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Architecture, City and Mathematics: The Lost Connection

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Abstract

The connection between architecture and sound based on mathematical relations has continued to develop since the rise of the Western classical civilization that originated in Ancient Greece. The mysterious Pythagorean cosmology pursued as secret esoteric knowledge was related to the search of rhythm, proportionality and harmony. Even somewhat earlier, Greek mysteries were based on a concord of music and form; accordingly architectural elements were to conform to musical notes. This line of reasoning can be traced as early as when the doctrines of Orphism emerged in early Greece to be followed by the concepts of Pythagoras and his followers and eventually by the philosophical school of Neo-Platonists. Early medieval Christian thinkers like St. Aurelius Augustine and Boethius revived and continued this ancient pagan tradition; they sustained and developed further the ideas of dependence between architecture and music (as well as mathematics). Their ideas were further elaborated by later Christian thinkers. Architectural principles practiced by the architects belonging to the Western tradition were passed further on, especially after the discovery of the writings by Vitruvius in the late medieval period. The Pythagorean tradition was still alive during the Renaissance and even Baroque.

This tradition was gradually marginalized and forgotten with the rise of scientific and eventually ascientistmentality developed during the post-Renaissance era. However, the roots of the application of mathematics and geometry to the design of urban settlements have survived. Such principles can be still observed while studying the early patterns of Western as well as non-Western civilizations, and thus one can speak about the universal mathematical/geometric character of early urban design.

Keywords: Architecture, urbanism, music, harmony, geometry, mathematical numbers, Pythagoreanism.

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1. Introduction

Having been invited to speak at the international conference Architecture and Mathematics held at University of Kashan, I felt ill at ease at least for a while. The discipline of mathematics belongs to the hard sciences and hard sciences produce predictions that are generally testable and are based on the performance of tightly controlled experiments, and in addition they rely on quantification of data and last but not least−application of mathematical models−they are believed to be based ona higher degree of accuracy and objectivity. Meanwhile, the humanities to which I belong are often suspected of subjectivity, lack of experimental character or of ambivalence in providing verifiable data and various other qualities that make 'soft' sciences quite a different stock of knowledge that often operate under a different logic than the so-called 'scientific method'. It took some time to find a way of how to approach the general theme of this important scientific as well as scholarly event focused on the historic and contemporary relations between two disciplines that are so different. While structuring the scope of my own presentation and struggling with the idea of how to approach the suggested topic meaningfully, I soon realized that I was not alone with this feeling of a certain awkwardness. More than two decades ago in his key-note address to the members of one of the Nexus conferences that explored relations between the *science of mathematics* and the *art of architecture*, Mario Salvadori, a renowned Italian mathematician and engineer spoke about his difficulty of looking into relations between "a science as abstract as mathematics and an art as concrete as architecture" and saw their territories as 'theoretically inconceivable'. However, after some reconsideration while musing on the issue he famously said that after exchanging his 'mathematical hat' into his 'engineering hat' his initial disquisition on the impossibility of relating mathematics and architecture gradually vanished, and speaking as a technologist he agreed with his learned audience 'that these relations are many and so important that if mathematics had not been invented, architects would have had to invent it themselves' [8]. Sharing the same initial concern and taking this important note for granted, I felt more comfortable in approaching the long and complex story of relations between architecture and mathematics and illuminating at least some aspects of their historical interactions. My own discussion of these connections that were once perhaps far closer and more metaphysically fundamental than we are ready to comprehend today, will hereby start from Antiquity. As I have already explored these relations to a certain degree elsewhere, I will also provide a brief summary of my earlier findings while researching some aspects of this issue.

2. The Contribution of the Greeks

In what is known these days as Western tradition, the Ancient Greek thinkers were very much concerned first and foremost with the nature of sound. The early Greek thinkers that left us traces of their reasoning, maintained that a cosmic sound [or drone] is generated by the seven planets revolving as they believed around the planet Earth. However, as they presumed−a human being was totally unable to hear it simply because of the imperfections of his sensory organs. It was during this period that esoteric, quasi-philosophical movements like Orphism and Pythagoreanism came into being and their members maintained that music *is a key in order to comprehend the mysteries of this world.* Orphism attributed a special mystical meaning to Orpheus−who was able as Greeks believed to reveal the secrets of the Gods to human beings with the help of music. Furthermore, he was even believed to have been able to compete with Gods in the power of prophetic insight [7].

It is speculated today that Orpheus might have lived in Thrace in the so-called post-Homeric period and that it was he who was the first to convey to people the meaning of the rituals of initiation. Perhaps that is why after his death when Orpheus became a symbol of wisdom and what is perhaps even more important−the object of a religious cult and following−he was placed on the same footing as his teaching. Since then he seems to have been worshipped as the son of Apollo−a divine and perfect truth, and Calypso−Muse of harmony and rhythm. Accordingly, Orphism was taken as a secret doctrine or otherwise Apollo revealed through music, i.e. the Muse by the name of Calypso. Even though the Orphic movement was not a mainstream phenomenon in Greece and might be more adequately termed as a kind of marginal cultural phenomenon, nevertheless some of its ideas and views entered other and somewhat more widely accepted doctrines, especially the cosmological teaching of Pythagoras who by the way, considered Orpheus his own guardian, and eventually some essential elements of Orphism penetrated into other schools of thought, including the Neo-Platonists.

It is well known that in the teachings of Pythagoras and his followers, special attention was given to the concept of the *world harmony* that was also called 'the music of the heavenly spheres'. Thus for Pythagoreans, cosmos meant an orderly or to be more precise − proportionate system that was treated as a *mere harmony.* Pythagoreans interpreted the universe as a gigantic musical instrument − the monochord − the upper part of its string stretching to the absolute spirit (i.e. heavens) and the lower part resting on absolute matter (i.e. the earth). Of course the monochord was not an instrument designed to be played. Instead it can be called a philosophical instrument that is believed to have been invented and constructed as a means for producing analogies between music (sound) and mathematics. According to George L. Hersey, it was used to demonstrate the relationship between the height of musical tones and the physical intervals of musical instruments. Although it is an ancient story which tells that the relationship between numbers and sounds was allegedly discovered by Pythagoras while he was walking through a smithy and realized that the hammers produced different sounds

– later it was dismissed as a true story, and is believed to be an old Asiatic legend (the same for that matter can be said of the story of Wilhelm Tell made famous by Friedrich von Schiller). It is nevertheless believed that Pythagoras conducted experiments with the monochord [3].

It is also difficult to say how many spheres Pythagorean cosmology in fact contained as authors still disagree on their exact number – ranging from seven to twelve, yet what Pythagoreans seem to have firmly believed was the idea that the movement of the planets produced sound. Consequently they reasoned that the lowest tone belongs to the planet of Saturn, while the highest – to the Moon. The distances between the individual planets were likened to musical intervals. When Pythagorean teaching was firmly consolidated, attempts were made to prove rationally that nature, man and music are governed by the same laws. These laws were no other than mathematical.

Moreover, since human senses are imperfect – only mathematical laws could be of any help. Thus a sequence of numbers was established to analyze the relations between the universe, the human body and music. This sequence was comprised from even numbers that were called female and odd numbers – treated as male. The numbers were believed to be ruled by a primeval non-divisible element of being – the monad.

For Pythagoreans, abstract numbers also represented a physical body: one meant a dot, two – a line, three – a triangle, four – a pyramid. According to the followers of Pythagoras teaching, the structure of the world matched the fifth and the sixth intervals of the octave – consonances. Thus Pythagoreans were perhaps the first school of thought in the history of Western civilization to suggest and to prove a relationship between musical intervals and simple proportions.

The traces of Pythagorean reasoning can be also found in the writings of Plato, especially the ones of his later period. Plato claimed that it was God who "devised the gift of sight for us that we might observe the movements which have been described by reason in heavens, and apply them to the motions of our own mind, which are akin to them so far that what is troubled can claim kinship to what is serene..."

Plotinus – another important Greek thinker of the so-called Hellenistic period took Platonic thinking seriously and insisted that the beauty of sounds should be measured by mathematical numbers. As the modes of sounds do not reach the human ear – thus by creating these modes that can be heard, the sensual sound should be measured with the help of numbers – i.e. mathematics was called into play. This particular line of reasoning proceeded further with Roman thinkers and the authors Cicero and Macrobius. The latter argued that while considering the movement of the spheres, the Soul of the Universe created tones separated by uneven but clearly settled intervals of proportions, corresponding to the primary material of creation. These proportioned intervals according to him gave birth to Harmony.

3. The Medieval Approach Toward Harmony

Vitruvius, or Marcus Vitruvius Pollio – the Roman architectural theorist who was especially esteemed by the scholars and authors during the Middle Ages as well as the Renaissance immediately after his writings were discovered in the late medieval period, seems to have accommodated the principles of Pythagorean harmony and applied them to architectural practice. In his writings, architecture was understood as a harmonious relation between the parts and the whole as well as the well-composed interaction of elements based on the rules of mathematics and geometry.

Medieval European thinkers, most of whom were Christian monks appealing to the authority of Vitruvius based their theories on the importance of proportions and mathematical numbers. According to the research done by the renowned scholar of Western medieval architecture Otto von Simpson, both – St. Aurelius Augustine and Boethius – i.e. canonical Christian authors of the period – treated music and architecture as sisters, because both were the 'daughters of number' – i.e. mathematics. Boethius ascribed music to sciences as sounds were subordinate to numbers. As von Simpson adequately emphasized, 'the most admirable ratio' according to Augustine is that of equality or symmetry, the ratio 1:1, since the union or consonance between the two parts is most intimate. Next in rank are the ratios 1:2, 2:3, 3:4 – the intervals of perfect consonances, octave, fifth and fourth. In the abbey of Fountenay (1130-47) – the best summary of Cistercian architecture – St. Bernard may himself be responsible for its plan – the octave ratio determines the great plan as well. Following Pythagorean line of thinking, Boethius claimed that 'the sense should be subjected to reason', i.e. mathematical laws should be followed, and he praised Pythagoras because it was he who rejected 'the estimation of the ear'. As Umberto Eco has noted this was the attitude of scientists – typical to the early medieval period. As I have already written elsewhere, 'The identification of music with reason was institutionalized: musical theory was taught in medieval university programs among the seven liberal arts (septemartisliberalisin Latin) as an essential discipline, equal to mathematics and astronomy. Boethius, in the same way as Aristotle before him, was convinced that music influences not just hearing, but also the human soul – certain rhythms and melodies help to reinforcethe morality of a human being. He concluded that the state of the human body and soul is ruled by the same proportions that unite and attune 'harmonious modulations'. For both St. Augustine and Boethius – authors who created the outlines of the medieval concept of beauty – music's dependence on number was a foundation of a quantitative aesthetical doctrine [2].

The analogy between music and architecture in the Middle Ages was expressed in philosophical metaphors: when medieval philosophers described God as an artistic architect (*elegansarchitectus* in Latin) who converted the whole universe into his palace and harmonized all the creation with the help of musical proportions – a specific architectural practice eventually followed, especially during the Gothic period [7].

As William Alexander McLung noted in The Architecture of Paradise, ‘That architecture mirrors divine forms mathematically is implicit in Augustinian aesthetics, which anchor the beauty in metaphysical reality, finding it chiefly in the art governed by the immutable numerical ratios of Pythagorean mysticism; music and architecture’.

4. Pythagoreanism in the Legacy of the Renaissance

The thinkers of the Renaissance era presented numerous attempts to develop a system of architectural proportions built on the principles of musical harmony. For example, arenowned figure of the period, Pico de Mirandola, discussed the bonds between Orphic and Pythagorean thinkers. The Pythagorean system seems to have been of utmost importance to Renaissance thinkers as well, as it was staunchly defended by the renowned musical theorist Vincenzo Galilei – the father of the great astronomer Galileo Galilei. The question, however, remains – how mathematical and musical principles were applied in practice? While reconsidering this particular issue, research done by the renowned art historian Ruddolf Wittkower remains instrumental to this very day. According to Wittkower, the architect Andrea Palladio was one of those practitioners that implemented the ideas of Pythagoras and his school: ‘In his exposition, Palladio sticks to the practical side of the meteier. Without mentioning what his proportions really signify. In actual fact, in his three examples the height of the room represents the arithmetic, geometric and harmonic means < . . . > These three types are attributed to Pythagoras and without them no rational system of the theory of proportions can be imagined [9]. A disciple of the spirit of the Renaissance, Andrea Palladio was deeply concerned with an old Pythagorean tradition. He agreed with Vitruvius’ statement that any edifice should meet at least three requirements: be useful, be durable and finally, be beautiful. The Palladian concept of beauty stated that the phenomenon of beauty consists of beautiful form and concord between the whole and its constituent parts. Thus he turned to geometry in order to explain the most beautiful and proportionate types of rooms and claimed that their form should be square or round and the length equal to the diagonal of the square and the width – to the edge of the square. Also a room considered beautiful could be a square plus one third of its original size and also it could be two squares. Palladio argued that the proportions of sound and space are closely related, thus it led him to believe in the universal value of harmonious system.

His fellow countryman Leone Battista Alberti indicated his indebtedness to musicians – i.e. those studying the laws of music – and described harmonious relationship as *finitio*. He further enumerated the numbers that condition those harmonious relations. According to Alberti, nature is based on the principle of the triad, the number five governs many things, the divine number seven indicates the seven planets, glory of creation, development and maturity. Among uneven numbers, nine is of great importance as it reflects the heavenly spheres. As Alberti seems to have firmly believed, even numbers also bear their own importance, because ‘some philosophers maintain that the four fold is consecrated to divinity < . . . > The six fold is one of the very few which is called ‘perfect’ because it is the sum of all its integral divisors. < . . . > eightfold exerts a great influence on Nature. Those born on eight month will not survive, we have observed except in Egypt. Among even numbers ten is the most important since Aristotle called it ‘the most perfect number of all’ [1].

The tradition of applying analogies between music, mathematics and architecture set up by Pythagoreans continued throughout the Renaissance and Baroque. As George L. Hersey has emphasized, during the reign of Baroque ‘geometrical principles and practices were gathered into a set of transcendent beliefs about the architecture of the cosmos. Human geometric activity, whether in building design or anything else, and whether dealing with effable shapes, effable sequences, the golden section, or the Fibonacci sequence, was encompassed and presided over by a gigantic vision of similar shapes and rations – those of the universe itself. In contrast to our current vision of space as a vast emptiness, people in the seventeenth century assumed that Earth and its fellow planets were embedded in translucent spheres and framed by other geometric presences, shapes that were invisible and impalpable but nonetheless real [3].

This worldview, however, was destined to give way to new scientific concepts of the universe and the Pythagorean legacy was eventually lost.

5. Geometry and the Historical City

The origins of the city are closely associated with certain geometric patterns that shaped cities in the early stage of their developments. One of the best students of urban history, Lewis Mumford, has observed that the early Greek colonies in Asia minor and Italy as early as the seventh century contain a checkerboard plan. According to Mumford, This Milesian planning introduced, almost automatically, two other elements: streets of uniform width and city blocks of fairly uniform dimensions. The city itself was composed of such standardized block units: their rectangular open spaces, used form agora or temple, were in turn simply empty blocks. If this formal order was broken by the presence of hill or a curved bay, there was no effort at adaptation by a change of pattern. With this plan goes a clarification of functions and a respect for convenience: so the agora shifted toward waterfront, to be near the incoming ships and warehouses. Geometric order, once established in the general plan of the city, penetrated its architectural conception as well. From Miletus, possibly through the work of Hippodamos, came a new type of agora, a formal rectangle, surrounded by a wall of shops on at least three sides. This geometric plan was not an easy one to apply to sites with an irregular topography; but it had one advantage that gave it currency in the sixth century and made it universal once more in the third century B.C.: it provided a simple and equitable method of dividing the land in a new city formed by colonization

[4].

Romans were particularly concerned with geometric shapes and applied them in urban planning practices. As Joseph Rykwert has accurately observed: 'Before the Roman cities assumed the gridiron pattern familiar to us, the idea of a regular city plan gad to be formed in their minds. The rectilinear city was not something at which they arrived by hit-or-miss experimentation, and explained afterwards' [5]. In his classical and monumental survey of the origins and development of world's cities, Lewis Mumford emphasizes the close connections between Hellenistic and Roman cities. Let me quote Mumford once again: 'in addition to its sacred outline, the Roman city was oriented to harmonize with cosmic order. The typical mark that distinguished it from Hellenistic cities of the same general character was the layout of its two principal streets, the *cardo*, running north and south, and the *decummannus*, running east and west. This axial type of town, with its two main streets crossing at right angles near the center, is an old form: Badawy finds the earliest recorded examples in the fortresses built on rocky islands or banks of the Nile during the Twelfth Dynasty. Fortress, camp, and city have a common base in military regimentation [4]. Furthermore, Mumford draws our attention to the fact that 'From the Hellenistic town the Roman received a pattern of esthetic order that rested on a practical base; and to each of the great institutions of Milesian planning − the formally enclosed agora, with its continuous structures, the broad unbroken street line with buildings, and the theater − the Roman gave a characteristic turn of his own, outdoing the original ornateness and magnificence. The places were these two streams of influence came together in the Roman mind were in the African and Syrian towns, often greatly developed as specialized manufacturing towns and trade centers, or in military colonization towns, founded to serve as holding points for the Empire, permanently stocked with legionaries who would be called back to action [4]. Classical Roman towns were most usually either circular or square in their form.

Geometric patterns played an important role in the lay-out of early cities in other regions as well. In China, the town originated at the same time as elsewhere in the world − two and three thousand years BC. The ancient Chinese city had a gridiron plan that was associated with agricultural society and division of fields into smaller squares. The square form was also related to the philosophical concept of a circular sky and a square earth, besides it had a direct relation to the balance between yin and yang −male and female elements as well as five essential elements that Chinese believed to be fundamental − i.e. water, fire, earth, wood and metal. The notion of dualism conditioned the central axis of towns and their symmetry. For example the city of Changan − the old Chinese capital (now Xian) had a sophisticated form and was laid-out as a chess-board. During later periods in Chinese history, the five elements and five numbers were related to a 9-part square. Number nine was associated with the turtle − an animal that was considered to be a universal symbol of order as the lower part of his back is square and the upper is vault-shaped. Because of these qualities the turtle became a universal cosmic symbol in Chinese civilization.

Early Indian cities had a complex geometrical character due to the application of the Vastu Purusha-Mandala. It was comprised of a square with the symbol of cosmic man who is pressedonit and subdivided by various deities. *Vastu* refers to a site (physical space), *purusha* indicated man, and meanwhile, the mandala represents a closed polygon. Though the mandala in principle can be derived from any polygon, the favorite shape is usually square as it is the most basic, rational and elementary among all known geometric forms. The town designed with the help of the mandala was supposed to be an exact rectangle [8]. On the outer borders of the mandala − the wall was usually erected. The mandala also provided detailed instructions about the network of streets. This can be seen from a vivid example of Jaipur, designed by the architect Vidyadhar who was also an astrologer and mathematician.

It must be emphasized, however, that the metaphysical and philosophical roots of the ancient cities were finally abandoned and the practice of urban design gave up the ancient wisdom even before the dawn of modernity. The urban design of the modern era still favored a gridiron plan; however, without any relation to former metaphysical reasoning. Several decades ago, the renowned architectural theorist and mathematician Christopher Alexander emphasized in his early seminal study *The City is not a Tree* that simplistic mathematical modeling causes more harm than it provides benefits, and more recently he explored mathematics more thoroughly in his monumental study, the *Order of Things* and *Pattern Language*− co-written with a group of his fellow researchers. More recently urban and architectural theorist and mathematician, Nikos Salingaros has demonstrated that if correctly applied mathematics can help to revitalize contemporary cities that suffer among many other things from simplistic planning principles. On numerous occasions he has referred to the work done by mathematician and architectural theorist Christian Alexander who has made an important contribution to understanding how mathematics is inherent to architecture. The discussion of their ideas and critique of modernist practice, however, goes beyond the scope of this essay. Nevertheless I would like to end this concise discussion with a quotation from Nikos Salingaros: "Historically, architecture was part of mathematics, and in many periods of the past, the two disciplines were indistinguishable. In the ancient world, mathematicians were architects whose constructions − the pyramids, ziggurats, temples, stadia and irrigation projects − were marvel at today. In Classical Greece and ancient Rome, architects were required to also be mathematicians. When the Byzantine emperor Justinian wanted an architect to build the Hagia Sophia as a building that surpassed everything ever built before, he turned to two professors of mathematics (geometers), Isidoros and Anthemisios, to do the job. This tradition continued into the Islamic civilization. Islamic architects created a wealth of two dimensional tiling patters centuries before western mathematicians gave a complete classification [6].

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