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Classification of Monogenic Ternary Semigroups

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Abstract

The aim of this paper is to classify all monogenic ternary semigroups, up to isomorphism. We divide them to two groups: finite and infinite. We show that every infinite monogenic ternary semigroup is isomorphic to the ternary semigroup $\mathbb{O}$, the odd positive integers with ordinary addition. Then we prove that all finite monogenic ternary semigroups with the same index and the same period are isomorphic. We also investigate structure of finite monogenic ternary semigroups and we prove that any finite monogenic ternary semigroup is isomorphic to a quotient ternary semigroup.

Keywords: Ternary semigroup, monogenic ternary semigroup, index, period.

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1. Introduction

The theory of ternary algebraic systems was introduced by D. H. Lehmer [3] in 1932, but before that (1904) such structures were studied by E. Kasner [1] who gave the idea of n-ary algebras. Lehmer studied certain ternary algebraic systems called triplexes, commutative ternary groups, in fact. Ternary structures and their generalization, the so called n-ary structures, are outstanding for their application in physics. The notion of ternary semigroup was known for the first time by S. Banach. By bringing an example, he showed that a ternary semigroup did not necessarily reduce to an ordinary semigroup ($T = \{-i, i\}$ is a ternary semigroup

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under the multiplication over complex numbers while T is not an ordinary semigroup under complex number multiplication). J. Los [4] studied some properties of ternary semigroups and he proved that every ternary semigroup could be embedded in an ordinary semigroup.

In this paper we investigate monogenic ternary semigroups and we prove that infinite monogenic ternary semigroups are isomorphic to the ternary semigroup $\mathbb{O}$, the odd positive integers with ordinary addition. We also show that there exists, up to isomorphism, exactly one finite monogenic ternary semigroup with constant index m and period r. We study some properties of the monogenic ternary semigroups. Then we characterize the structure of ideals of a finite monogenic ternary semigroup. We also prove that any finite monogenic ternary semigroup is isomorphic to a quotient ternary semigroup of a finite monogenic ternary semigroup.

2. Results

The first we express some primary notions that we need them in the next sections. A non-empty set T is called a ternary semigroup if there exists a ternary operation $T \times T \times T \to T$, written as $(a, b, c) \to abc$ satisfying the statement $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in T$.

Remark 2.1. Let T be a ternary semigroup and $m, n \in \mathbb{N}$ $(m \leq n)$ and x_1, x_2, $\ldots$, $x_{2n+1} \in T$. Then we can write

$$(x_1 x_2 ... x_{2n+1}) = (x_1 ... ((x_m x_{m+1} x_{m+2}) x_{m+3} x_{m+4}) ... x_{2n+1}).$$

Example 2.2. Let $\mathbb{N}$, $\mathbb{E}$ and $\mathbb{O}$ be the set of positive integers, even positive integers and odd positive integers, respectively. Then with the usual ternary addition of integers, $\mathbb{N}$ ($\mathbb{E}$, $\mathbb{O}$) forms a ternary semigroup.

An element e of a ternary semigroup T is said to be idempotent if $e^3 = e$. A ternary semigroup T is said to be commutative if $x_1 x_2 x_3 = x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$ for every permutation σ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in T$. Let T be a ternary semigroup. For non-empty subsets A, B and C of T, let $ABC := \{abc \mid a \in A, b \in B, c \in C\}$. A non-empty subset S of T is called a ternary subsemigroup if $SSS \subseteq S$. If A is a non-empty subset of T, then the smallest ternary subsemigroup of T containing A (the intersection of all ternary subsemigroups of T containing A) is called the ternary subsemigroup of T generated by A and it is denoted by $\langle A \rangle$. If $A = \{a\}$, then we denote $\langle \{a\} \rangle$ by $\langle a \rangle$ and we call it the monogenic ternary subsemigroup of T generated by a. It is clear that: $< A >= \{a_1 a_2 ... a_n |$ n is an odd number, $a_i \in A$ for all $1 \leq i \leq n\}$ and $< a >= \{a^n |$ n is odd number$\}$. The order of the element a is defined, as in group theory, as the order of the ternary subsemigroup $\langle a \rangle$ and we denote by $|a|$. If T is a ternary semigroup in which there exists an element a such that $T = \langle a \rangle$, then T is said to be a monogenic ternary semigroup.

A non-empty subset A of a ternary semigroup T is called, (i) a left ideal if $TTA \subseteq A$; (ii) a right ideal if $ATT \subseteq A$; (iii) a lateral ideal if $TAT \subseteq A$; (iv) a two-sided ideal if it is a left and right ideal; (v) an ideal if it is a left, right and lateral ideal.

A non-empty set Q of a ternary semigroup T is called a quasi-ideal of T, if $TTQ \cap (TQT \cup TTQTT) \cap QTT \subseteq Q$. A ternary semigroup T is called a ternary group if for every $a, b, c \in T$, the equations $abx = c$, $axb = c$ and $xab = c$ have solutions in T. An equivalence relation ρ on a ternary semigroup T is said to be a congruence if for all $a, b, c, a', b', c' \in T$, $a\rho a', b\rho b', c\rho c'$ imply $abc\, \rho\, a'b'c'$.

If ρ is a congruence on a ternary semigroup T, then for every $a \in T$ we denote the equivalent class contains a by $[a]_\rho$ and we define a ternary operation on the quotient set T/ρ, the set of all equivalent classes, by $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in T$. Note that with this ternary operation, T/ρ forms a ternary semigroup.

Let S and T be two ternary semigroups. Then the map $f : S \to T$ is called a homomorphism if $f(abc) = f(a)f(b)f(c)$ for all $a, b, c \in S$. A homomorphism $f : S \to T$ is called a monomorphism (an epimorphism) if it is one to one (onto). Also f is called an isomorphism if it is both one to one and onto. In this case we say that ternary semigroups S and T are isomorphic and we denote it by $S \simeq T$.

Theorem 2.3. *Let S and T be two ternary semigroups. Also let $f : S \to T$ be a homomorphism. Then $kerf = \{(a, b) \in S \times S \mid f(a) = f(b)\}$ is a congruence on S and $\frac{S}{kerf} \simeq Imf$.*

Proof. See [2, Theorem 3.5]. □

Now we try to classify all monogenic ternary semigroups. Let a be an element of a ternary semigroup T. Then we have two cases:

1. There are no repetitions in the list $a, a^3, a^5, \ldots$ i.e. for every odd numbers m, n, $a^m = a^n$ implies $m = n$. In this case $\varphi : (\mathbb{O}, +) \to \langle a \rangle$ by $\varphi(n) = a^n$ for every $n \in \mathbb{O}$ is an isomorphism and we say that $\langle a \rangle$, the monogenic ternary subsemigroup generated by a, is an infinite monogenic ternary semigroup and that a has infinite order in T.

2. There are $m, n \in \mathbb{O}$ such that $a^m = a^n$. Then the set
$$\{x \in \mathbb{O} \mid \exists y \in \mathbb{O}\, (y \neq x), a^x = a^y\}$$
is non-empty and so it has a least element. Let us denote this least element by m and we call it the index of the element a. Now the set $\{x \in \mathbb{E} \mid a^{m+x} = a^m\}$ is non-empty and so it too has a least element r, which we call it the period of a. We shall also refer to m and r as the index and period, respectively, of the monogenic ternary semigroup $\langle a \rangle$.

It is obvious that if a is an element of a ternary semigroup T with index m and period r then $a^m = a^{m+qr}$ for every $q \in \mathbb{N}$.

Proposition 2.4. *Let a be an element of a ternary semigroup T with index m and period r. Then,*

(i) for every $u, v \in \mathbb{E}$, $a^{m+u} = a^{m+v}$ *if and only if* $u \equiv v\,(mod\, r)$,
(ii) $< a >= \{a, a^3, \ldots, a^m, a^{m+2}, \ldots, a^{m+r-2}\}$ *and* $|a| = (m + r - 1)/2$.

Proof. By the minimality of m and r, we deduce that the elements $a, a^3, \ldots, a^m$, $\ldots, a^{m+r-2}$ are all distinct. Let s be an odd number such that $s > m$. Then $s = m + k$ for some $k \in \mathbb{E}$. Now we can, by division algorithm, write $k = qr + u$, where $q \geq 0$ and $0 \leq u \leq r-1$. It then follows that $a^s = a^{m+qr}a^u = a^m a^u = a^{m+u}$. Also since $k, r \in \mathbb{E}$, then $u \in \mathbb{E}$. Thus $a^{m+u} = a^{m+v}$ if and only if $u \equiv v\,(mod\, r)$. Therefore $< a >= \{a, a^3, \ldots, a^m, a^{m+2} \ldots a^{m+r-2}\}$ and $|a| = (m + r - 1)/2$. □

Remark 2.5. An ordinary monogenic semigroup is finite if and only if it contains one idempotent but there is no need any finite monogenic ternary semigroup to have an idempotent. However we have the following proposition.

Proposition 2.6. *A finite monogenic ternary semigroup contains an idempotent if and only if it has period* $r = 2k$ *where* k *is an odd number.*

Proof. Let $T = \langle a \rangle$ be a finite monogenic ternary semigroup with index m and period $r = 2k\,(k \in \mathbb{N})$. Also suppose that T contains idempotent element t. It is clear that $t = a^{m+p}\,(\,p \in \{0, 2, 4, \ldots, r-2\}\,)$. Now $2m + 3p \equiv p\,(mod\, r)$ since $a^{m+p} = a^{3(m+p)}$. Hence $r|\,2(m+p)$ and so $k|\,m+p$. Therefore k is an odd number since $m+p$ is an odd number. Conversely suppose that k is an odd number. Also suppose that n is an odd number such that $n \geq 3$ and $nk \geq m$. We prove that $t = a^{nk}$ is an idempotent element of T. Since $r|\,2nk$, so $r|\,(3nk - m) - (nk - m)$. Hence $3nk - m \equiv nk - m\,(mod\, r)$. Therefore $a^{3nk} = a^{nk}$ by Proposition 2.4. □

Clearly every infinite monogenic ternary semigroup does not have any idempotent. Therefore there are infinite ternary semigroups that have no idempotent. There are also infinite semigroups that every element of them is an idempotent. My favorite example of this so far is $\mathbb{Z}$, the set of integers, with ternary operation $abc = min\{a, b, c\}$.

Remark 2.7. Every finite monogenic ternary semigroup $T = \langle a \rangle$ is homomorphic image of an infinite monogenic ternary semigroup (consider the epimorphism $\varphi : \mathbb{O} \longrightarrow T$ by $\varphi(k) = a^k$ for every $k \in \mathbb{O}$). Also it is easy to see that all the homomorphic images of an infinite monogenic ternary semigroup, which are not isomorphic to it, will be finite monogenic ternary semigroup.

Let $T = \langle a \rangle$ be a finite monogenic ternary semigroup with index m and period r. Then for every $t \in \{1, 3, 5, \ldots, m\}$ we denote subset $\{a^t, a^{t+2}, \ldots, a^{m+r-2}\}$ of T by I_t.

Proposition 2.8. *Let* $T = \langle a \rangle$ *be a finite monogenic ternary semigroup with index* m *and period* r. *Then the subset* I *of* T *is an ideal of* T *if and only if* $I = I_t$ *for some* $t \in \{1, 3, 5, \ldots, m\}$.

Proof. It is clear that I_t is an ideal of T for every $t \in \{1, 3, 5, \ldots, m\}$. Conversely suppose that t is the least element of $\{k \in \mathbb{O} \,|\, 1 \leq k \leq m + r - 2, a^k \in I\}$. It is easy to verify $I = I_t$. Note that $t \leq m$ because if $t = m + s$ ($s \in \mathbb{E}$, $2 \leq s \leq r - 2$), then $a^m = a^{m+r} = a^t a^{r-s} \in I$. Therefore $a^m \in I_t$ and it is a contradiction. $\square$

Since every monogenic ternary semigroup is a commutative ternary semigroup we have the following proposition:

Proposition 2.9. *Let $T = \langle a \rangle$ be a finite monogenic ternary semigroup with index m and period r. Then the subset I of T is an left (right, lateral) ideal of T if and only if $I = I_t$ for some $t \in \{1, 3, 5, \ldots, m\}$.*

Remark 2.10. Every left (right, lateral) ideal of a ternary semigroup T is a quasi-ideal of T but the converse is not true generally. However in a monogenic ternary semigroup these concepts coincide.

Let $T = \langle a \rangle$ be a finite monogenic ternary semigroup with index m and period r. Then from Proposition 2.9, the subset $K_a = \{a^m, a^{m+2}, \ldots, a^{m+r-2}\}$ of T is a minimal ideal of T. We call K_a, the kernel of $\langle a \rangle$.

Proposition 2.11. *Let $T = \langle a \rangle$ be a finite monogenic ternary semigroup with index m and period r. Then K_a, the kernel of $\langle a \rangle$, is a maximal subgroup of T.*

Proof. Clearly K_a is a ternary semigroup. Suppose that a^{m+u}, a^{m+v} and a^{m+w} are arbitrary elements of K_a. Choose $x \in \mathbb{E}$ such that $x \equiv w - u - v - 2m \, (mod\, r)$ and $0 \leq x \leq r-2$. So, we have $a^{m+u}\, a^{m+v}\, a^{m+x} = a^{m+u}\, a^{m+x}\, a^{m+v} = a^{m+x}\, a^{m+u}\, a^{m+v}$ $= a^{m+w}$. Therefore K_a is a ternary subgroup of T. Now let L be a subgroup of T such that $K \subseteq L$. Also let a^t is an arbitrary element of L. Then there exists $k \in \mathbb{O}$ such that $a^t a^t a^k = a^t$. So $a^{2t+k} = a^t$. Hence $t \geq m$ and $K = L$. Therefore K is a maximal subgroup. $\square$

Example 2.12. $\mathbb{Z}_n$ with the usual multiplication of $\mathbb{Z}_n$ is a ternary semigroup for every $n \in \mathbb{N}$. $\bar{2} \in \mathbb{Z}_8$ has index 3 and period 2 but $\bar{2}$ as an element of $\mathbb{Z}_{20}$ has index 3 and period 4.

Let X be a non-empty set. Then we denote the set of all maps from X into X by T_X. T_X with ternary composition of maps is a ternary semigroup. Ternary semigroup T_X is called the full transformation ternary semigroup on X. If T is a ternary subsemigroup of T_X, then we say that T is a transformation ternary semigroup.

Example 2.13. Let $X = \{1, 2, \ldots, 7\}$ and consider the element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 5 \end{pmatrix}$$

of the ternary semigroup T_X. Then it is easy to calculate that:

$$\begin{aligned}
\alpha^3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 5 & 6 & 7 \end{pmatrix}, \\
\alpha^5 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 5 & 6 & 7 & 5 & 6 \end{pmatrix}, \\
\alpha^7 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 5 & 6 & 7 & 5 \end{pmatrix}, \\
\alpha^9 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 6 & 7 & 5 & 6 & 7 \end{pmatrix}, \\
\alpha^{11} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 5 & 6 & 7 & 5 & 6 \end{pmatrix}.
\end{aligned}$$

So $\alpha^5 = \alpha^{11}$. Thus α has index 5 and period 6. Also $K_\alpha = \{\alpha^5, \alpha^7, \alpha^9\}$.

Proposition 2.14. *For every $m \in \mathbb{O}$ and $r \in \mathbb{E}$, there exists a ternary semigroup T containing an element a of index m and period r.*

Proof. It is easy to verify that the element

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & m+1 & \dots & m+r-1 & m+r \\ 2 & 3 & \dots & m+2 & \dots & m+r & m+1 \end{pmatrix}$$

of the semigroup T_X has index m and period r, where $X = \{1, 2, \dots, m+r\}$. $\square$

Theorem 2.15. *Let a and b be elements of finite order in the same or different ternary semigroup. Then $\langle a \rangle \simeq \langle b \rangle$ if and only if a and b have the same index and the same period.*

Proof. Suppose that a and b have the same index m and period r. Then $\langle a \rangle = \{a, a^3, \dots, a^m, \dots, a^{m+r-2}\}$ and $\langle b \rangle = \{b, b^3, \dots, b^m, \dots, b^{m+r-2}\}$. It is clear that for every $i \in \mathbb{O}\,(1 \le i \le m+r-2)$, $\varphi : \langle a \rangle \to \langle b \rangle$ by $\varphi(a^i) = b^i$ is an isomorphism. Therefore $\langle a \rangle \simeq \langle b \rangle$.

Conversely, let $\varphi : \langle a \rangle \to \langle b \rangle$ be an isomorphism. Also let a has index m and period r and b has index m' and period r'. Then $m + r = m' + r'$ since $|\langle a \rangle| = |\langle b \rangle|$. Moreover suppose that $\varphi(a) = b^t$ and $\varphi^{-1}(b) = a^s(\,t, s \in \mathbb{O},\, 1 \leqslant t \leqslant m'+r'-2, 1 \leqslant s \leqslant m+r-2\,)$. Then we have $a = \varphi^{-1}\varphi(a) = \varphi^{-1}(b^t) = (a^s)^t = a^{st}$ and $b = \varphi\varphi^{-1}(b) = \varphi(a^s) = b^{st}$. Thus either $m = m' = 1$ or $s = t = 1$. Therefore $\varphi(a) = b$ and $\varphi^{-1}(b) = a$. Now $\varphi(a^{m'}) = b^{m'} = b^{m'+r'} = \varphi(a^{m'+r'})$ and $\varphi^{-1}(b^m) = a^m = a^{m+r} = \varphi^{-1}(b^{m+r})$. Hence $a^{m'} = a^{m'+r'}$ and $b^m = b^{m+r}$ and consequently $m \le m'$ and $m' \le m$. Therefore $m = m'$ and consequently $r = r'$. $\square$

By attention to previous theorem, for every $m \in \mathbb{O}$ and $r \in \mathbb{E}$ there exists, up to isomorphism, exactly one monogenic ternary semigroup. Let us we denote this monogenic ternary semigroup by $MT(m, r)$.

By attention to Proposition 2.16, $MT(1,r)$ is a ternary group for every $r \in \mathbb{E}$ which we call it cyclic ternary group.

Proposition 2.16. *For every $m \in \mathbb{O}$, $m \neq 1$ and $r \in \mathbb{E}$, $T = MT(m,r)$ has exactly one generator.*

Proof. Suppose that $T = \langle a \rangle = \langle b \rangle$. Then $T = \{a, a^3, \dots, a^m, \dots, a^{m+r-2}\}$. We prove that $T \setminus T^3 = \{a\}$. We have $a^3 \in T^3, a^5 = a^3aa \in T^3, \dots, a^{m+r-2} = a^{m+r-4}aa \in T^3$. Now if $a \in T^3$, then $a = a^k$ for some $k \in \mathbb{O}$. Thus $m = 1$ that is a contradiction. Hence, $T \setminus T^3 = \{a\}$. By the same way $T \setminus T^3 = \{b\}$. Therefore $a = b$. $\square$

Let T be a commutative ternary semigroup. Then for every $n \in \mathbb{E}$ define the relation θ_n^T on T by $a\ \theta_n^T\ b$ if and only if $x_1x_2\dots x_na = x_1x_2\dots x_nb$ for every $x_1, x_2, \dots, x_n \in T$. It is clear that θ_n^T is a congruence on T and $\theta_2^T \subseteq \theta_4^T \subseteq \theta_6^T \subseteq \dots$. For every $n \in \mathbb{E}$, we denote ternary semigroup $\frac{T}{\theta_n^T}$ by T_n.

Lemma 2.17. *Let T be a commutative ternary semigroup. Then $T_n \simeq \frac{T_{n-2}}{\theta_2^{T_{n-2}}}$.*

Proof. Define $\varphi : T_{n-2} \longrightarrow T_n$ by $\varphi([a]_{\theta_{n-2}}) = [a]_{\theta_n}$ for every $a \in T$. If $[a]_{\theta_{n-2}} = [b]_{\theta_{n-2}}$ then $(a,b) \in \theta_{n-2} \subseteq \theta_n$. Hence $[a]_{\theta_n} = [b]_{\theta_n}$. So φ is well-defined . Also, φ is an epimorphism clearly. Now if $a, b \in T$ then

$$\begin{aligned}
[a]_{\theta_{n-2}}\ \theta_2^{T_{n-2}}\ [b]_{\theta_{n-2}} &\Longleftrightarrow [xya]_{\theta_{n-2}} = [xyb]_{\theta_{n-2}}\,(\,\forall x, y \in T\,) \\
&\Longleftrightarrow x_1\dots x_{n-2}xya = x_1\dots x_{n-2}xyb\,(\,\forall x_1, \dots, x_{n-2}, x, y \in T\,) \\
&\Longleftrightarrow a\,\theta_n\,b \\
&\Longleftrightarrow \varphi([a]_{\theta_{n-2}}) = \varphi([b]_{\theta_{n-2}}) \\
&\Longleftrightarrow [a]_{\theta_{n-2}}\ ker\varphi\ [b]_{\theta_{n-2}}.
\end{aligned}$$

Therefore ker $\varphi = \theta_2^{T_{n-2}}$ and consequently $T_n \simeq \frac{T_{n-2}}{\theta_2^{T_{n-2}}}$. $\square$

Theorem 2.18. *Let $m \in \mathbb{O}, r \in \mathbb{E}\,(m \neq 1)$. Then for every $n \in \mathbb{E}\,(n < m)$,*

$$\frac{MT(m,r)}{\theta_n^{MT(m,r)}} \simeq MT(m-n,r).$$

Proof. We first prove that $\frac{MT(m,r)}{\theta_2^{MT(m,r)}} \simeq MT(m-2,r)$. Suppose that $T = MT(m,r)$ $= \langle a \rangle = \{a, a^3, \dots, a^{m-2}, a^m, a^{m+2}, \dots, a^{m+r-2}\}$ and $[a^{n_1}]_{\theta_2} = [a^{n_2}]_{\theta_2}$ for some $a^{n_1}, a^{n_2} \in MT(m,r)$. Then $aaa^{n_1} = aaa^{n_2}$. So $a^{n_1+2} = a^{n_2+2}$. If $n_1 \neq m+r-2$ and $n_2 \neq m+r-2$ then $n_1 = n_2$. On the other hand, $[a^{m-2}]_{\theta_2} = [a^{m+r-2}]_{\theta_2}$ since $a^{t_1}a^{t_2}a^{m-2} = a^{t_1}a^{t_2-2}a^m = a^{t_1}a^{t_2-2}a^{m+r} = a^{t_1}a^{t_2}a^{m+r-2}$ for every $a^{t_1}, a^{t_2} \in MT(m,r)$. This shows that $\frac{MT(m,r)}{\theta_2^{MT(m,r)}} = \{[a], [a]^3, \dots, [a]^{m-2}, [a]^m, \dots, [a]^{m+r-4}\}$ $\simeq MT(m-2,r)$. Now $\frac{T}{\theta_4^T} = T_4 \simeq \frac{T_2}{\theta_2^{T_2}} \simeq \frac{MT(m-2,r)}{\theta_2^{MT(m-2,r)}} \simeq MT(m-4,r)$ and by induction we deduce that $\frac{MT(m,r)}{\theta_n^{MT(m,r)}} \simeq MT(m-n,r)$ for every $n \in \mathbb{E}$. $\square$

Conflicts of Interest. The author declares that there is no conflicts of interest regarding the publication of this article.

References

[1] E. Kasner, An extension of the group concept (reported by L.G. Weld), *Bull. Amer. Math. Soc.* **10**(6) (1904) 290 – 291.

[2] S. Kar and B. K. Maity, Congruences on ternary semigroups, *J. Chungcheong Math. Soc.* **20**(3) (2007) 191 – 201.

[3] D. H. Lehmer, A ternary analogue of abelian groups, *Amer. J. Math.* **54**(2) (1932) 329 – 338.

[4] J. Los, On the extending of models (I), *Fund. Math.* **42** (1955) 38 – 54.

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