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On Edge-Decomposition of Cubic Graphs into Copies of the Double-Star with Four Edges

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Abstract

A tree containing exactly two non-pendant vertices is called a double-star. Let k_1 and k_2 be two positive integers. The double-star with degree sequence $(k_1+1, k_2+1, 1, \ldots, 1)$ is denoted by S_{k_1,k_2}. It is known that a cubic graph has an $S_{1,1}$-decomposition if and only if it contains a perfect matching. In this paper, we study the $S_{1,2}$-decomposition of cubic graphs. We present some necessary and some sufficient conditions for the existence of an $S_{1,2}$-decomposition in cubic graphs.

Keywords: Edge-decomposition, double-star, cubic graph, regular graph, bipartite graph.

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1. Introduction

Let G be a graph and $V(G)$, $E(G)$ denote the vertex set and the edge set of G, respectively. Suppose that $v \in V(G)$. We denote the set of neighbors of v by $N(v)$ and for $X \subseteq V(G)$, $N(X) = \cup_{x \in X} N(x)$. Also, for $S \subseteq V(G)$, let $N_S(X) = N(X) \cap S$. We denote $|N(v)|$ and $|N_S(v)|$ by $d(v)$ and $d_S(v)$, respectively.

An *independent set* is a set of vertices in a graph in which no two vertices are adjacent. The *independence number*, $\alpha(G)$, is the size of a largest independent set in G. A *dominating set* of G is a subset D such that every vertex not in D is

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adjacent to at least one vertex in D. The *domination number*, $\gamma(G)$, is the size of a smallest dominating set in G.

A subset $M \subseteq E(G)$ is called a *matching*, if no two edges of M are incident. A matching M is called a *perfect matching*, if every vertex of G is incident with some edge in M. Hall proved that a bipartite graph $G = (A, B)$ has a matching which covers every vertex in A if and only if for every $S \subseteq A$ we have $|N_B(S)| \geq |S|$, see [3, Theorem 16.4].

Let $A \subseteq V(G)$. Then the induced subgraph of G with vertex set A is denoted by $G[A]$. Given a graph H, the graph G is called *H-free*, if it contains no induced subgraph isomorphic to H. A graph G has an *H-decomposition*, if edges of G can be decomposed into subgraphs isomorphic to H. If G has an H-decomposition, then we say that G is *H-decomposable*. A tree with exactly two non-pendant vertices is called a *double-star*. Let k_1 and k_2 be two positive integers. The double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, \ldots, 1)$ is denoted by S_{k_1,k_2}. A vertex of degree i is called an *i-vertex*. If G is an r-regular graph with an $S_{1,r-1}$-decomposition and $S \subseteq V(G)$ is the set of r-vertices of this decomposition, then we say that G is *$(S_{1,r-1}, S)$-decomposable*.

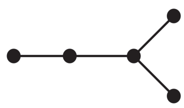

Figure 1: $S_{1,2}$.

Tree decompositions of highly connected graphs have been extensively studied by several authors, see [1], [5] and [6]. Barát and Gerbner [1] showed that every 191-edge-connected graph, whose size is divisible by 4 has an $S_{1,2}$-decomposition. Recently, Bensmail et al. [2] claimed that they have proved Barát-Thomassen conjecture.

In this paper, we study the double-star decomposition of cubic graphs. Let G be a cubic graph. If G is S-decomposable and S is a double-star, then S is isomorphic to $S_{1,1}$, $S_{1,2}$ or $S_{2,2}$, otherwise S has a vertex of degree at least four. Kötzig [4] proved that a cubic graph has an $S_{1,1}$-decomposition if and only if it contains a perfect matching. We study the edge-decomposition of cubic graphs into copies of $S_{1,2}$. Also, we obtain some results on $S_{1,r-1}$-decomposition of r-regular graphs.

2. Results

Let G be an r-regular graph and $S \subseteq V(G)$. The question is whether G is $(S_{1,r-1}, S)$-decomposable or not? For giving a response to this question, we use a new bipartite graph $H = (S, L)$, in which S is the set of r-vertices of $S_{1,r-1}$-trees and for each edge $e \in E(G \setminus S)$, we put a vertex u_e in L. Two vertices s_i and u_{e_j}

are adjacent in H if and only if we can obtain an $S_{1,r-1}$ by adding e_j to a $K_{1,r-1}$ containing s_i as a central vertex. We have the following remarks which follow by the fact that the vertices of degree r are not adjacent.

Remark 1. *Let G be an r-regular graph of order n. Then G is* $(S_{1,r-1}, S)$*-decomposable if and only if* $|S| = \frac{rn}{2(r+1)}$ *and* $H = (S, L)$ *has a perfect matching.*

Remark 2. *Let G be an r-regular graph of order n which is* $S_{1,r-1}$*-decomposable. Then* $\alpha(G) \geq \frac{rn}{2(r+1)}$.

In the following lemma we provide some necessary conditions for the existence of an $S_{1,2}$-decomposition in cubic graphs.

Lemma 2.1. *Let G be a cubic graph of order n which has an* $S_{1,2}$*-decomposition. Then there exists an independent set* $S \subset V(G)$ *of size* $\frac{3n}{8}$ *such that:*

1. *Each component of* $G \setminus S$ *is either a cycle or a tree,*
2. *No component of* $G \setminus S$ *has two* 3*-vertices.*

Proof. Let $S \subseteq V(G)$ be the set of 3-vertices in an $S_{1,2}$-decomposition and $H = (S, L)$ be the bipartite graph defined before Remark 1. Also, suppose that F is a given component of $G \setminus S$. If F is neither a tree nor a cycle, then it contains a cycle $C : v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ and an edge $e = v_i w$, where $1 \leq i \leq t$ and $w \in V(F)$. Let $A = E(C)$, then $|N_H(A)| \leq |A| - 1$. Now, by Hall's theorem, H has no perfect matching. Then by Remark 1, G has no $S_{1,2}$-decomposition, a contradiction.

If there exist two 3-vertices u and v in some component F, then there exists a (u, v)-path P in F. Now, let $A = E(P)$. Then $|N_H(A)| \leq |A| - 1$, a contradiction. □

These necessary conditions are not sufficient. Some examples are given in Figure 2.

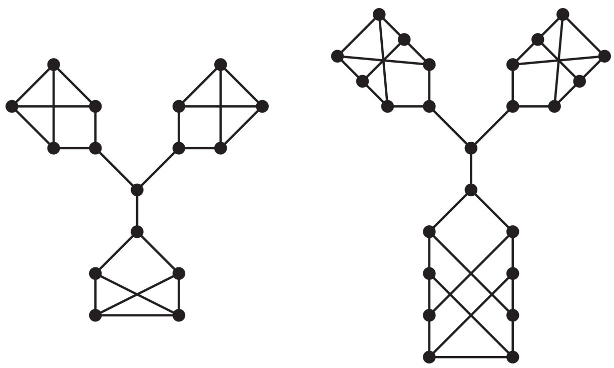

Figure 2: Cubic graphs with no $S_{1,2}$-decomposition.

By Lemma 2.1, if G is an $S_{1,2}$-decomposable cubic graph, then $\alpha(G) \geq \frac{3n}{8}$. We consider the case that $\alpha(G) = \frac{3n}{8}$ and find two sufficient conditions for the existence of an $S_{1,2}$-decomposition in this case.

First, we prove the following theorem. Note that if $\alpha(G) = \frac{3n}{8}$ and S is an independent set with $|S| = \frac{3n}{8}$, then each components of $G \setminus S$ is a cycle, path or an isolated vertex. In the following we consider the case in which each component is either a cycle or an isolated vertex.

Theorem 2.2. *Let G be a cubic graph of order n with $\alpha(G) = \frac{3n}{8}$. Suppose that there exists an independent set $S \subseteq V(G)$ such that $|S| = \frac{3n}{8}$ and each component of $G \setminus S$ is a cycle or an isolated vertex. If no vertex of S is contained in a triangle, then G is $(S_{1,2}, S)$-decomposable.*

Proof. Suppose that $C_1, \ldots, C_t$ are cycle components of $G \setminus S$ and let $W = V(C_1) \bigcup \ldots \bigcup V(C_t)$. We claim that $N_S(\{u\}) \neq N_S(\{v\})$, for every two vertices $u, v \in W$. If u and v are adjacent, then since no vertex of S is contained in a triangle, we are done. So we may assume that u and v are not adjacent. If $N_S(\{u\}) = N_S(\{v\}) = \{s\}$, then $S' = (S \setminus \{s\}) \cup \{u, v\}$ is an independent set and $|S'| > \frac{3n}{8}$, a contradiction.

This implies that every $w \in W$ is adjacent to one vertex in S and no vertex of S is adjacent to two vertices in W. Hence, $H = (S, L)$ defined before Remark 1 is a 2-regular graph and has a perfect matching. This yields that G is $(S_{1,2}, S)$-decomposable. □

Now, we prove the following theorem.

Theorem 2.3. *Let G be a cubic graph of order n with $\alpha(G) = \frac{3n}{8}$ and S be an independent set of size $\frac{3n}{8}$. Let I be the set of isolated vertices in $G \setminus S$. If there exists no cycle of length 3, 5 or 7 in $G[V(G) \setminus I]$ which contains a vertex of S, then G has a $S_{1,2}$-decomposition.*

Proof. First note that there exists a graph with the conditions of this theorem, see Figure 3. We divide the proof into four claims.

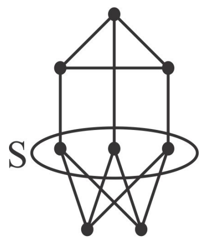

Figure 3: A graph with the conditions of Theorem 2.3.

Claim 1. Each component of $G \setminus S$ is a path or a cycle.

If there exists a vertex of degree 3 in $G \setminus S$, then by adding this vertex to S we obtain an independent set S' such that $|S'| = \frac{3n}{8} + 1$, a contradiction.

Claim 2. Let $u, v \in V(G \setminus S)$ be two 2-vertices in $G \setminus S$. Then $N_S(\{u\}) \neq N_S(\{v\})$.

Let u and v be two vertices in $G \setminus S$ such that $d_{G \setminus S}(u) = d_{G \setminus S}(v) = 2$ and $N_S(\{u\}) = N_S(\{v\}) = \{s\}$. If u and v are adjacent, then s is contained in a triangle in $G[V(G) \setminus I]$, a contradiction. Also, if u and v are not adjacent, then $S' = (S \setminus \{s\}) \cup \{u, v\}$ is an independent set and $|S'| = \frac{3n}{8} + 1$, a contradiction. So, the claim is proved.

Now, we check the condition of Hall's theorem for the edges of $G \setminus S$. Suppose that $L = \{e_1, \ldots, e_l\} \subseteq E(G \setminus S)$. Let $P_1, \ldots, P_k$ be path components of $G \setminus S$. Let $\langle L \rangle$ be an induced subgraph with edge set L. Now, we consider two cases:

Case 1. No P_i is contained in $\langle L \rangle$.

Note that for each edge $e \in L$, one of its end points has degree two in $G \setminus S$. Because if both end points are of degree one in $G \setminus S$, then the induced subgraph on this edge is a path component of $G \setminus S$. Now, we show that for each edge $e_i \in L$, one can find $v_{e_i} \in V(G \setminus S)$ such that $d_{G \setminus S}(v_{e_i}) = 2$, v_{e_i} is an end point of e_i and if $i \neq j$, then $v_{e_i} \neq v_{e_j}$.

Suppose that $P_i = v_1 v_2 \ldots v_t$ has some edge in $\langle L \rangle$, where $1 \leq i \leq k$. Let $e_{i_1}, e_{i_2}, \ldots, e_{i_t}$ be edges of P_i contained in $\langle L \rangle$ such that $i_1 < i_2 < \ldots < i_t$ and $e_{i_j} = v_{i_j} v_{i_j+1}$. If $e_1 \notin \langle L \rangle$, then let $v_{e_{i_j}} = v_{i_j - 1}$ and we are done. Also, if $e_{t-1} \notin \langle L \rangle$, then let $v_{e_{i_j}} = v_{i_j+1}$. So, we may assume that $e_1, e_{t-1} \in \langle L \rangle$. Since P_i is not contained in $\langle L \rangle$, there exists $1 < l < t-1$, such that $e_l \notin \langle L \rangle$. Now, if $i_j < l$, let $v_{e_{i_j}} = v_{i_j+1}$ and if $i_j > l$, let $v_{e_{i_j}} = v_{i_j - 1}$. By repeating this procedure for $P_1, P_2, \ldots, P_k$, we are done.

Now, Claim 2 implies that for each $e \in L$ there exists a distinct vertex in S which is adjacent to v_e and so in this case, the condition of Hall's theorem holds.

Case 2. There exist $i_1, \ldots, i_t$ such that $1 \leq i_1, \ldots, i_t \leq k$ and $P_{i_1}, \ldots, P_{i_t}$ are path components of $G \setminus S$ contained in $\langle L \rangle$. Let $W_j \subseteq V(G \setminus S)$ be the set of vertices of degree j in $G \setminus S$, for $j = 0, 1, 2$. We have the following.

Claim 3. Let $v \in V(G \setminus S)$ such that $d_{G \setminus S}(v) = 1$ and $N_S(\{v\}) = \{x, y\}$. Then $N_{G \setminus S}(\{x\}) \bigcap W_2 = \varnothing$ or $N_{G \setminus S}(\{y\}) \bigcap W_2 = \varnothing$.

By contrary, suppose that x and y are adjacent to v_x and v_y in $G \setminus S$, respectively, and $d_{G \setminus S}(v_x) = d_{G \setminus S}(v_y) = 2$. Note that v is not adjacent to v_x and v_y, since otherwise there exists a triangle containing v, a contradiction. Now, if v_x and v_y are adjacent, then $C : v, x, v_x, v_y, y, v$ is a cycle of length 5 containing v, a contradiction. These imply that $\{v, v_x, v_y\}$ is an independent set. Now, let $S' = (S \setminus \{x, y\}) \cup \{v, v_x, v_y\}$. It can be easily seen that S' is an independent set and $|S'| = \frac{3n}{8} + 1$, a contradiction.

Now, we can prove that in the second case, L satisfies the condition of Hall's theorem. It suffices to show that the edges of $P_{i_1}, \ldots, P_{i_t}$ satisfy Hall's condition.

Because, similar to the proof of the first case, one can see that other edges have distinct neighbors in S and we are done. Now, Claim 2 implies that we can find $\sum_{j=1}^{t}(|E(P_{i_j})|-1)$ vertices in S which are adjacent to the vertices of degree two in $P_{i_1}, P_{i_2}, \ldots, P_{i_t}$. Let $T \subseteq S$ be the set of vertices in S which are adjacent to the end vertices of $P_{i_1} \ldots, P_{i_t}$ and to no vertex of degree two in $G \setminus S$. It suffices to show that $|T| \geq t$.

By contrary, suppose that $|T| \leq t-1$. Then Claim 3 implies that each end vertex of the paths has a neighbor in T. Let A be the set of end vertices of paths that have one neighbor in T and let B be the set of end vertices which have two neighbors in T. We have the following.

$$|A| + |B| = 2t \ , \ |A| + 2|B| \leq 3t - 3.$$

Hence, we conclude that $|A| \geq t + 3$. Now, we prove the following claim.

Claim 4. If $u, v \in A$, then $N_T(\{u\}) \cap N_T(\{v\}) = \varnothing$.

First, note that if u and v are adjacent, then we are done. So, we may assume that u and v are not adjacent. Let $N_T(\{u\}) = N_T(\{v\}) = \{w\}$. Suppose that $N_S(\{u\}) = \{w, x\}$ and $N_S(\{v\}) = \{w, y\}$. By the definition of T, we conclude that x and y are adjacent to some vertices of degree 2 in $G \setminus S$, say v_x and v_y, respectively. First, suppose that $x = y$. If $\{u, v, v_x\}$ is not independent set, then one can find a triangle contains a vertex of S, a contradiction, see Figure 4. Thus, $\{u, v, v_x\}$ is an independent set. Now, $S' = (S \setminus \{w, x\}) \cup \{u, v, v_x\}$ is an independent set of size $\frac{3n}{8} + 1$, a contradiction.

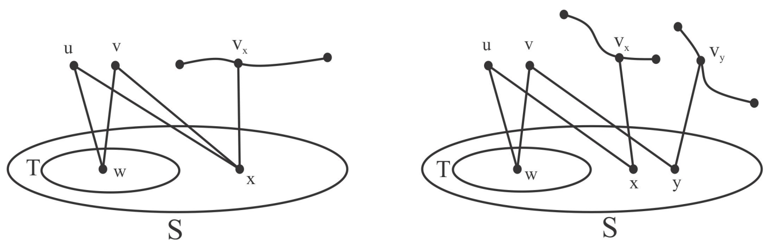

Figure 4: The case $x = y$ (left side) and the case $x \neq y$ (right side).

So, we may assume that $x \neq y$. We show that $\{u, v, v_x, v_y\}$ is an independent set.

Since no vertex of S is contained in a triangle in $G[V(G) \setminus I]$, we conclude that u and v_x are not adjacent (similarly, v and v_y are not adjacent). So, suppose that v and v_x are adjacent. Then $C : u, x, v_x, v, w, u$ is a cycle of length 5 in $G[V(G) \setminus I]$ which contains $w \in S$, a contradiction, see Figure 4. Also, note that v_x and v_y are not adjacent. Since, otherwise $C : u, x, v_x, v_y, y, v, w, u$ is a cycle of length 7 in $G[V(G) \setminus I]$ which contains $w \in S$, a contradiction, see Figure 4. These imply that $\{u, v, v_x, v_y\}$ is an independent set. Now, let $S' = (S \setminus \{x, y, w\}) \cup \{u, v, v_x, v_y\}$.

Then S' is an independent set and $|S'| > \frac{3n}{8}$, a contradiction and this completes the proof of the claim.

Now, Claim 4 implies that for every $v \in A$ we have a distinct neighbor $t_v \in T$ and this implies that $|T| \geq t + 3$, a contradiction. This completes the proof. $\square$

We have the following result in regular bipartite graphs which makes a connection between the domination number and the existence of a $S_{1,r-1}$-decomposition.

Theorem 2.4. *If $G = (A, B)$ is a bipartite r-regular graph of order n such that $r + 1|n$ and $\gamma(G) = \frac{n}{r+1}$, then G is $S_{1,r-1}$-decomposable.*

Proof. Let D be a dominating set of G of size $\frac{n}{r+1}$. Then vertices of D have no common neighbor in $V(G) \setminus D$. Now, let $D_A = D \cap A$ and $D_B = D \cap B$ and $|D_A| = a, |D_B| = b$. Since D is a dominating set of size $\frac{n}{r+1}$ we have:

$$a + b = \frac{n}{r+1} \ , \ ra + b = \frac{n}{2}.$$

Then $a = b = \frac{n}{2(r+1)}$. Now, let $S = N(D_A)$. We show that G has a $(S_{1,r-1}, S)$-decomposition. Clearly, $|S| = \frac{rn}{2(r+1)}$ and $E(G \setminus S)$ is exactly the set of edges between D_B and $N(D_B)$. Note that if $v \in N(D_B)$, then $d_S(v) = r - 1$. Now, it is not hard to see that the graph $H = (S, L)$, defined in Remark 1, is a $(r - 1)$-regular bipartite graph and hence it has a perfect matching. So, by Remark 1, G is $(S_{1,r-1}, S)$-decomposable. $\square$

In bipartite cubic graphs we can find a better result.

Theorem 2.5. *Let $G = (A, B)$ be a cubic bipartite graph of order n such that $8|n$. Then $\gamma(G) = \frac{n}{4}$ if and only if there exists $S \subseteq A$ of size $\frac{3n}{8}$ such that G is both $(S_{1,2}, S)$-decomposable and $(S_{1,2}, N(A \setminus S))$-decomposable.*

Proof. Note that if $\gamma(G) = \frac{n}{4}$, then by Theorem 2.4, we are done.

For converse, first notice that if G is a cubic graph, then $\gamma(G) \geq \frac{n}{4}$. Suppose that there exists $S \subseteq A$ such that G is both $(S_{1,2}, S)$-decomposable and $(S_{1,2}, N(A \setminus S))$-decomposable. Note that each vertex in $A \setminus S$ is a 3-vertex in $G \setminus S$. Now, Lemma 2.1 implies that each of them is in a different component of $G \setminus S$ and so they have no common neighbors. By a similar method, one can show that the vertices of $B \setminus N(A \setminus S)$ have no common neighbors. Now, $D = (A \setminus S) \cup (B \setminus N(A \setminus S))$ is a dominating set of size $\frac{n}{4}$ and this completes the proof. $\square$

Now, we provide another sufficient condition for the existence of an $S_{1,2}$-decomposition in bipartite cubic graphs.

Theorem 2.6. *Let $G = (A, B)$ be a bipartite cubic graph of order n and $S \subseteq A$ be of size $\frac{3n}{8}$. Then G is $(S_{1,2}, S)$-decomposable if and only if there exists a perfect matching between S and $N(A \setminus S)$.*

Proof. First, suppose that G is $(S_{1,2}, S)$-decomposable. Then the second part of Lemma 2.1 indicates that no component of $G \setminus S$ has two 3-vertices. This implies that no two vertices of $A \setminus S$ have a common neighbor in B. So, $|N(A \setminus S)| = \frac{3n}{8}$. Now, note that if Hall's condition does not hold for S and $N(A \setminus S)$, then Hall's condition does not hold in $H = (S, L)$, too. This is a contradiction and this completes the proof.

Now, suppose that there exists a perfect matching between S and $N(A \setminus S)$. Then $|N(A \setminus S)| = \frac{3n}{8}$ which implies that no two vertices of $A \setminus S$ have a common neighbor in B. For each vertex $v \in N(A \setminus S)$, there exists a unique edge $e_v \in E(G \setminus S)$ in which v is one of its end points. Let $M = \{(u_i, v_i) | \; i = 1, 2, \dots, \frac{3n}{8}\}$ be a matching between S and $N(A \setminus S)$. Then by adding the edge e_{v_i} to a claw containing u_i as a 3-vertex, one can obtain an $S_{1,2}$-decomposition. □

We close the paper with the following questions.

Question 1. *Does there exist a triangle-free 2-connected cubic graph of order divisible by 8 which has no $S_{1,2}$-decomposition?*

Question 2. *Is it true that every bipartite cubic graph of order divisible by 8 is $S_{1,2}$-decomposable?*

Conflicts of Interest. The author declares that there is no conflicts of interest regarding the publication of this article.

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