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$(c, 1, \ldots, 1)$ Polynilpotent Multiplier of some Nilpotent Products of Groups

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Abstract

In this paper we determine the structure of $(c, 1, \ldots, 1)$ polynilpotent multiplier of certain class of groups. The method is based on the characterizing an explicit structure for the Baer invariant of a free nilpotent group with respect to the variety of polynilpotent groups of class row $(c, 1, \ldots, 1)$.

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1. Introduction and Preliminaries

There have been several papers, on structure of the well-known notion of the Schur multiplier and its varietal generalization, the Baer invariant, of some famous products of groups, such as direct products, free products, nilpotent products and regular products.

I. Schur [12] in 1907 and J. Wiegold [13] in 1971 obtained the structure of the Schur multiplier of the direct product of two finite groups as follows:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus \frac{[A,B]}{[A,B,A*B]}, \text{ where } \frac{[A,B]}{[A,B,A*B]} \cong A_{ab} \otimes B_{ab}.$$

In 1979, M. R. R. Moghaddam [8] and in 1998, G. Ellis [1], generalized the above result and obtained the structure of the c-nilpotent multiplier of the direct product of two groups, $\mathfrak{N}_c M(A \times B)$. Also in 1997 M. R. R. Moghaddam in a

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joint paper [9] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group.

W. Haebich In 1972 [2] presented a formula for the Schur multiplier of a regular product of a family of groups. It is familiar that the regular product is a generalization of the nilpotent product and the last one is a generalization of the direct product, so Haebich's result is an interesting generalization of the Schur-Wiegold result.

In 2001, Mashayekhy [5] found a structure similar to Haebich's type for the c-nilpotent multiplier of a nilpotent product of a family of cyclic groups. Also the c-nilpotent multiplier of a free product of some cyclic groups was studied by Mashayekhy [6] in 2002. Mashayekhy and Parvizi [7] concentrated on the Baer invariant with respect to the variety of polynilpotent groups, and succeeded to present an explicit formula for the polynilpotent multipliers of finitely generated abelian groups. They also [11] obtained an explicit formula for some polynilpotent multipliers of an n-th nilpotent product of some infinite cyclic groups, that is

$$\mathfrak{N}_{c,1}M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \dots \overset{n}{*} \mathbb{Z}), \text{ for all } c > 2n-2.$$

Recently Hokmabadi, Mashayekhy and Mohammadzadeh [3, 4] studied polynilpotent multipliers of nilpotent products of cyclic groups.

Let $\mathfrak{N}_{c,1,\dots,1}$ be the variety of polynilpotent groups of class row $(c, 1, \dots, 1)$, and G be an arbitrary group with a free presentation

$$1 \to R \to F \to G \to 1.$$

The Baer invariant of G with respect to this variety is:

$$\mathfrak{N}_{c,\underbrace{1,\dots,1}_{i-times}}M(G) \cong \frac{R \cap \overbrace{\gamma_2(\dots\gamma_2}^{i-times}(\gamma_{c+1}(F))\dots)}{[R, {}_cF, \gamma_{c+1}(F), \gamma_2(\gamma_{c+1}(F)), \dots, \underbrace{\gamma_2(\dots\gamma_2}_{(i-1)-times}(\gamma_{c+1}(F))\dots)]},$$

where $\gamma_2(\dots\gamma_2(\gamma_{c+1}(F))\dots)$ is the term of iterated lower central series of F. The Baer invariant of G with respect to this variety, is called a $(c, 1, \dots, 1)$ *polynilpotent multiplier.*

The paper is organized as follows. In the present section we establish the concepts and preliminary theorems which are used throughout the paper. Section 2 is devoted to investigate the structure of $(c, 1, \dots, 1)$ polynilpotent multiplier of a free nilpotent group of class at most n and rank m for all $2n-2 < c$. The result of this section generalizes the work of [11]. The structure of Baer invariant of certain class of groups with respect to the variety of polynilpotent groups of class row $(c, 1, \dots, 1)$, is determined in section 3. The results in this paper are a part of MSc dissertation of the first author at the Ferdowsi University of Mashhad.

Definition 1.1. Let X be an independent subset of a free group, and select an arbitrary total order for X. We define the basic commutators on X, their weight wt, and the ordering among them as follows:

(1) The elements of X are basic commutators of weight one, ordered according to the total order previously chosen.

(2) Having defined the basic commutators of weight less than n, the basic commutators of weight n are the $c_k = [c_i, c_j]$, where:

(a) c_i and c_j are basic commutators and $wt(c_i) + wt(c_j) = n$, and

(b) $c_i > c_j$, and if $c_i = [c_s, c_t]$ then $c_j \geq c_t$.

(3) The basic commutators of weight n follow those of weight less than n. The basic commutators of weight n are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] \leq [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

The next two theorems are vital in our investigation.

Theorem 1.2. *(Hall, 1959). Let $F = \langle x_1, x_2, \ldots, x_d \rangle$ be a free group, then*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}\ , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

Theorem 1.3. *(Witt Formula). The number of basic commutators of weight n on d generators is given by the following formula:*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

where $\mu(m)$ is the Möbius function, which is defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \ldots p_s, \end{cases}$$

where the p_i, $1 \leq i \leq k$, are the distinct primes dividing m.

Proof. See Hall (1959). □

2. The structure of $\mathfrak{N}_{c,1,\ldots,1} M(\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z})$

In this section we obtain an explicit structure for the $(c, 1, \ldots, 1)$ polynilpotent multiplier of a free nilpotent group. The method we use to determine the explicit structure of the mentioned Baer invariant is described in the following.

Let G be a free nilpotent group of class n and rank m, so G has the following free presentation

$$1 \to \gamma_{n+1}(F) \to F \to G \to 1,$$

where $F = \langle x_1, \ldots, x_m \mid \emptyset \rangle$ is the free group generated by the set $\{x_1, \cdots, x_m\}$. It can be shown that $G \cong \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}$, where $\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}$ is an $n-$ nilpotent product of m infinite cyclic groups. Assume that $G^0 = \gamma_{c+1}(F)$ and $G^i = [G^{i-1}, G^{i-1}]$ for each $1 \leq i$. Since $2n - 2 < c$ we have $\gamma_{n+1}(F) \cap G^i = G^i$. Therefore

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(G) \cong \frac{\gamma_{n+1}(F) \cap G^i}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]} = \frac{G^i}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}.$$

Now in order to compute the quotient group

$$\frac{G^i}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}$$

we introduce suitable sets of basic commutators; A, B say; with the properties $G^i \equiv \langle A \rangle \pmod{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}$ and $[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}] \equiv \langle B \rangle$ $(\bmod\ \gamma_k(F))$ for some $k \in \mathbb{N}$ and $A \cap B = \emptyset$. This implies

$$\frac{G^i}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}$$

to be the free abelian group on the set

$$\bar{A} = \{a[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}] | a \in A\}.$$

To do this some preliminary concepts and theorems are needed which are listed below.

Lemma 2.4. *With the above notations, if $n - 1 \leq c$ then*

$$G^i \subseteq [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-2}] \subseteq G^{i-1}.$$

Proof. Straightforward. □

The following definition and theorem gives the first set of basic commutators which have claimed to introduce.

Theorem 2.5. *Let V_0 be the set of all basic commutators on X of weights $c + 1, \cdots, c + n$. Let V_i is defined and its elements proved to be basic commutators on X, define $V_{i+1} = \{[a, b] | a, b \in V_i, a > b\}$. Then each element of V_i is a basic commutator on X.*

Proof. The proof follows by induction on i. For $i = 1$ see [10, Lemma 2.2]. Let v_{i+1} be an element of V_{i+1}, so $v_{i+1} = [a, b]$ in which $a, b \in V_i$ and $b < a$. By induction hypothesis both a and b are basic commutators on X. Also, if $a = [a_1, a_2]$ then $a_2 \in V_{i-1}$ thus $a_2 < b$. □

The following theorem shows the importance of the sets V_i.

Theorem 2.6. *If $n - 1 \leq c$ then*

$$G^i \equiv \langle V_i \rangle \pmod{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}.$$

Proof. We use induction on i. For $i = 1$ see [10, Lemma 2.3]. Let α be a generator of G^i, so $\alpha = [a, b]$ in which $a, b \in G^{i-1}$. Induction hypothesis implies that

$$a = a_1^{\alpha_1} \ldots a_r^{\alpha_r} \mu \quad , \quad b = b_1^{\beta_1} \ldots b_s^{\beta_s} \eta$$

in which $a_j, b_k \in V_{i-1}, \alpha_j, \beta_k = \pm 1$ and $\mu, \eta \in [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-2}]$.

Now a simple computation shows that α can be written as a product of elements of the forms $([a_j, b_k]^{f_{jk}})^{\lambda_{jk}}$, $([a_j, \eta]^{h_j})^{\alpha_j}$, $([\mu, b_k]^{g_k})^{\beta_k}$ and $[\mu, \eta]^l$ where $\lambda_{jk} = \alpha_j \beta_k$ and $f_{jk}, h_j, g_k, l \in G^{i-1}$. By Lemma 2.4, it is easy to see that

$$[a_j, b_k, f_{jk}], [a_j, \eta], [\mu, b_k], [\mu, \eta] \in [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}].$$

Therefore $\alpha \equiv \prod_{j,k} [a_j, b_k]^{\lambda_{jk}} \pmod{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}$. □

Now the set B can be introduced. In fact we introduce some sets of basic commutators, U_i for which the following holds.

$[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}] \subseteq \langle U_i \rangle \pmod{\gamma_{2^i c + 2^i n + 2^i}(F)}$ and $U_i \cap V_i = \emptyset$.

In order to attain such sets we prove the following lemmas from which the first one is in [11].

Lemma 2.7. *Let*

$$W = \{\ [a, b] \mid a \text{ and } b \text{ are basic commutators on } X \text{ such that } c + n + 1 \leq w(a),\ c + 1 \leq w(b),\ w(a) + w(b) \leq 2c + 2n + 1\}.$$

If $2n - 2 < c$, then

(*i*) *every element of W is a basic commutator on X,*

(*ii*) $[\gamma_{c+n+1}(F), \gamma_{c+1}(F)] \equiv \langle W \rangle \quad (mod\ \gamma_{2c+2n+2}(F))$.

Lemma 2.8. *There exists a set of basic commutators on X, U_i say; with*

$$[\gamma_{c+n+1}(F), \gamma_{c+1}(F), \ldots, \gamma_{2^{i-1}c+2^{i-1}}(F)] \subseteq \langle U_i \rangle \quad mod(\gamma_{2^i c + 2^i n + 2^i}(F))$$

Proof. This is also done by induction on i. For $i = 1$, let $U_1 = W$. Let α be a generator of $[\gamma_{c+n+1}(F), \gamma_{c+1}(F), \ldots, \gamma_{2^{i-1}c+2^{i-1}}(F)]$, so $\alpha = [a, b]$ in which $a \in [\gamma_{c+n+1}(F), \gamma_{c+1}(F), \ldots, \gamma_{2^{i-2}c+2^{i-2}}(F)]$ and $b \in \gamma_{2^{i-1}c+2^{i-1}}(F)$. Induction hypothesis implies that $a = u_1^{\alpha_1} \ldots u_r^{\alpha_r} \mu$, in which $u_j \in U_{i-1}, \alpha_j = \pm 1, \mu \in \gamma_{2^{i-1}c+2^{i-1}n+2^{i-1}}(F)$. By Hall's Theorem we have

$$b = b_1^{\beta_1} \dots b_s^{\beta_s} \eta \qquad \text{and} \qquad \mu = v_1^{\epsilon_1} \dots v_t^{\epsilon_t} \lambda,$$

in which b_k and v_l are basic commutators on X, such that

$$2^{i-1}c + 2^{i-1} \leq w(b_k) < 2^{i-1}c + 2^i n + 2^{i-1},$$

$$2^{i-1}c + 2^{i-1}n + 2^{i-1} \leq w(v_l) < 2^{i-1}c + 2^i n + 2^i,$$

and $\eta \in \gamma_{2^{i-1}c+2^i n+2^{i-1}}(F)$ and $\lambda \in \gamma_{2^{i-1}c+2^i n+2^i}(F)$.

Therefore $[a, b] = [u_1^{\alpha_1} \dots u_r^{\alpha_r} v_1^{\epsilon_1} \dots v_t^{\epsilon_t} \lambda, b_1^{\beta_1} \dots b_s^{\beta_s} \eta]$. By using commutator manipulations, it is easy to see that

$$[a, b] \equiv \prod_{j,k} [u_j, b_k]^{\alpha_j \beta_k} \prod_{l,k} [v_l, b_k]^{\epsilon_l \beta_k} \pmod{\gamma_{2^i c+2^i n+2^i}(F)}.$$

With respect to the recent relation, if let

$$\begin{gathered} L = \{l \mid l \text{ is a basic commutator on } X \text{ such that} \\ 2^{i-1}c + 2^{i-1} \leq w(l) < 2^{i-1}c + 2^i n + 2^{i-1}\}, \\ V = \{v \mid v \text{ is a basic commutator on } X \text{ such that} \\ 2^{i-1}c + 2^{i-1}n + 2^{i-1} \leq w(v) < 2^{i-1}c + 2^i n + 2^i\}, \\ M = \{[u_j, l]^\alpha \mid u_j \in U_{i-1}, l \in L, w(u_j) + w(l) < 2^i c + 2^i n + 2^i \\ \text{and} \\ \alpha = 1 \text{ if } l < u_j \text{ otherwise } \alpha = -1\}, \\ N = \{[v, l] \mid v \in V, l \in L, w(v) + w(l) < 2^i c + 2^i n + 2^i\}, \end{gathered}$$

and $U_i = M \cup N$. Now it is enough to show that every element of U_i is a basic commutator on X. Let $[v, l] \in N$. Then $v > l$ and also if $v = [v_1, v_2]$ then $w(v_2) \leq \frac{1}{2} w(v) < \frac{1}{2}(2^{i-1}c + 2^i n + 2^i) \leq 2^{i-1}c + 2^{i-1} \leq w(l)$. So each element of N is a basic commutator on X. Now we prove the claim for M. First assume that $u_i > l$ and $u_i = [a, b]$, it is enough to show that $w(b) < w(l)$. But $w(b) \leq \frac{1}{2} w(u_i) < \frac{1}{2}(2^{i-1}c + 2^{i-1}n + 2^{i-1}) \leq 2^{i-1}c + 2^{i-1} \leq w(l)$. Similarly the case $l > u_i$ can be checked. □

Lemma 2.9. *With the above notations and assumption, $U_i \cap V_i = \emptyset$.*

Proof. This is also done by induction on i. Clearly $U_1 \cap V_1 = \emptyset$. Let a be an arbitrary element of U_i. If $a \in M$, since $U_{i-1} \cap V_{i-1} = \emptyset$, then $a \neq v_i$ for each $v_i \in V_i$. Thus $A \cap V_i = \emptyset$. Clearly $N \cap V_i = \emptyset$, therefore $U_i \cap V_i = \emptyset$. □

It is clear that

$$[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}] \subseteq [\gamma_{c+n+1}(F), \gamma_{c+1}(F), \dots, \gamma_{2^{i-1}c+2^{i-1}}(F)],$$

therefore we have proved the following theorem.

Theorem 2.10. *There exists a set of basic commutators on X, U_i say; with $[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}] \subseteq \langle U_i \rangle$ modulo $\gamma_{2^i c+2^i n+2^i}(F)$, and $U_i \cap V_i = \emptyset$.*

We are now ready to prove the main result of this section. Note that $\mathfrak{S}_l$, the variety of solvable groups of derived length at most l is in fact the variety of polynilpotent groups of class row $(\underbrace{1,\ldots,1}_{l-times})$.

Theorem 2.11. *Let G be the group $\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}$ the nth nilpotent product of m copies of $\mathbb{Z}$, then*

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(G) \cong \bigoplus_{k=1}^{p} \mathbb{Z}$$

in which $p = \underbrace{\chi_2(\ldots(\chi_2}_{i-times}(\sum_{j=1}^{n} \chi_{c+j}(m)))\ldots)$. In particular

$$\mathfrak{S}_l M(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \cong \bigoplus_{j=1}^{t} \mathbb{Z}$$

where $t = \underbrace{\chi_2(\ldots\chi_2}_{l-times}(m)\ldots)$.

Proof. We have proved that

$$G^i \equiv \langle V_i \rangle \pmod{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}.$$

It is easy to see that $|V_i| = p$, so it is enough to show that $\bar{V}_i$ is linearly independent.

Let $\sum \alpha_j \bar{v}_j = \bar{0}$, in the group $\mathfrak{N}_{c,1,\ldots,1} M(G)$ where $v_j \in V_i$ and $\alpha_j \in \mathbb{Z}$. Therefore $\sum \alpha_j v_j \in [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]$. Considering Theorem 2.10, we have

$$\sum \alpha_j v_j \equiv \sum \beta_j u_j \pmod{\gamma_{2^i c+2^i n+2^i}(F)},$$

for some $u_j \in U_i$ and $\beta_j \in \mathbb{Z}$. So $\sum \alpha_j v_j - \sum \beta_j u_j \equiv 0$. On the other hand $V_i \cap U_i = \emptyset$, and also $V_i \cup U_i$ forms a part of a basis of the free abelian group

$$\frac{\gamma_{2^i c+2^i}(F)}{\gamma_{2^i c+2^i n+2^i}(F)}.$$

Therefore $\alpha_j = 0$ and the result holds. □

The above theorem generalizes the work of [11].

3. The structure of $\mathfrak{N}_{c,1,\ldots,1} M(\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s)$

In this section, using the results of section 2, we can find a formula for the Baer invariant of an *nth* nilpotent product of two finite cyclic groups under some conditions, with respect to the variety of polynilpotent groups of class row $(c, 1, \ldots, 1)$.

Let

$$1 \to \langle x^r \rangle^{F_1} \to F_1 = \langle x \rangle \to \mathbb{Z}_r = \langle x \mid x^r = 1 \rangle \to 1$$

and

$$1 \to \langle y^s \rangle^{F_2} \to F_2 = \langle y \rangle \to \mathbb{Z}_s = \langle y \mid y^s = 1 \rangle \to 1$$

be free presentations for $\mathbb{Z}_r$ and $\mathbb{Z}_s$, respectively. Clearly

$$1 \to R \to F \to \mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s \to 1$$

is a free presentation for $\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s$, where

$$F = F_1 * F_2 \qquad , \qquad S = \langle x^r, y^s \rangle^F \qquad \textit{and} \qquad R = S\gamma_{n+1}(F)$$

Now put $\rho_{c+1}(S) = [S, {}_c F]$. It is easy to see that

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s) \cong \frac{S\gamma_{n+1}(F) \cap G^i}{[S\gamma_{n+1}(F), {}_c F, G^0, \ldots, G^{i-1}]},$$

in which $G^0 = \gamma_{c+1}(F)$ and $G^i = [G^{i-1}, G^{i-1}]$ for each $1 \le i$. On the other hand

$$[S\gamma_{n+1}(F), {}_c F, G^0, \ldots, G^{i-1}] = [\rho_{c+1}(S), G^0, \ldots, G^{i-1}][\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}],$$

also $\gamma_{n+1}(F) \cap G^i = G^i$, if $2n - 2 < c$.

Replacing the recent relations in the above formula we get:

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s) \cong \frac{G^i/[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}{[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]/[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}.$$

In the previous section we showed that

$$\frac{G^i}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}$$

is a free abelian group of rank $\chi_2(\mid V_{i-1} \mid)$, where V_i is defined in Theorem 2.5. One notes that the main problem is to determine the structure of the factor group

$$\frac{[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}{[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]}.$$

In order to find this structure, we need some notations and lemmas.

The following lemma has an important role in this section.

Lemma 3.12. *Let D be the set of all dth powers of the basic commutators in V_0, in which V_0 be the set defined in Theorem 2.5. Then we have*

$$\rho_{c+1}(S) \equiv \langle D\rangle \qquad (mod \quad \gamma_{c+n+1}(F)),$$

for each r,s and n in which satisfy in the following conditions.
If $n = 1$ then r and s are nonnegative integers.
If $n = 2$ then r and s are odd .
If $n = 3, 4$ then r and s should not be divisible by 2 and 3.

Lemma 3.13. *With the above notations*

$$[\rho_{c+1}(S)\gamma_{c+n+1}(F), \gamma_{c+1}(F)] \ \equiv\ \langle B_1\rangle \quad (\mathrm{mod}\ [\gamma_{c+n+1}(F), \gamma_{c+1}(F)]),$$

where $B_1 = \{v^d \mid v \in V_1\}$.

Proof. Let α be a generator of $[\rho_{c+1}(S)\gamma_{c+n+1}(F), \gamma_{c+1}(F)]$, so $\alpha = [a, b]$ in which $a \in \rho_{c+1}(S)\gamma_{c+n+1}(F)$ and $b \in \gamma_{c+1}(F)$. Lemma 3.12 and Hall's Theorem 1.2 imply that $a = a_1^{\alpha_1} \ldots a_r^{\alpha_r}\mu$ and $b = b_1^{\beta_1} \ldots b_s^{\beta_s}\eta$, where $a_j \in D,\ \ b_k \in V_0,\ \ \alpha_j, \beta_k = \pm 1$ and $\mu, \eta \in \gamma_{c+n+1}(F)$. But one can see that

$$[a, b] \equiv \prod_{j,k}[a_j, b_k]^{\epsilon_{jk}} \quad (\mathrm{mod}\ [\gamma_{c+n+1}(F), \gamma_{c+1}(F)]),$$

in which $\epsilon_{jk} = \alpha_j\beta_k$. Now since $a_j \in D$ then there exists $v_j \in V_0$ such that $a_j = v_j^d$. On the other hand $[v_j^d, b_k] \equiv [v_j, b_k]^d$ modulo $[\gamma_{c+n+1}(F), \gamma_{c+1}(F)]$. Therefore $[a, b] \equiv \prod_{j,k}([v_j, b_k]^d)^{\epsilon_{jk}}$ modulo $[\gamma_{c+n+1}(F), \gamma_{c+1}(F)]$, and the result holds. □

Theorem 3.14. *With the above notations,*

$$[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}] \ \equiv\ \langle B_i\rangle \quad (\mathrm{mod}\ [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]),$$

in which $B_i = \{v_i^d \mid v_i \in V_i\}$.

Proof. We use induction on i. For $i = 1$ follows from Lemma 3.13. Let

$$[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-2}] \ \equiv\ \langle B_{i-1}\rangle$$

modulo

$$[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-2}].$$

For the claim, we first show that

$$[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}] \ \equiv\ \langle L_i\rangle \quad (\mathrm{mod}\ [\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]),$$

where $L_i = \{[b, c] \mid b \in\ B_{i-1}, c \in V_{i-1}\}$. Then we prove that $\langle L_i\rangle\ \equiv\ \langle B_i\rangle$ modulo $[\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]$.

Let λ be a generator of $[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-1}]$, so $\lambda = [\alpha, \beta]$ such that $\alpha \in [\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \ldots, G^{i-2}]$ and $\beta \in G^{i-1}$.
By induction hypothesis and Theorem 2.6, we have

$$\alpha = b_1^{\alpha_1} \dots b_r^{\alpha_r} \eta \qquad \text{and} \qquad \beta = c_1^{\beta_1} \dots c_t^{\beta_t} \mu$$

in which $b_j \in B_{i-1}$, $c_k \in V_{i-1}$, $\alpha_j, \beta_k = \pm 1$ and $\mu, \eta \in [\gamma_{c+n+1}(F), G^0, \dots, G^{i-2}]$.

Using Lemma 2.4, it is routine to show that

$$\lambda = [\alpha, \beta] \equiv \prod_{j,k} [b_j, c_k]^{\epsilon_{jk}} \pmod{[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]},$$

where $\epsilon_{jk} = \alpha_j \beta_k$. Therefore

$$[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}] \equiv \langle L_i \rangle \pmod{[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]}.$$

Now, suppose that $l \in L_i$, thus $l = [b, c]$ such that $b \in B_{i-1}$, and $c \in V_{i-1}$. Since $b \in B_{i-1}$ then there exists $c_0 \in V_{i-1}$ such that $b = c_0^d$. But

$$[c_0^d, c] \equiv [c_0, c]^d \qquad (mod \quad [\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]).$$

Therefore $\langle L_i \rangle \equiv \langle B_i \rangle$ modulo $[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]$. □

The following corollary can be deduced from Theorems 2.10 and 3.14 and has a similar proof of Theorem 2.11.

Corollary 3.15. *With the above notations, if* $2n - 2 < c$ *then*

$$\frac{[\rho_{c+1}(S)\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]}{[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}]},$$

is a free abelian group with the following basis

$$\overline{B}_i = \{b_j[\gamma_{c+n+1}(F), G^0, \dots, G^{i-1}] \mid b_j \in B_i\}.$$

Now the following interesting results can be deduced.

Theorem 3.16. *Let* r *and* s *be arbitrary positive integers. Then for any* $c \geq 1$

$$\mathfrak{N}_{c,\underbrace{1,\dots,1}_{i-times}} M(\mathbb{Z}_r \times \mathbb{Z}_s) \cong \mathbb{Z}_d \oplus \dots \oplus \mathbb{Z}_d \qquad (\underbrace{\chi_2(\dots\chi_2}_{i-times}(\chi_{c+1}(2))\dots) - copies),$$

in which $d = (r, s)$. *In particular if* $l > 1$

$$\mathfrak{S}_l M(\mathbb{Z}_r \times \mathbb{Z}_s) \cong \langle 1 \rangle,$$

where $\mathfrak{S}_l$ *is the variety of all solvable groups of derived length at most* l.

Proof. We know that

$$\mathfrak{N}_{c,\underbrace{1,\dots,1}_{i-times}} M(\mathbb{Z}_r \overset{1}{*} \mathbb{Z}_s) \cong \frac{G^i}{[\rho_{c+1}(S)\gamma_{c+2}(F), G^0, \dots, G^{i-1}]}.$$

On the other hand $\mathbb{Z}_r \times \mathbb{Z}_s \cong \mathbb{Z}_r \overset{1}{*} \mathbb{Z}_s$, therefore using Theorem 2.10 and Corollary 3.15 we have

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \times \mathbb{Z}_s) \cong \frac{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}{d\mathbb{Z} \oplus \ldots \oplus d\mathbb{Z}} \cong \bigoplus_{k=1}^{p} \mathbb{Z}_d.$$

□

Theorem 3.17. *If $(r, s) = 1$ then for any $1 \leq n$ and $2n - 2 < c$ we have*

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s) \cong \langle 1 \rangle$$

Proof. If $(r, s) = 1$, then we have $\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s \cong \mathbb{Z}_r \overset{1}{*} \mathbb{Z}_s$ (for this isomorphism see [10]). Therefore

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{n}{*} \mathbb{Z}_s) \cong \mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{1}{*} \mathbb{Z}_s)$$

Now Theorem 3.16 gives the result. □

Using Theorem 2.11 and Corollary 3.15, we obtain the following theorem.

Theorem 3.18. *Using the notations at the previous sections,*
(i) For any odd integers r and s , and all $c > 2$

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{2}{*} \mathbb{Z}_s) \cong \mathbb{Z}_d \oplus \ldots \oplus \mathbb{Z}_d \qquad (\underbrace{\chi_2(\ldots\chi_2}_{i-times}(\sum_{i=1}^{2} \chi_{c+i}(2))\ldots) - copies).$$

(ii)For all non negative integers r and s , which are not divisible by 2 and 3 , then

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{3}{*} \mathbb{Z}_s) \cong \mathbb{Z}_d \oplus \ldots \oplus \mathbb{Z}_d \qquad (\underbrace{\chi_2(\ldots\chi_2}_{i-times}(\sum_{i=1}^{3} \chi_{c+i}(2))\ldots) - copies),$$

where $c > 4$ and

$$\mathfrak{N}_{c,\underbrace{1,\ldots,1}_{i-times}} M(\mathbb{Z}_r \overset{4}{*} \mathbb{Z}_s) \cong \mathbb{Z}_d \oplus \ldots \oplus \mathbb{Z}_d \qquad (\underbrace{\chi_2(\ldots\chi_2}_{i-times}(\sum_{i=1}^{4} \chi_{c+i}(2))\ldots) - copies),$$

where $c > 6$.

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