^n(9n^2m - 10n^2 + 4m - 6 + 12nm - 16n),$$

$$M_1(Q_n \boxtimes C_m) = (|V(C_m)| + 4|E(C_m)|)M_1(Q_n) + (|V(Q_n)| + 4|E(Q_n)|)M_1(C_m) \\ + M_1(Q_n)M_1(C_m) + 8|E(Q_n)||E(C_m)| \\ = m2^n(9n^2 + 12n + 4).$$

Theorem 2.8. [20] *For the graphs G and H , we have*

$$M_2(G \boxtimes H) = 3|E(H)|M_1(G) + 3|E(G)|M_1(H) \\ + 3M_1(G)M_1(H) + 2M_2(G)M_2(H) \\ + (6|E(H)| + 3M_1(H) + |V(H)|)M_2(G) \\ + (6|E(G)| + 3M_1(G) + |V(G)|)M_2(H).$$

By the previous theorem,

$$M_2(P_n \boxtimes C_m) = 3|E(C_m)|M_1(P_n) + 3|E(P_n)|M_1(C_m) \\ + 3M_1(P_n)M_1(C_m) + 2M_2(P_n)M_2(C_m) \\ + (6|E(C_m)| + 3M_1(C_m) + |V(C_m)|)M_2(P_n) \\ + (6|E(P_n)| + 3M_1(P_n) + |V(P_n)|)M_2(C_m) \\ = 256mn - 414m, (m > 2).$$

Consider Q_n on $n \geq 1$, then

$$\begin{aligned}
M_2(Q_n \boxtimes P_m) &= 3|E(P_m)|M_1(Q_n) + 3|E(Q_n)|M_1(P_m) \\
&\quad + 3M_1(Q_n)M_1(P_m) + 2M_2(Q_n)M_2(P_m) \\
&\quad + (6|E(P_m)| + 3M_1(P_m) + |V(P_m)|)M_2(Q_n) \\
&\quad + (6|E(Q_n)| + 3M_1(Q_n) + |V(Q_n)|)M_2(P_m) \\
&= 2^n(27mn^2 - 45n^2 + 18mn - 33n + \frac{27}{2}n^3m - 20n^3 + 4m - 8),
\end{aligned} \tag{7}$$

$$\begin{aligned}
M_2(Q_n \boxtimes C_m) &= 3|E(C_m)|M_1(Q_n) + 3|E(Q_n)|M_1(C_m) \\
&\quad + 3M_1(Q_n)M_1(C_m) + 2M_2(Q_n)M_2(C_m) \\
&\quad + (6|E(C_m)| + 3M_1(C_m) + |V(C_m)|)M_2(Q_n) \\
&\quad + (6|E(Q_n)| + 3M_1(Q_n) + |V(Q_n)|)M_2(C_m) \\
&= m2^n(\frac{27}{2}n^3 + 27n^2 + 18n + 4).
\end{aligned} \tag{8}$$

By replacing Q_n with Q_1 in the relations (7) and (8), we have M_2 of open and closed fences, as follow:

$$\begin{aligned}
M_2(Q_1 \boxtimes P_m) &= M_2(K_2 \boxtimes P_m) = 2(21m - 36 + 24m - 42 \\
&\quad + \frac{27}{2}m - 20 + 4m - 8) = 125m - 212, \\
M_2(Q_1 \boxtimes C_m) &= M_2(K_2 \boxtimes C_m) = 2m(\frac{27}{2} + 27 + 18 + 4) = 125m.
\end{aligned}$$

A connected graph is called a **self-centered** graph if all of its vertices have the same eccentricity. Then a connected graph G is self-centered if and only if $r(G) = D(G)$.

Theorem 2.9. [20] *Let G and H be self-centered graphs that $D(H) \leq D(G)$. Then*

$$\xi^c(G \boxtimes H) = 2r(G)(|E(G)||V(H)| + |E(H)||V(G)| + 2|E(G)||E(H)|).$$

One can see that $r(C_n) = \lceil \frac{n}{2} \rceil$. So, if $n \geq m$, then

$$\begin{aligned}
\xi^c(C_n \boxtimes C_m) &= 2r(C_n)(|E(C_n)||V(C_m)| + |E(C_m)||V(C_n)| + 2|E(C_n)||E(C_m)|) \\
&= 8nm\lceil \frac{n}{2} \rceil.
\end{aligned}$$

Clearly, $r(Q_n) = n$, $|E(Q_n)| = n2^{n-1}$. Therefore,

$$\begin{aligned}\xi^c(Q_n \boxtimes C_m) &= 2r(Q_n)(|E(Q_n)||V(C_m)| + |E(C_m)||V(Q_n)| + 2|E(Q_n)||E(C_m)|) \\ &= n2^n(3mn + 2m) \quad \text{if } n \geq \lfloor \frac{m}{2} \rfloor,\end{aligned}\quad (9)$$

$$\begin{aligned}\xi^c(Q_n \boxtimes C_m) &= 2r(C_m)(|E(Q_n)||V(C_m)| + |E(C_m)||V(Q_n)| + 2|E(Q_n)||E(C_m)|) \\ &= \lfloor \frac{m}{2} \rfloor 2^n(3mn + 2m) \quad \text{if } n \leq \lfloor \frac{m}{2} \rfloor.\end{aligned}\quad (10)$$

By replacing n with 1 in the relations (9) and (10), we have ξ^c of closed fence, as follow:

$$\xi^c(Q_1 \boxtimes C_m) = \xi^c(K_2 \boxtimes C_m) = \begin{cases} 10m & \text{if } \lfloor \frac{m}{2} \rfloor \leq 1, \\ 10m \lfloor \frac{m}{2} \rfloor & \text{if } \lfloor \frac{m}{2} \rfloor \geq 1. \end{cases}$$

3. Open Problem

We found the exact value of $\xi^c(G \boxtimes H)$, where G and H are self-centered graphs. A natural question arises here is if G and H are arbitrary graphs, then what is the value of $\xi^c(G \boxtimes H)$. If someone can find the answer, can calculate values of $\xi^c(Q_n \boxtimes P_m)$, $\xi^c(P_n \boxtimes P_m)$ and $\xi^c(P_n \boxtimes C_m)$ as a result.

Acknowledgement. The author would like to thank reviewer for careful reading and suggestions, which improved the manuscript.

Conflicts of Interest. The authors declare that they have no conflicts of interest.

References

- [1] A. R. Ashrafi, T. Došlić, A. Hamzeh, Extremal graphs with respect to the Zagreb coindices, *MATCH Commun. Math. Comput. Chem.* **65**(1) (2011) 85–92.
- [2] A. R. Ashrafi, T. Došlić, M. Saheli, The eccentric connectivity index of $TUC_4C_8(R)$ nanotubes, *MATCH Commun. Math. Comput. Chem.* **65**(1) (2011) 221–230.
- [3] A. R. Ashrafi, M. Saheli, M. Ghorbani, The eccentric connectivity index of nanotubes and nanotori, *J. Comput. Appl. Math.* **235** (2011) 4561–4566.
- [4] G. G. Cash, Polynomial expressions for the hyper-Wiener index of extended hydrocarbon networks, *Comput. Chem.* **25** (2001) 577–582.
- [5] G. G. Cash, Relationship between the Hosoya polynomial and the hyper-Wiener index, *Appl. Math. Lett.* **15** (2002) 893–895.

- [6] A. A. Dobrymin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.* **66** (2001) 211–249.
- [7] A. A. Dobrymin, I. Gutman, S. Klavšar, P. Žigert, Wiener index of hexagonal systems, *Acta Appl. Math.* **72** (2002) 247–294.
- [8] S. Gupta, M. Singh, A. K. Madan, Eccentric distance sum: a novel graph invariant for predicting biological and physical properties, *J. Math. Anal. Appl.* **275** (2002) 386–401.
- [9] I. Gutman, Relation between hyper-Wiener and Wiener index, *Chem. Phys. Lett.* **364** (2002) 352–356.
- [10] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total φ -electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
- [11] R. Hammack, W. Imrich, S. Klavšar, *Handbook of Product Graphs*, Second edition, CRC Press, Boca Raton, FL, (2011).
- [12] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, *Bull. Chem. Soc. Jpn.* **44** (1971) 2332–2339.
- [13] A. Ilić, I. Gutman, Eccentric connectivity index of chemical trees, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 731–744.
- [14] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.* **56** (2008) 1402–1407.
- [15] S. Klavšar, P. Žigert, I. Gutman, An algorithm for the calculation of the hyper-Wiener index of benzenoid hydrocarbons, *Comput. Chem.* **24** (2000) 229–233.
- [16] D. J. Klein, I. Lukovits, I. Gutman, On the definition of the hyper-Wiener index for cycle-containing structures, *J. Chem. Inf. Comput. Sci.* **35** (1995) 50–52.
- [17] M. J. Morgan, S. Mukwembi, H. C. Swart, On the eccentric connectivity index of a graph, *Discrete Math.* **311** (2011) 1229–1234.
- [18] B. E. Sagan, Y. -N. Yeh, P. Zhang, The Wiener polynomial of a graph, *Int. J. Quant. Chem.* **60**(5) (1996) 959–969.
- [19] V. Sharma, R. Goswami, A. K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure property and structure activity studies, *J. Chem. Inf. Comput. Sci.* **37** (1997) 273–282.

-
- [20] M. Tavakoli, F. Rahbarnia, A. R. Ashrafi, Note on strong product of graphs, *Kragujevac J. Math.* **37**(1) (2013) 187–193.
- [21] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.
- [22] G. Yu, L. Feng, A. Ilić, On the eccentric distance sum of trees and unicyclic graphs, *J. Math. Anal. Appl.* **375** (2011) 99–107.
- [23] B. Zhou, Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **52** (2004) 113–118.
- [24] B. Zhou, Z. Du, On eccentric connectivity index, *MATCH Commun. Math. Comput. Chem.* **63** (2010) 181–198.
- [25] B. Zhou, I. Gutman, Relations between Wiener, hyper-Wiener and Zagreb indices, *Chem. Phys. Lett.* **394** (2004) 93–95.

Mostafa Tavakoli
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R. Iran
E-mail: m_tavakoli@um.ac.ir

Freydoon Rahbarnia
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R. Iran
E-mail: rahbarnia@um.ac.ir

Irandozht Rezaee Abdolhosseinzadeh
Department of Applied Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, I. R. Iran
E-mail: ir_rezaee899@stu.um.ac.ir