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Note

# On the Regular Power Graph on the Conjugacy Classes of Finite Groups

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**Abstract**

*The (undirected) power graph on the conjugacy classes* $\mathcal{P}_{\mathcal{C}}(G)$ of a group $G$ is a simple graph in which the vertices are the conjugacy classes of $G$ and two distinct vertices $C$ and $C'$ are adjacent in $\mathcal{P}_{\mathcal{C}}(G)$ if one is a subset of a power of the other. In this paper, we describe groups whose associated graphs are $k$-regular for $k = 5, 6$.

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## 1. Introduction

Let $G$ be a finite group. In [5], we defined the *(undirected) power graph on the conjugacy classes* $\mathcal{P}_{\mathcal{C}}(G)$ of $G$ with conjugacy classes of $G$ as the vertex set, in which two distinct vertices are adjacent if one is a subset of a power of the other. Moreover, we described the algebraic structure of groups whose associated graphs are complete graphs, bipartite graphs, star graph, wheel graph, and $k$-regular graphs for $k \leq 4$. (See Theorems 1-4 of Section 2).

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It is clear that if a group $G$ is abelian, then the power graph on conjugacy classes $\mathcal{P}_{\mathcal{C}}(G)$ coincides with the power graph $\mathcal{P}(G)$ defined in [1]. In this paper, we describe groups whose associated graphs are $k$-regular for $k = 5, 6$.

We summarize our notations. $A : B$ denotes a Frobenius group with kernel $A$ and complement $B$ and $Dic_{12}$ denotes the dicyclic group of order 12.

## 2. Main Results

In this section, before stating our main result, we mention some results about this graph.

**Theorem 2.1** (Theorem 2.3 of [5]). *Let $G$ be a finite group. The graph $\mathcal{P}_{\mathcal{C}}(G)$ is complete if and only if $G$ has a unique chief series and every normal subgroup in this series is generated by a conjugacy class of $G$.*

Recall that a regular graph is a graph where each vertex has the same number of neighbors.

**Theorem 2.2** (Theorem 2.4 of [5]). *Let $G$ be a finite group. The graph $\mathcal{P}_{\mathcal{C}}(G)$ is $k$-regular if and only if the graph $\mathcal{P}_{\mathcal{C}}(G)$ is the complete graph with $k+1$ vertices. Furthermore the following are true.*

*i) for $k = 2$, $G \cong C_3$ or $S_3$,*

*ii) for $k = 3$, $G \cong C_4, A_4,$ or $D_{10}$,*

*iii) for $k = 4$, $G \cong C_5, A_5, S_4, C_5 \rtimes C_4,$ or $C_7 \rtimes C_3$.*

A star in an undirected graph is a tree in which at most one vertex has degree larger than one.

**Theorem 2.3** (Theorem 3.1 of [5]). *Let $G$ be a finite group. The graph $\mathcal{P}_{\mathcal{C}}(G)$ is bipartite if and only if the graph $\mathcal{P}_{\mathcal{C}}(G)$ is a star if and only if $G$ is an elementary abelian 2-group.*

A wheel is a graph in which one vertex, called the hub, is joined to each of the other vertices by an edge, all these other vertices forming a cycle, called the rim.

**Theorem 2.4** (Theorem 4.1 of [5]). *Let $G$ be a finite group. The graph $\mathcal{P}_{\mathcal{C}}(G)$ is a wheel graph if and only if $G \cong C_4, A_4,$ or $D_{10}$.*

Now, we are ready to prove the main Theorem of this paper. In the next proof, we will find conjugacy classes of groups using section 12 of [3] and Theorem 13.8 of [2].

**Theorem 2.5.** *Let $G$ be a finite group. The graph $\mathcal{P}_{\mathcal{C}}(G)$ is $k$-regular if and only if* *i) for* $k = 5$, $G \cong Dic_{12}$, $D_{18}$, $(C_3 \times C_3) : C_4$ *or* $PSL(2, 7)$,

*ii) for* $k = 6$, $G \cong C_7$, $D_{22}$, $SL(2, 3)$, $C_{13} : C_3$, $C_{13} : C_4$, $C_{11} : C_5$, $S_5$, *or* $A_6$.

*Proof.* The graph $\mathcal{P}_{\mathcal{C}}(G)$ is $k$-regular if and only if $\mathcal{P}_{\mathcal{C}}(G)$ is a complete graph with $k+1$ vertices, since each conjugacy class of $G$ and $1_G$ are adjacent. By Theorem 2.1, the graph $\mathcal{P}_{\mathcal{C}}(G)$ is complete if and only if $G$ has a unique chief series and every normal subgroup in this series is generated by a conjugacy class of $G$.

i) For $k = 5$, since $G$ has 6 conjugacy classes then by table 1 of [4], $G \cong C_6$, $C_2 \times S_3$, $Dic_{12}$, $D_{18}$, $(C_3 \times C_3) : C_2$, $(C_3 \times C_3) : C_4$, $(C_3 \times C_3) : Q_8$ or $PSL(2, 7)$.

   Suppose that $G$ is isomorphic to $C_6 \cong \langle a \rangle$, $C_2 \times S_3 \cong \langle a \rangle \times S_3$, $(C_3 \times C_3) : C_2 \cong (\langle a \rangle \times \langle b \rangle) : \langle c \rangle$, or $(C_3 \times C_3) : Q_8 \cong (\langle c \rangle \times \langle d \rangle) : Q_8$ in which $Q_8 = \langle a, b | a^5 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Since $\langle a^2 \rangle \cap \langle a^3 \rangle = \{1_G\}$ in $C_6$, neither $\langle (1, cl((1\,2))) \rangle \nsubseteq \langle (a, (cl((1\,2))) \rangle$ nor $\langle (1, cl((1\,2))) \rangle \nsubseteq \langle (a, cl((1\,2))) \rangle$ in $C_2 \times S_3$, $\langle cl(a) \rangle \cap \langle cl(b) \rangle = \{1_G\}$ in $(C_3 \times C_3) : C_2$, $cl(x) = cl(x)(C_3 \times C_3)$ for all $x \in Q_8$ and so neither $\langle cl(a) \rangle \subseteq \langle cl(b) \rangle$ nor $\langle cl(b) \rangle \subseteq \langle cl(a) \rangle$ in $(C_3 \times C_3) : Q_8$ , then by Theorem 2.1, $\mathcal{P}_{\mathcal{C}}(G)$ is not complete.

   On the other hand, if $G$ is isomorphic to $Dic_{12} \cong \langle a, x | a^6 = 1, x^2 = a^3, xax^{-1} = a^{-1} \rangle$, $D_{18} \cong \langle a, b | a^9 = 1, b^2 = 1, bab = a^{-1} \rangle$, $(C_3 \times C_3) : C_4 \cong (\langle a \rangle \times \langle b \rangle) : \langle c \rangle$, or $PSL(2, 7)$, since $1_G \subset \langle cl(x^2) \rangle \subset \langle cl(a) \rangle \subset \langle cl(x) \rangle = Dic_{12}$, $1_G \subset \langle cl(a^3) \rangle \subset \langle cl(a) \rangle \subset \langle cl(b) \rangle = D_{18}$, $1_G \subset \langle cl(x) \rangle \subset \langle cl(a^2) \rangle \subset \langle cl(a) \rangle = (C_3 \times C_3) : C_4$, and $1_G \subset PSL(2, 7)$ are the unique chief series of corresponding groups, by Theorem 2.1, $\mathcal{P}_{\mathcal{C}}(G)$ is complete.

ii) For $k = 6$, since $G$ has 6 conjugacy classes then by table 1 of [4], $G \cong C_7$, $D_{16}$, $Q_{16}$, $SD_{16}$, $D_{22}$, $SL(2, 3)$, $C_{13} : C_3$, $C_7 : C_6$, $C_{13} : C_4$, $C_{11} : C_5$, $S_5$, or $A_6$.

   First, assume that $G$ is isomorphic to $D_{16} \cong \langle a, b | a^8 = 1, b^2 = 1, bab = a^{-1} \rangle$, $Q_{16} \cong \langle a, b | a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1} \rangle$, or $SD_{16} = \langle a, b | a^8 = 1, b^2 = 1, bab = a^3 \rangle$. Since neither $\langle cl(b) \rangle \subseteq \langle cl(a) \rangle$ nor $\langle cl(a) \rangle \subseteq \langle cl(b) \rangle$ in these groups, then by Theorem 2.1, $\mathcal{P}_{\mathcal{C}}(G)$ is not complete.
   Moreover, since $\langle cl(b^2) \rangle \nsubseteq \langle cl(b^3) \rangle$ and $\langle cl(b^3) \rangle \nsubseteq \langle cl(b^2) \rangle$ in $C_7 : C_6 \cong \langle a \rangle : \langle b \rangle$, then the associated graph also is not complete.

   Now, assume that $G$ is isomorphic to $C_7$, $D_{22} = \langle a, b | a^1 1 = 1, b^2 = 1, bab = a^{-1} \rangle$, $SL(2, 3)$, $C_{13} : C_3 \cong \langle a \rangle : \langle b \rangle$, $C_{13} : C_4 \cong \langle a \rangle : \langle b \rangle$, $C_{11} : C_5 \cong \langle a \rangle : \langle b \rangle$, $S_5$, or $A_6$. Since $1_G \subset C_7$, $1_G \subset \langle cl(a) \rangle \subset \langle cl(b) \rangle = D_{22}$, $1_G \subset \langle cl(b) \rangle \subset \langle cl(a^2) \rangle \subset \langle cl(a) \rangle = C_{13} : C_4$, $1_G \subset \langle cl(b) \rangle \subset \langle cl(a) \rangle = C_{13} : C_3$, $1_G \subset \langle cl(b) \rangle \subset \langle cl(a) \rangle = C_{11} : C_5$, $1_G \subset Z(SL(2, 3) \subset SL(2, 3)' \subset SL(2, 3)$, $1 \subset A_5 \subset S_5$, and $1 \subset A_6$ are the unique chief series of corresponding groups, then by Theorem 2.1, $\mathcal{P}_{\mathcal{C}}(G)$ is complete.

This completes the proof. □

**Conflicts of Interest.** The author declares that there is no conflicts of interest regarding the publication of this article.

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