On Edge-Decomposition of Cubic Graphs into Copies of the Double-Star with Four Edges

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Abstract

A tree containing exactly two non-pendant vertices is called a double-star. Let $k_1$ and $k_2$ be two positive integers. The double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, \ldots, 1)$ is denoted by $S_{k_1,k_2}$. It is known that a cubic graph has an $S_{1,1}$-decomposition if and only if it contains a perfect matching. In this paper, we study the $S_{1,2}$-decomposition of cubic graphs. We present some necessary and some sufficient conditions for the existence of an $S_{1,2}$-decomposition in cubic graphs.

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1. Introduction

Let $G$ be a graph and $V(G)$, $E(G)$ denote the vertex set and the edge set of $G$, respectively. Suppose that $v \in V(G)$. We denote the set of neighbors of $v$ by $N(v)$ and for $X \subseteq V(G)$, $N(X) = \bigcup_{x \in X} N(x)$. Also, for $S \subseteq V(G)$, let $N_S(X) = N(X) \cap S$. We denote $|N(v)|$ and $|N_S(v)|$ by $d(v)$ and $d_S(v)$, respectively.

An independent set is a set of vertices in a graph in which no two vertices are adjacent. The independence number, $\alpha(G)$, is the size of a largest independent set in $G$. A dominating set of $G$ is a subset $D$ such that every vertex not in $D$ is
adjacent to at least one vertex in $D$. The domination number, $\gamma(G)$, is the size of a smallest dominating set in $G$.

A subset $M \subseteq E(G)$ is called a matching, if no two edges of $M$ are incident. A matching $M$ is called a perfect matching, if every vertex of $G$ is incident with some edge in $M$. Hall proved that a bipartite graph $G = (A, B)$ has a matching which covers every vertex in $A$ if and only if for every $S \subseteq A$ we have $|N_B(S)| \geq |S|$, see [3, Theorem 16.4].

Let $A \subseteq V(G)$. Then the induced subgraph of $G$ with vertex set $A$ is denoted by $G[A]$. Given a graph $H$, the graph $G$ is called $H$-free, if it contains no induced subgraph isomorphic to $H$. A graph $G$ has an $H$-decomposition, if edges of $G$ can be decomposed into subgraphs isomorphic to $H$. If $G$ has an $H$-decomposition, then we say that $G$ is $H$-decomposable. A tree with exactly two non-pendant vertices is called a double-star. Let $k_1$ and $k_2$ be two positive integers. The double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, \ldots , 1)$ is denoted by $S_{k_1,k_2}$. A vertex of degree $i$ is called an $i$-vertex. If $G$ is an $r$-regular graph with an $S_{1,r-1}$-decomposition and $S \subseteq V(G)$ is the set of $r$-vertices of this decomposition, then we say that $G$ is $(S_{1,r-1},S)$-decomposable.

Figure 1: $S_{1,2}$.

Tree decompositions of highly connected graphs have been extensively studied by several authors, see [1], [5] and [6]. Barát and Gerbner [1] showed that every 191-edge-connected graph, whose size is divisible by 4 has an $S_{1,2}$-decomposition. Recently, Bensmail et al. [2] claimed that they have proved Barát-Thomassen conjecture.

In this paper, we study the double-star decomposition of cubic graphs. Let $G$ be a cubic graph. If $G$ is $S$-decomposable and $S$ is a double-star, then $S$ is isomorphic to $S_{1,1}$, $S_{1,2}$ or $S_{2,2}$, otherwise $S$ has a vertex of degree at least four. Kotzig [4] proved that a cubic graph has an $S_{1,1}$-decomposition if and only if it contains a perfect matching. We study the edge-decomposition of cubic graphs into copies of $S_{1,2}$. Also, we obtain some results on $S_{1,r-1}$-decomposition of $r$-regular graphs.

2. Results

Let $G$ be an $r$-regular graph and $S \subseteq V(G)$. The question is whether $G$ is $(S_{1,r-1},S)$-decomposable or not? For giving a response to this question, we use a new bipartite graph $H = (S,L)$, in which $S$ is the set of $r$-vertices of $S_{1,r-1}$-trees and for each edge $e \in E(G \setminus S)$, we put a vertex $u_e$ in $L$. Two vertices $s_i$ and $u_e$,
are adjacent in $H$ if and only if we can obtain an $S_{1,r-1}$ by adding $e_j$ to a $K_{1,r-1}$ containing $s_i$ as a central vertex. We have the following remarks which follow by the fact that the vertices of degree $r$ are not adjacent.

**Remark 1.** Let $G$ be an $r$-regular graph of order $n$. Then $G$ is $(S_{1,r-1}, S)$-decomposable if and only if $|S| = \frac{rn}{2(r+1)}$ and $H = (S, L)$ has a perfect matching.

**Remark 2.** Let $G$ be an $r$-regular graph of order $n$ which is $S_{1,r-1}$-decomposable. Then $\alpha(G) \geq \frac{rn}{2(r+1)}$.

In the following lemma we provide some necessary conditions for the existence of an $S_{1,2}$-decomposition in cubic graphs.

**Lemma 2.1.** Let $G$ be a cubic graph of order $n$ which has an $S_{1,2}$-decomposition. Then there exists an independent set $S \subset V(G)$ of size $\frac{3n}{8}$ such that:

1. Each component of $G \setminus S$ is either a cycle or a tree,

2. No component of $G \setminus S$ has two 3-vertices.

**Proof.** Let $S \subseteq V(G)$ be the set of 3-vertices in an $S_{1,2}$-decomposition and $H = (S, L)$ be the bipartite graph defined before Remark 1. Also, suppose that $F$ is a given component of $G \setminus S$. If $F$ is neither a tree nor a cycle, then it contains a cycle $C : v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ and an edge $e = v_iw$, where $1 \leq i \leq t$ and $w \in V(F)$. Let $A = E(C)$, then $|N_H(A)| \leq |A| - 1$. Now, by Hall’s theorem, $H$ has no perfect matching. Then by Remark 1, $G$ has no $S_{1,2}$-decomposition, a contradiction.

If there exist two 3-vertices $u$ and $v$ in some component $F$, then there exists a $(u, v)$-path $P$ in $F$. Now, let $A = E(P)$. Then $|N_H(A)| \leq |A| - 1$, a contradiction. \hfill \Box

These necessary conditions are not sufficient. Some examples are given in Figure 2.

![Figure 2: Cubic graphs with no $S_{1,2}$-decomposition.](image-url)
By Lemma 2.1, if $G$ is an $S_{1,2}$-decomposable cubic graph, then $\alpha(G) \geq \frac{3n}{8}$. We consider the case that $\alpha(G) = \frac{3n}{8}$ and find two sufficient conditions for the existence of an $S_{1,2}$-decomposition in this case.

First, we prove the following theorem. Note that if $\alpha(G) = \frac{3n}{8}$ and $S$ is an independent set with $|S| = \frac{3n}{8}$, then each components of $G\setminus S$ is a cycle, path or an isolated vertex. In the following we consider the case in which each component is either a cycle or an isolated vertex.

**Theorem 2.2.** Let $G$ be a cubic graph of order $n$ with $\alpha(G) = \frac{3n}{8}$. Suppose that there exists an independent set $S \subseteq V(G)$ such that $|S| = \frac{3n}{8}$ and each component of $G\setminus S$ is a cycle or an isolated vertex. If no vertex of $S$ is contained in a triangle, then $G$ is $(S_{1,2}, S)$-decomposable.

**Proof.** Suppose that $C_1, \ldots, C_t$ are cycle components of $G\setminus S$ and let $W = V(C_1) \cup \ldots \cup V(C_t)$. We claim that $N_S(\{u\}) \neq N_S(\{v\})$, for every two vertices $u, v \in W$. If $u$ and $v$ are adjacent, then since no vertex of $S$ is contained in a triangle, we are done. So we may assume that $u$ and $v$ are not adjacent. If $N_S(\{u\}) = N_S(\{v\}) = \{s\}$, then $S' = (S \setminus \{s\}) \cup \{u, v\}$ is an independent set and $|S'| > \frac{3n}{8}$, a contradiction.

This implies that every $w \in W$ is adjacent to one vertex in $S$ and no vertex of $S$ is adjacent to two vertices in $W$. Hence, $H = (S, L)$ defined before Remark 1 is a 2-regular graph and has a perfect matching. This yields that $G$ is $(S_{1,2}, S)$-decomposable. \hfill $\square$

Now, we prove the following theorem.

**Theorem 2.3.** Let $G$ be a cubic graph of order $n$ with $\alpha(G) = \frac{3n}{8}$ and $S$ be an independent set of size $\frac{3n}{8}$. Let $I$ be the set of isolated vertices in $G\setminus S$. If there exists no cycle of length 3, 5 or 7 in $G[V(G) \setminus I]$ which contains a vertex of $S$, then $G$ has a $S_{1,2}$-decomposition.

**Proof.** First note that there exists a graph with the conditions of this theorem, see Figure 3. We divide the proof into four claims.

![Figure 3: A graph with the conditions of Theorem 2.3.](image)

**Claim 1.** Each component of $G \setminus S$ is a path or a cycle.
If there exists a vertex of degree 3 in \( G \setminus S \), then by adding this vertex to \( S \) we obtain an independent set \( S' \) such that \( |S'| = \frac{3n}{8} + 1 \), a contradiction.

**Claim 2.** Let \( u, v \in V(G \setminus S) \) be two 2-vertices in \( G \setminus S \). Then \( N_S(\{u\}) \neq N_S(\{v\}) \).

Let \( u \) and \( v \) be two vertices in \( G \setminus S \) such that \( d_{G \setminus S}(u) = d_{G \setminus S}(v) = 2 \) and \( N_S(\{u\}) = N_S(\{v\}) = \{s\} \). If \( u \) and \( v \) are adjacent, then \( s \) is contained in a triangle in \( G[V(G) \setminus \{s\}] \), a contradiction. Also, if \( u \) and \( v \) are not adjacent, then \( S' = (S \setminus \{s\}) \cup \{u, v\} \) is an independent set and \( |S'| = \frac{3n}{8} + 1 \), a contradiction. So, the claim is proved.

Now, we check the condition of Hall's theorem for the edges of \( G \setminus S \). Suppose that \( L = \{e_1, \ldots, e_t\} \subseteq E(G \setminus S) \). Let \( P_1, \ldots, P_k \) be path components of \( G \setminus S \). Let \( \langle L \rangle \) be an induced subgraph with edge set \( L \). Now, we consider two cases:

**Case 1.** No \( P_i \) is contained in \( \langle L \rangle \).

Note that for each edge \( e \in L \), one of its end points has degree two in \( G \setminus S \). Because if both end points are of degree one in \( G \setminus S \), then the induced subgraph on this edge is a path component of \( G \setminus S \). Now, we show that for each edge \( e_i \in L \), one can find \( v_{e_i} \in V(G \setminus S) \) such that \( d_{G \setminus S}(v_{e_i}) = 2 \), \( v_{e_i} \) is an end point of \( e_i \) and if \( i \neq j \), then \( v_{e_i} \neq v_{e_j} \).

Suppose that \( P_i = v_1v_2 \ldots v_l \) has some edge in \( \langle L \rangle \), where \( 1 \leq i \leq k \). Let \( e_{i_1}, e_{i_2}, \ldots, e_{i_t} \) be edges of \( P_i \) contained in \( \langle L \rangle \) such that \( i_1 < i_2 < \ldots < i_t \) and \( e_{i_j} = v_{i_j}v_{i_j+1} \). If \( e_{i-1} \notin \langle L \rangle \), then let \( v_{e_{i_j}} = v_{i_j-1} \) and we are done. Also, if \( e_{t+1} \notin \langle L \rangle \), then let \( v_{e_{i_j}} = v_{i_{j+1}} \). So, we may assume that \( e_1, e_{t+1} \in \langle L \rangle \). Since \( P_t \) is not contained in \( \langle L \rangle \), there exists \( 1 < l < l-1 \), such that \( e_l \notin \langle L \rangle \). Now, if \( i_j < l \), let \( v_{e_{i_j}} = v_{i_j-1} \) and if \( i_j > l \), let \( v_{e_{i_j}} = v_{i_j+1} \). By repeating this procedure for \( P_1, P_2, \ldots, P_k \), we are done.

Now, Claim 2 implies that for each \( e \in L \) there exists a distinct vertex in \( S \) which is adjacent to \( v \), and so in this case, the condition of Hall's theorem holds.

**Case 2.** There exist \( i_1, \ldots, i_t \) such that \( 1 \leq i_1, \ldots, i_t \leq k \) and \( P_{i_1}, \ldots, P_{i_t} \) are path components of \( G \setminus S \) contained in \( \langle L \rangle \). Let \( W_2 \subseteq V(G \setminus S) \) be the set of vertices of degree \( j \) in \( G \setminus S \), for \( j = 0, 1, 2 \). We have the following.

**Claim 3.** Let \( v \in V(G \setminus S) \) such that \( d_{G \setminus S}(v) = 1 \) and \( N_S(\{v\}) = \{x, y\} \). Then \( N_{G \setminus S}(\{x\}) \cap W_2 = \emptyset \) or \( N_{G \setminus S}(\{y\}) \cap W_2 = \emptyset \).

By contrary, suppose that \( x \) and \( y \) are adjacent to \( v_x \) and \( v_y \) in \( G \setminus S \), respectively, and \( d_{G \setminus S}(v_x) = d_{G \setminus S}(v_y) = 2 \). Note that \( v \) is not adjacent to \( v_x \) and \( v_y \), since otherwise there exists a triangle containing \( v \), a contradiction. Now, if \( v_x \) and \( v_y \) are adjacent, then \( C : v, x, v_x, v_y, y, v \) is a cycle of length 5 containing \( v \), a contradiction. These imply that \( \{v, v_x, v_y\} \) is an independent set. Now, let \( S' = (S \setminus \{x, y\}) \cup \{v, v_x, v_y\} \). It can be easily seen that \( S' \) is an independent set and \( |S'| = \frac{3n}{8} + 1 \), a contradiction.

Now, we can prove that in the second case, \( L \) satisfies the condition of Hall's theorem. It suffices to show that the edges of \( P_1, \ldots, P_t \) satisfy Hall's condition.
Because, similar to the proof of the first case, one can see that other edges have distinct neighbors in $S$ and we are done. Now, Claim 2 implies that we can find $\sum_{j=1}^{t}(|E(P_j)| - 1)$ vertices in $S$ which are adjacent to the vertices of degree two in $P_1, P_2, \ldots, P_t$. Let $T \subseteq S$ be the set of vertices in $S$ which are adjacent to the end vertices of $P_1, \ldots, P_t$, and to no vertex of degree two in $G \setminus S$. It suffices to show that $|T| \geq t$.

By contrary, suppose that $|T| \leq t - 1$. Then Claim 3 implies that each end vertex of the paths has a neighbor in $T$. Let $A$ be the set of end vertices of paths that have one neighbor in $T$ and let $B$ be the set of end vertices which have two neighbors in $T$. We have the following.

$$|A| + |B| = 2t, \quad |A| + 2|B| \leq 3t - 3.$$ 

Hence, we conclude that $|A| \geq t + 3$. Now, we prove the following claim.

**Claim 4.** If $u, v \in A$, then $N_T(\{u\}) \cap N_T(\{v\}) = \emptyset$.

First, note that if $u$ and $v$ are adjacent, then we are done. So, we may assume that $u$ and $v$ are not adjacent. Let $N_T(\{u\}) = N_T(\{v\}) = \{w\}$. Suppose that $N_S(\{u\}) = \{w, x\}$ and $N_S(\{v\}) = \{w, y\}$. By the definition of $T$, we conclude that $x$ and $y$ are adjacent to some vertices of degree 2 in $G \setminus S$, say $v_x$ and $v_y$, respectively. First, suppose that $x = y$. If $\{u, v, v_x\}$ is not independent set, then one can find a triangle contains a vertex of $S$, a contradiction, see Figure 4. Thus, $\{u, v, v_x\}$ is an independent set. Now, $S' = (S \setminus \{w, x\}) \cup \{u, v, v_x\}$ is an independent set of size $\frac{2n}{3} + 1$, a contradiction.

Figure 4: The case $x = y$ (left side) and the case $x \neq y$ (right side).

So, we may assume that $x \neq y$. We show that $\{u, v, v_x, v_y\}$ is an independent set.

Since no vertex of $S$ is contained in a triangle in $G[V(G) \setminus I]$, we conclude that $u$ and $v_x$ are not adjacent (similarly, $v$ and $v_y$ are not adjacent). So, suppose that $v$ and $v_x$ are adjacent. Then $C: u, v, v_x, v, w, u$ is a cycle of length 5 in $G[V(G) \setminus I]$ which contains $w \in S$, a contradiction, see Figure 4. Also, note that $v_x$ and $v_y$ are not adjacent. Since, otherwise $C: u, x, v_x, v_y, y, v, w, u$ is a cycle of length 7 in $G[V(G) \setminus I]$ which contains $w \in S$, a contradiction, see Figure 4. These imply that $\{u, v, v_x, v_y\}$ is an independent set. Now, let $S' = (S \setminus \{x, y, w\}) \cup \{u, v, v_x, v_y\}$. 


Then $S'$ is an independent set and $|S'| > \frac{3n}{8}$, a contradiction and this completes the proof of the claim.

Now, Claim 4 implies that for every $v \in A$ we have a distinct neighbor $t_v \in T$ and this implies that $|T| \geq t + 3$, a contradiction. This completes the proof. \hfill \qed

We have the following result in regular bipartite graphs which makes a connection between the domination number and the existence of a $S_{1,r-1}$-decomposition.

**Theorem 2.4.** If $G = (A, B)$ is a bipartite $r$-regular graph of order $n$ such that $r + 1 \mid n$ and $\gamma(G) = \frac{n}{r+1}$, then $G$ is $S_{1,r-1}$-decomposable.

**Proof.** Let $D$ be a dominating set of $G$ of size $\frac{n}{r+1}$. Then vertices of $D$ have no common neighbor in $V(G) \setminus D$. Now, let $D_A = D \cap A$ and $D_B = D \cap B$ and $|D_A| = a, |D_B| = b$. Since $D$ is a dominating set of size $\frac{n}{r+1}$ we have:

$$a + b = \frac{n}{r+1}, \quad ra + b = \frac{n}{2}.$$

Then $a = b = \frac{n}{2(r+1)}$. Now, let $S = N(D_A)$. We show that $G$ has a $(S_{1,r-1}, S)$-decomposition. Clearly, $|S| = \frac{ra}{2(r+1)}$ and $E(G \setminus S)$ is exactly the set of edges between $D_B$ and $N(D_B)$. Note that if $v \in N(D_B)$, then $d_{S}(v) = r - 1$. Now, it is not hard to see that the graph $H = (S, L)$, defined in Remark 1, is a $(r-1)$-regular bipartite graph and hence it has a perfect matching. So, by Remark 1, $G$ is $(S_{1,r-1}, S)$-decomposable. \hfill \qed

In bipartite cubic graphs we can find a better result.

**Theorem 2.5.** Let $G = (A, B)$ be a cubic bipartite graph of order $n$ such that $8 \mid n$. Then $\gamma(G) = \frac{n}{4}$ if and only if there exists $S \subseteq A$ of size $\frac{3n}{8}$ such that $G$ is both $(S_{1,2}, S)$-decomposable and $(S_{1,2}, N(A \setminus S))$-decomposable.

**Proof.** Note that if $\gamma(G) = \frac{n}{4}$, then by Theorem 2.4, we are done. For converse, first notice that if $G$ is a cubic graph, then $\gamma(G) \geq \frac{n}{4}$. Suppose that there exists $S \subseteq A$ such that $G$ is both $(S_{1,2}, S)$-decomposable and $(S_{1,2}, N(A \setminus S))$-decomposable. Note that each vertex in $A \setminus S$ is a 3-vertex in $G \setminus S$. Now, Lemma 2.1 implies that each of them is in a different component of $G \setminus S$ and so they have no common neighbors. By a similar method, one can show that the vertices of $B \setminus N(A \setminus S)$ have no common neighbors. Now, $D = (A \setminus S) \cup (B \setminus N(A \setminus S))$ is a dominating set of size $\frac{n}{4}$ and this completes the proof. \hfill \qed

Now, we provide another sufficient condition for the existence of an $S_{1,2}$-decomposition in bipartite cubic graphs.

**Theorem 2.6.** Let $G = (A, B)$ be a bipartite cubic graph of order $n$ and $S \subseteq A$ be of size $\frac{3n}{8}$. Then $G$ is $(S_{1,2}, S)$-decomposable if and only if there exists a perfect matching between $S$ and $N(A \setminus S)$. 
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Proof. First, suppose that $G$ is $(S_{1,2}, S)$-decomposable. Then the second part of Lemma 2.1 indicates that no component of $G \setminus S$ has two 3-vertices. This implies that no two vertices of $A \setminus S$ have a common neighbor in $B$. So, $|N(A \setminus S)| = \frac{3n}{8}$. Now, note that if Hall’s condition does not hold for $S$ and $N(A \setminus S)$, then Hall’s condition does not hold in $H = (S, L)$, too. This is a contradiction and this completes the proof.

Now, suppose that there exists a perfect matching between $S$ and $N(A \setminus S)$. Then $|N(A \setminus S)| = \frac{3n}{8}$ which implies that no two vertices of $A \setminus S$ have a common neighbor in $B$. For each vertex $v \in N(A \setminus S)$, there exists a unique edge $e_v \in E(G \setminus S)$ in which $v$ is one of its end points. Let $M = \{(u_i, v_i) | i = 1, 2, \ldots, \frac{3n}{8}\}$ be a matching between $S$ and $N(A \setminus S)$. Then by adding the edge $e_v$, to a claw containing $u_i$ as a 3-vertex, one can obtain an $S_{1,2}$-decomposition.

We close the paper with the following questions.

Question 1. Does there exist a triangle-free 2-connected cubic graph of order divisible by 8 which has no $S_{1,2}$-decomposition?

Question 2. Is it true that every bipartite cubic graph of order divisible by 8 is $S_{1,2}$-decomposable?

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References


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