Mathematics Interdisciplinary Research **3** (2018) 109 − 115

Groups with Two Extreme Character Degrees and their Minimal Faithful Representations

Houshang Behravesh and Mahtab Delfani ⋆

Abstract

For a finite group G, we denote by $p(G)$ the minimal degree of faithful permutation representations of G, and denote by $c(G)$, the minimal degree of faithful representation of G by quasi-permutation matrices over the complex field $\mathbb{C}$. In this paper we will assume that, G is a p-group of exponent p and class 2, where p is prime and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Then we will show that $c(G) \leq |G : Z(G)|^{1/2}c(Z(G))$ and $p(G) \leq |G : Z(G)|^{1/2}p(Z(G))$.

Keywords: Quasi-permutation, linear character, non-linear character.

2010 Mathematics Subject Classification: Primary: 20C15, Secondary: 20D15.

How to cite this article
H. Behravesh and M. Delfani, Groups with two extreme character degrees and their minimal faithful representations, *Math. Interdisc. Res.* **3** (2018) 109 - 115.

1. Introduction

The study of the structure of a finite group G by imposing conditions on the set $\mathrm{cd}(G)$ of the degrees of its complex irreducible characters has been considered in many research papers. For example, it has been shown that groups having just two different character degrees are solvable and these groups have been thoroughly investigated by Isaacs and Passman in [8, 9] and [7, Chapter 12].

In this paper, let G be a finite p-group of exponent p, such that $G' \leq Z(G)$ and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. By [7, Theorem 12.5], it is known that G is the direct product of a p-group and an abelian group. Also Moreto in [6], for p-group

⋆Corresponding author (E-mail: ma.delfani@gmail.com)
Academic Editor: Mohammad Reza Darafsheh
Received 10 June 2017, Accepted 5 September 2017
DOI: 10.22052/mir.2019.186347.1133

G has proven that, the normal subgroups of G either contain G' or are contained in $Z(G)$.

For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G. Various interesting results have been obtained about $p(G)$. For example, in [2], it has been shown that if $A = A_1 \times \cdots \times A_r$ is an abelian group with each A_i cyclic of prime power order a_i, then $p(G) = a_1 + \cdots + a_r$. More generally, if H and K are nontrivial nilpotent groups, then $p(H \times K) = p(H) + p(K)$, which emphasizes the importance of studying $p(G)$ for finite p-groups [10].

In a parallel direction, one may define other degree corresponding to embeddings of a finite group G in special types of matrix groups. In this way, we obtain degree $c(G)$ which can be completely determined from the character table of G and often give best possible lower bounds for $p(G)$. We introduce the notion of a quasi-permutation matrix. By a quasi-permutation matrix over a subfield F of complex field $\mathbb{C}$, we simply mean a square matrix over F with non-negative integral trace. For a finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. Notice that every permutation matrix is a quasi-permutation matrix. Evidently, we have $c(G) \leqslant p(G)$.

The quantity $c(G)$ was introduced in [1] and has been studied in [2, 3]. For example, in [2, 3], $c(G)$ and $p(G)$ were calculated for abelian and metacyclic groups. In fact, using the above notation we have $c(A) = a_1 + \cdots + a_r - n$ for an abelian group A, where n is the largest integer such that C_6^n is a direct summand of A. In [1], it has been shown that for a finite p-group G of class 2 with cyclic center, $c(G) = p(G) = |Z(G)||G : Z(G)|^{1/2}$. Moreover, in [4], it has been shown that if p is odd prime, then $c(G) = p(G)$. Our goal is to find a sharp bound for $c(G)$:

Theorem A. Let G be a finite p-group of exponent p and $G' \leq Z(G)$. Also assume that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Then

(i) $c(G) \leq |G : Z(G)|^{1/2} c(Z(G))$.

(ii) $p(G) \leq |G : Z(G)|^{1/2} p(Z(G))$.

2. Results

Theorem 2.1. [1, Theorem 2.2] *Let G be a finite group. Then*

$$p(G) = \min\left\{ \sum_{i=1}^{n} |G : H_i| : H_i \leq G \ for \ i = 1, 2, ..., n \ and \ \bigcap_{i=1}^{n} (H_i)_G = 1 \right\}.$$

Corollary 2.2. [1, Corollary 2.4] *Let G be a finite group with a unique minimal normal subgroup. Then $p(G)$ is the smallest index of a subgroup with trivial core.*

Notation. Assume that G is a finite group and χ is a character of G. We denote by $\Gamma(\chi)$ the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$.

Remark 2.3. Let $\mathcal{C}_i$ for $0 \leq i \leq r$ be the Galois conjugacy classes over $\mathbb{Q}$ of irreducible complex characters of the group G. For $0 \leq i \leq r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$ and $K_i = \ker\psi_i$. For $I \subseteq \{0, 1, \ldots, r\}$, put $K_I = \bigcap_{i \in I} K_i$.

Definition 2.4. Let G be a finite group. Let χ be an irreducible complex character of G. Then define

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$.

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ |\min\{\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha(g) : g \in G\}| & \text{otherwise} \end{cases}$,

(3) $c(\chi) = \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha + m(\chi)1_G$.

Theorem 2.5. [1, Theorem 3.6] *Let G be a finite group. Then in the above notation*

$$c(G) = \min \{c(\chi)(1) : \chi \text{ is faithful irreducible complex character of } G\}.$$

Theorem 2.6. [1, Theorem 4.6] *Let G be a finite p-group with a unique minimal normal subgroup. Then there exists a faithful irreducible character χ. Suppose that all faithful irreducible character of G have degree $\chi(1)$ and $\chi^2(1) = |G : Z(G)|$. Then $c(G) = \chi(1)|Z(G)| = |Z(G)||G : Z(G)|^{1/2}$.*

Remark 2.7. Let $G \cong \prod_{i=1}^{r} C_{p_i}$ where p_i is a prime power. As in [2], define $T(G) = \sum_{i=1}^{r} p_i$, when $G = \{1\}$ let $T(G) = 1$. In [2] it is proved that $p(G) = T(G)$ and $c(G) = T(G) - n$ for an abelian group G, where n is the largest integer such that, C_6^n is a direct summand of G.

Lemma 2.8. *Let G be a finite group and* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. *Also let χ be a non-linear irreducible character of G. Then $Z(G) = Z(\chi)$.*

Proof. By [7, Corollary 2.28], $Z(G) \leq Z(\chi)$. Then by [7, Corollary 2.30],

$$\chi^2(1) \leq |G : Z(\chi)| \leq |G : Z(G)| = \chi^2(1).$$

Therefore $|G : Z(\chi)| = |G : Z(G)|$. Hence $|Z(\chi)| = |Z(G)|$ and $Z(\chi) = Z(G)$. $\square$

Lemma 2.9. *Let G be a finite group and* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$*. Then G has $|G/G'|$ linear irreducible characters and $|Z(G)| - |Z(G)/G'|$ non-linear irreducible characters.*

Proof. By [7, Corollary 2.23], the number of linear character of G is equal to $|G : G'|$. Also by [7, Corollary 2.7], we have

$$|G| = \sum_{\chi \in Irr(G)} \chi(1)^2 = |G : G'| + \alpha |G : Z(G)|,$$

where α is the number of non-linear irreducible character. Therefore $\alpha = |Z(G)| - |\frac{Z(G)}{G'}|$. □

Remark 2.10. Let G be a finite group of class 2, that is, $G' \leq Z(G)$. Also assume that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Let χ be a non-linear character of G, then by [7, Lemma 2.27(c) and Corollary 2.30], for all $g \in G$ and $\chi \in \mathrm{Irr}(G)$ and $\chi(1) \neq 1$, we have

$$\chi(g) = \begin{cases} 0 & \text{if } g \in G \setminus Z(G) \\ |G : Z(G)|^{1/2} \lambda(g) & \text{if } g \in Z(G) \text{ and for some } \lambda \in \mathrm{Irr}(Z(G)) \end{cases}.$$

Lemma 2.11. *Let G be a finite group and* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$*. Let*

$$\chi(g) = \begin{cases} 0 & \text{if } g \in G \setminus Z(G) \\ |G : Z(G)|^{1/2} \lambda(g) & \text{if } g \in Z(G) \text{ and } \lambda \in \mathrm{Irr}(Z(G)) \end{cases}.$$

Then χ is a class function and $[\chi, \chi] = 1$.

Proof. Let λ be an class function of $Z(G)$, then by [7, Corollary 2.17], $[\lambda, \lambda] = 1$. So

$$[\lambda, \lambda] = \frac{1}{|Z(G)|} \sum_{g \in Z(G)} \lambda(g)\overline{\lambda(g)} = 1 \Longrightarrow \sum_{g \in Z(G)} \lambda(g)\overline{\lambda(g)} = |Z(G)|.$$

Now let $\chi = |G : Z(G)|^{1/2} \lambda$, where λ is an class function of $Z(G)$. Then χ is a class function and

$$\begin{aligned} [\chi, \chi] &= [\lambda |G : Z(G)|^{1/2}, \lambda |G : Z(G)|^{1/2}] = |G : Z(G)| \frac{1}{|G|} \sum_{g \in G} \lambda(g)\overline{\lambda(g)} \\ &= |G : Z(G)| \big(\frac{1}{|G|} (\sum_{g \in Z(G)} \lambda(g)\overline{\lambda(g)} + \sum_{g \in G \setminus Z(G)} \lambda(g)\overline{\lambda(g)}) \big) \\ &= |G : Z(G)| \frac{1}{|G|} \sum_{g \in Z(G)} \lambda(g)\overline{\lambda(g)} = |G : Z(G)| \frac{1}{|G|} |Z(G)| = 1, \end{aligned}$$

proving the lemma. □

Remark 2.12. Let G be a finite group and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Let $\chi \in \mathrm{Irr}(G)$ and $\lambda \in \mathrm{Irr}(Z(G))$. Also by Remark 2.10, we know that χ is equal to a multiple of an irreducible character of $Z(G)$ and this factor is $|G : Z(G)|^{1/2}$. Then there are two cases to be considered here.
Case 1: If $\chi = |G : Z(G)|^{1/2}\lambda$. Then

$$[\chi, \lambda] = [|G : Z(G)|^{1/2}\lambda, \lambda] = |G : Z(G)|^{1/2}[\lambda, \lambda] = |G : Z(G)|^{1/2}.$$

Case 2: If $|G : Z(G)|^{1/2}\lambda \neq \chi$. Since $\frac{1}{|G:Z(G)|^{1/2}}\chi$ and λ are irreducible characters of $Z(G)$. So $\frac{1}{|G:Z(G)|^{1/2}}\chi \neq \lambda$ and $[\frac{1}{|G:Z(G)|^{1/2}}\chi, \lambda] = 0$. Therefore $[\chi, \lambda] = 0$.

Theorem 2.13. *Let G be a nilpotent group of class 2 and $G' = Z(G)$. Also assume that* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. *Then* $c(G) = |G : Z(G)|^{1/2}c(Z(G))$.

Proof. Since $G' = Z(G)$, so by Lemma 2.9, there are exactly $|G/G'| + |Z(G)| - 1$ irreducible characters for G. Therefore by Theorem 2.5, Remark 2.10 and Lemma 2.11, we can calculated $c(G)$, only by using irreducible characters of $Z(G)$. Hence the result follows. □

Lemma 2.14. *Let G be a finite group. Also, let χ be a linear character of G. Then* $\chi|_{Z(G)} \in \mathrm{Irr}(Z(G))$.

Proof. Let χ be a linear character of G. We know that linear characters of G are homomorphisms from G to $\mathbb{C}^*$. Therefore $\chi|_{Z(G)}$ is also a homomorphism from $Z(G)$ to $\mathbb{C}^*$. Hence $\chi|_{Z(G)} \in \mathrm{Irr}(Z(G))$. □

Remark 2.15. Let $H \cong C_p$ and $\varepsilon = e^{2\pi i/p}$, where p is prime. Also let θ be an irreducible character of H. Then for $l \in \mathbb{Z}$,

$$\Gamma(\theta) = \mathrm{Gal}(\mathbb{Q}(\varepsilon^l) : \mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\varepsilon) : \mathbb{Q}).$$

Therefore $|\Gamma(\theta)| = \varphi(p)$, where φ is the Euler function.

Now let $G = H \times K \cong C_p \times C_p$ and χ, θ and ψ are irreducible characters of G, H and K, respectively. Then for $h, k \in \mathbb{Z}$,

$$\Gamma(\chi) = \Gamma(\theta\psi) = \mathrm{Gal}(\mathbb{Q}(\varepsilon^h\varepsilon^k) : \mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\varepsilon) : \mathbb{Q}).$$

Therefore $|\Gamma(\chi)| = \varphi(p)$.

Hence in general if $G \cong C_p \times \ldots \times C_p$, then in order to calculate $c(G)$, we have to select irreducible characters of G, say χ_i's, such that $\bigcap \ker\chi_i = 1$. Also note that different faithful characters have same degree and its minimals are also the same numbers.

Theorem 2.16. *Let G be a finite p-group of exponent p and $G' < Z(G)$. Also assume that* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. *Then* $c(G) \leq |G : Z(G)|^{1/2}c(Z(G))$.

Proof. By Lemma 2.14, if χ is a linear character of G, then $\chi|_{Z(G)} \in \mathrm{Irr}(Z(G))$. Now if χ is a non-linear character of G and $\chi(1) \neq 1$, then by Remark 2.10, $\chi|_{Z(G)} = |G : Z(G)|^{1/2}\lambda$, where $\lambda \in \mathrm{Irr}(Z(G))$.

Since G is a p-group of exponent p, so $Z(G) \cong C_p \times \ldots \times C_p$. Hence by Remark 2.15, it is easy to see that $c(G) \leq |G : Z(G)|^{1/2} c(Z(G))$. □

The following examples show that if G is a p-group of exponent p and $G' \leq Z(G)$ and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ the bound $|G : Z(G)|^{1/2} c(Z(G))$ is a sharp bound.

Example 2.17. (a) Let

$$G = \langle x, y, z : \ x^p = y^p = z^p = [x, y] = [x, z] = 1, z^{-1}yz = yx\rangle.$$

Then $G' = Z(G) = \langle x\rangle$ and $\mathrm{cd}(G) = \{1, p\}$. Hence by Remark 2.7, $c(Z(G)) = p$ and by Theorem 2.6, $c(G) = |G : Z(G)|^{1/2} c(Z(G)) = p^2$.

(b) Let

$$G = \langle x, y, z, t : \ x^p = y^p = z^p = t^p = [x, z] = [x, t] = [y, z] = [y, t] = [z, t] = 1, [x, y] = t\rangle.$$

Then $G' = \langle t\rangle$, $Z(G) = \langle z, t\rangle$ and $\mathrm{cd}(G) = \{1, p\}$. Hence by [5, Theorem 4.1], $c(G) = p(G) = p^2 + p$. Also we have $|G : Z(G)|^{1/2} c(Z(G)) = p(p + p) = p^2 + p^2$. Hence $c(G) \leq |G : Z(G)|^{1/2} c(Z(G))$.

Lemma 2.18. *Let G be finite group. Then $p(G) \leq |G : Z(G)| p(Z(G))$.*

Proof. Let $Z(G) \cong C_{p_1^{m_1}} \times \cdots \times C_{p_r^{m_r}}$, where p is a prime and let

$$H_i \cong C_{p_1^{m_1}} \times \cdots \times C_{p_{i-1}^{m_{i-1}}} \times C_{p_{i+1}^{m_{i+1}}} \times \cdots \times C_{p_r^{m_r}}.$$

Then $H_i \lhd G$ and also $\bigcap\limits_{i=1}^{n} H_i = 1$. So by Lemma 2.1, we have:

$$p(G) \leq \sum_{i=1}^{r} |\frac{G}{H_i}| = |G| \frac{\sum_{i=1}^{r} |m_i|}{|Z(G)|} = |G : Z(G)| \sum_{i=1}^{r} |m_i|.$$

But by Remark 2.7, $\sum_{i=1}^{r} |m_i| = p(Z(G))$. Hence the result follows. □

Corollary 2.19. *Let G be a finite p-group of exponent p, where p is an odd prime and $G' \leq Z(G)$. Let $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Then*

$$p(G) \leq |G : Z(G)|^{1/2} p(Z(G)).$$

Proof. Since G is a p-group, p is odd prime, $p(G) = c(G)$ and $c(Z(G)) = p(Z(G))$. Hence by Theorem 2.16,

$$p(G) = c(G) \leq |G : Z(G)|^{1/2} c(Z(G)) = |G : Z(G)|^{1/2} p(Z(G)),$$

as desired. □

Conflicts of Interest. The authors declare that there is no conflicts of interest regarding the publication of this article.

References

[1] H. Behravesh, Quasi-permutation representations of p-groups of class 2, *J. London Math. Soc. (2)* **55**(2) (1997) 251 − 260.

[2] H. Behravesh, The minimal degree of a faithful quasi-permutation representation of an abelian group, *Glasgow Math. J.* **39**(1) (1997) 51 − 57.

[3] H. Behravesh, Quasi-permutation representations of metacyclic p-groups, *J. Sci. I. R. Iran* **9**(3) (1998) 258 − 264.

[4] H. Behravesh and G. Ghaffarzadeh, Minimal degree of faithful quasi-permutation representations of p-groups, *Algerba Colloq.* **18** (2011) 843 − 846.

[5] H. Behravesh and H. Mousavi, A note on p-groups of order $\leq p^4$, *Proc. Indian Acad. Sci. Math. Sci.* **119**(2) (2009) 137 − 143.

[6] G. A. Fernandez-Alcober and A. Moreto, Groups with two extreme character degrees and their normal subgroups, *Trans. Amer. Math. Soc.* **353**(6) (2001) 2171 − 2192.

[7] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[8] I. M. Isaacs and D. S. Passman, A characterization of groups in terms of the degrees of their characters, *Pacific J. Math.* **15** (1965) 877 − 903.

[9] I. M. Isaacs and D. S. Passman, A characterization of groups in terms of the degrees of their characters II, *Pacific J. Math.* **24** (1968) 467 − 510.

[10] D. Wright, Degree of minimal embeddings for some direct products, *Amer. J. Math.* **97**(4) (1975) 897 − 903.

Houshang Behravesh
Department of Mathematics,
Urmia University,
Urmia, I. R. Iran
E-mail: h.behravesh@urmia.ac.ir

Mahtab Delfani
Department of Mathematics,
Urmia University,
Urmia, I. R. Iran
E-mail: ma.delfani@gmail.com