Mathematics Interdisciplinary Research **5** (2020) 225 – 238

# Numerical Solution of System of Nonlinear Integro-Differential Equations Using Hybrid of Legendre Polynomials and Block-Pulse Functions

*Mehdi Sabzevari* ⋆ *and Fatemeh Molaei*

**Abstract**

In this paper, numerical techniques are presented for solving system of nonlinear integro-differential equations. The method is implemented by applying hybrid of Legendre polynomials and Block-Pulse functions. The operational matrix of integration and the integration of the cross product of two hybrid function vectors are derived in order to transform the system of nonlinear integro-differential equations into a system of algebraic equations. Finally, the accuracy of the method is illustrated through some numerical examples and the corresponding results are presented.

Keywords: Integro-differential equations; Hybrid functions; Block-Pulse functions; Legendre polynomials; Operational matrix.

2010 Mathematics Subject Classification: 65R20, 47G20.

**How to cite this article**
M. Sabzevari and F. Molaei, Numerical solution of system of nonlinear integro-differential equations using hybrid of Legendre polynomials and block-pulse functions, *Math. Interdisc. Res.* **5** (2020) 225 – 238.

## 1. Introduction

Integral and differential equations play an important role in different branches of sciences and engineering. A large class of initial and boundary value problems which are appeared in control, mechanics, economics, electrical engineering, medicine, etc., can be converted to integral, differential or specially integro-differential equations [1].

⋆Corresponding author (E-mail: sabzevari@kashanu.ac.ir)
Academic Editor: Ali Reza Ashrafi
Received 26 February 2019, Accepted 3 July 2019
DOI: 10.22052/mir.2019.173293.1119

Many kinds of basic functions have been employed in numerical solutions of different types of integral and integro-differential equations. Orthogonal functions are a large class of these basic functions, which can be classified into three families [4]:

**i)** Piecewise constant orthogonal functions, such as Block-Pulse, Haar, Walsh, etc. [5, 6].

**ii)** Orthogonal polynomials, such as Legendre, Chebyshev, Hermite, etc. [13].

**iii)** Sine-Cosine (Fourier) functions [7].

In recent years, hybrid functions have been applied extensively by many authors, such as hybrid of Taylor and block-pulse functions [14], hybrid of Chebyshev and block-bulse functions [17], hybrid of Bernstein and block-bulse functions [2], etc. The high accuracy of these basic functions in the numerical solutions of different types of integral and integro-differential equations, is one of the biggest advantages of hybrid functions.

Maleknejad and Tavassoli Kajani [11] were the first to use the Hybrid of Legendre and Block-Pulse (HLBP) functions for the solution of integral equations. More specifically, they used these basic functions to estimate the solution of the linear Fredholm integral equations of the second kind. Afterward, the authors in [15] used the HLBP functions for the solution of linear Fredholm Fuzzy integral equations, as well as these basic functions have been used in [8] for numerical solution of Fredholm and Volterra integral equations. Also, numerical solution of nonlinear integro-differential equations and system of nonlinear Fredholm-Hammerstein integral equations, using HLBP functions, were considered in [12] and [16], respectively.

In this paper, by using hybrid of Legendre polynomials and block-pulse functions, we propose a numerical approach for solving a system of nonlinear integro-differential equations of the second kind as in (1) below:

$$\begin{cases} \displaystyle\sum_{j=1}^{l} \alpha_{ij} u_j(x) + \sum_{j=1}^{l} \beta_{ij} u_j^{'}(x) = f_i(x) \\ \qquad\qquad\qquad \displaystyle + \sum_{j=1}^{l} \int_0^1 k_{ij}(x,t)\varphi_{ij}(t, u_j(t))\mathrm{d}t, \quad i = 1, 2, \ldots, l, \\ u_j(0) = u_{j0}, \quad j = 1, 2, \ldots, l, \end{cases} \tag{1}$$

where the functions $f_i(x) \in L^2[0,1)$ and $k_{ij}(x,t)$ and $\varphi_{ij}(t, u_j(t))$ belong to $L^2\left([0,1) \times [0,1)\right)$ are known for $i, j = 1, 2, \ldots, l$. $u_j(x)$ are unknown functions for $j = 1, 2, \ldots, l$. The operational matrix of integration and the integration of the cross product of two hybrid function vectors will be derived in order to transform the nonlinear integro-differential equations into a system of algebraic equations. Finally, some numerical examples and the corresponding results will be presented to illustrate the accuracy of the method.

# 2. Properties of Hybrid Functions

## 2.1 Hybrid of Legendre and Block-Pulse Functions

Block-pulse functions are a set of orthogonal functions with piecewise constant values which are usually applied as a useful tool in the analysis, synthesis, identification and other problems of control and systems science. As proposed by many authors, a set of block-pulse functions is usually defined as follows [10]:

**Definition 2.1.** A set of block-pulse functions are defined on the interval $[0, 1)$ for $n = 1, 2, \dots N$ as in Equation (2) below:

$$b_n(x) = \begin{cases} 1 & \frac{n-1}{N} \leq x < \frac{n}{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

These functions are disjoint on the interval [0, 1). It means for $n, m = 1, 2, \dots, N$, we have $b_n(x)b_m(x) = \delta_{nm}b_n(x)$. Also, the block-pulse functions are orthogonal and complete set in $L^2[0, 1)$.

On the other hand, Legendre polynomials obey the following three-term recurrence relation known as Bonnet's recursion formula on the interval $[-1, 1]$:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= x, \\ L_m(x) &= \frac{2m-1}{m} x L_{m-1}(x) - \frac{m-1}{m} L_{m-2}(x), \quad m = 2, 3, \dots. \end{aligned}$$

The set of these polynomials is complete and orthogonal in the Hilbert space $L^2[-1, 1]$. However, the hybrid of Legendre polynomials and block-pulse functions is defined as follows:

**Definition 2.2.** For $m = 0, 1\dots, M-1$ and $n = 1, 2, \dots, N$, the hybrid of Legendre polynomials and block-pulse functions are defined on the interval $[0, 1)$ as in (3) below:

$$b_{nm}(x) = \begin{cases} L_m(2Nx - 2n + 1) & \frac{n-1}{N} \leq x < \frac{n}{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here, $M$ and $N$ are the order of Legendre polynomials and block-pulse functions, respectively. The set of these hybrid functions is complete and orthogonal in $L^2[0, 1)$.

## 2.2 Function Approximation

Any function $u(x) \in L^2[0,1)$ can be expanded by the hybrid of Legendre and block-pulse functions as:

$$u(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{nm} b_{nm}(x), \tag{4}$$

where,

$$u_{nm} = \frac{(u(x), b_{nm}(x))}{(b_{nm}(x), b_{nm}(x))} = \frac{\int_0^1 u(x) b_{nm}(x)\mathrm{d}x}{\int_0^1 b_{nm}(x) b_{nm}(x)\mathrm{d}x},$$

which $(\cdot,\cdot)$ denotes the inner product. If the infinite series in (4) is truncated at some values of $M$ and $N$, then this equation can be written as:

$$u(x) \simeq u_{NM}(x) = \sum_{n=1}^{N}\sum_{m=0}^{M-1} u_{nm} b_{nm}(x) = B^T(x)\, U, \tag{5}$$

where $U$ and $B(x)$ are $MN \times 1$ vectors in the form of:

$$U = [u_{10}, u_{11}, \ldots, u_{1(M-1)}, u_{20}, \ldots, u_{N(M-1)}]^T,$$

$$B(x) = [b_{10}(x), b_{11}(x), \ldots, b_{1(M-1)}(x), b_{20}(x), \ldots, b_{N(M-1)}(x)]^T. \tag{6}$$

Similarly, bivariate function $k(x,t) \in L^2([0,1) \times [0,1))$, can be approximated by the hybrid of Legendre and block-pulse functions as:

$$k(x,t) \simeq k_{NM}(x,t) = B^T(x) K B(t), \tag{7}$$

where $K$ is a $MN \times MN$ matrix which its elements for $i,j = 1,2,\ldots,MN$ are computed as in (8) below:

$$K_{ij} = \frac{(B_i(x), (k(x,t), B_j(t)))}{(B_i(x), B_i(x))(B_j(t), B_j(t))}. \tag{8}$$

## 2.3 Operational Matrix of Integration

Consider the $MN \times 1$ vector $B(t)$ on the interval $[0,1)$, defined in (6), which its elements are Hybrid Legendre polynomials and block-pulse functions. The integration of the vector is given by:

$$\int_0^x B(t)\mathrm{d}t \quad \simeq PB(x),$$

where $P$ is the $MN \times MN$ and we call it operational matrix for integration of hybrid Legendre polynomials and block-pulse functions. It appears as [3]:

$$P = \begin{pmatrix} E & H & \cdots & H \\ 0 & E & \cdots & H \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \end{pmatrix},$$

which $H$ and $E$ are $M \times M$ matrices and defined as:

$$H = \frac{1}{N}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

$$E = \frac{1}{2N}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2M-1} & 0 \end{pmatrix}.$$

Also we define the matrix $D$, the integration of the cross product of two hybrid function vectors $B(x)$ in (6), as follows:

$$D = \int_0^1 B(t)B^T(t)\mathrm{d}t. \tag{9}$$

The matrix $D$ can be obtained as:

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_N \end{pmatrix},$$

where $D_i$ is the $M \times M$ diagonal matrix and independent of $i$, that is given by:

$$D_i = \frac{1}{N}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2M-1} \end{pmatrix}.$$

We use the matrix $D$ to convert the Fredholm part of integro-differential equations to an algebraic equation.

## 3. Solution of System of Nonlinear Integro-Differential Equations

The set of hybrid Legendre polynomials and block-pulse functions is complete and orthogonal in $L^2[0,1)$. Hence, we should consider the systems of integro-differential equations in the interval $[0,1)$. Nevertheless, if this assumption is not satisfied, we can convert the system into the mentioned interval by a simple change of variables. Consider the following system of nonlinear integro-differential equations of the second kind:

$$\begin{cases} \displaystyle\sum_{j=1}^{l} \alpha_{ij} u_j(x) + \sum_{j=1}^{l} \beta_{ij} u_j^{'}(x) = f_i(x) \\ \qquad\qquad\qquad \displaystyle + \sum_{j=1}^{l} \int_0^1 k_{ij}(x,t)\varphi_{ij}(t,u_j(t))\mathrm{d}t, \quad i = 1,2,\ldots,l, \\ u_j(0) = u_{j0}, \quad j = 1,2,\ldots,l, \end{cases} \tag{10}$$

where the functions $f_i(x) \in L^2[0,1)$ and $k_{ij}(x,t)$ and $\varphi_{ij}(t,u_j(t))$ belong to $L^2([0,1) \times [0,1))$ are known for $i,j = 1,2,\ldots,l$. $u_j(x)$ are unknown functions for $j = 1,2,\ldots,l$.

Let's suppose

$$\varphi_{ij}(t,u_j(t)) = \psi_{ij}(t), \quad 0 \le t < 1. \tag{11}$$

Now by considering Equations (5) and (7), each function approximation in the set defined as the following:

$$\begin{aligned} u_j(x) &\simeq B^T(x)\, U_j, \\ f_i(x) &\simeq B^T(x)\, F_i, \\ k_{ij}(x,t) &\simeq B^T(x) K_{ij} B(t), \\ \psi_{ij}(t) &\simeq B^T(t)\, \Psi_{ij}. \end{aligned} \tag{12}$$

In above Equation (12), $U_j$ and $\Psi_{ij}$ are $MN \times 1$ vectors with unknown elements, $F_i$ are $MN \times 1$ vectors and $K_{ij}$ are $MN \times MN$ matrices with known elements, for $i,j = 1,2,\ldots,l$.

Now, let us approximate $u_j'(x)$ as in (13) below:

$$u_j'(x) \simeq U_j'^T B(x). \tag{13}$$

We attempt to evaluate $U_j'$ in term of $U_j$. Hence,

$$u_j(x) - u_j(0) = \int_0^x u_j'(t)\mathrm{d}t \simeq \int_0^x U_j'^T B(t)\mathrm{d}t \simeq U_j'^T P B(x). \tag{14}$$

If we approximate $u_j(0)$ as a function by the hybrid functions, i.e. $u_j(0) = U_{j0}^T B(x)$, then $U_{j0}$ can be obtained as the following:

$$U_{j0} = \big[\underbrace{\overbrace{u_j(0), 0, \ldots, 0}^{M}, \overbrace{u_j(0), 0, \ldots, 0}^{M}, \ldots, \overbrace{u_j(0), 0, \ldots, 0}^{M}}_{NM}\big]^T.$$

Consequently, it follows from Equation (14) that,

$$U_j^T B(x) - U_{j0}^T B(x) \simeq U_j'^T P B(x),$$

and hence,

$$U_j - U_{j0} \simeq P^T U_j'.$$

Now, substituting each of approximation functions (12) and (13) into the system of integro-differential (10), we have:

$$\begin{aligned}\sum_{j=1}^{l} \alpha_{ij} B^T(x) U_j + \sum_{j=1}^{l} \beta_{ij} B^T(x) U_j' &= B^T(x) F_i \\ &+ \sum_{j=1}^{l} \int_0^1 B^T(x) K_{ij} B(t) B^T(t) \Psi_{ij} \mathrm{d}t,\end{aligned} \tag{15}$$

Applying Equations (9) and (15) becomes:

$$B^T(x) \sum_{j=1}^{l} \alpha_{ij} U_j + B^T(x) \sum_{j=1}^{l} \beta_{ij} U_j' = B^T(x) F_i + B^T(x) \sum_{j=1}^{l} K_{ij} D \Psi_{ij},$$

where $i = 1, 2, \ldots, l$. Therefore,

$$\sum_{j=1}^{l} \alpha_{ij} U_j + \sum_{j=1}^{l} \beta_{ij} U_j' \quad = F_i + \textstyle\sum_{j=1}^{l} K_{ij} D \Psi_{ij}, \quad i = 1, 2, \ldots, l. \tag{16}$$

Finally, by multiplying $P^T$ in both sides of Equation (16) from the left and using Equation (15) we get:

$$\sum_{j=1}^{l} \alpha_{ij} P^T U_j + \sum_{j=1}^{l} \beta_{ij} (U_j - U_{j0}) = P^T F_i + \sum_{j=1}^{l} P^T K_{ij} D \Psi_{ij}, \quad i = 1, 2, \ldots, l. \tag{17}$$

The above Equation (17) is a system of $NMl$ equations and $NM(l^2+l)$ unknowns. These unknowns are the elements of $U_j$ and $\Psi_{ij}$ vectors defined in Equation (12). To calculate the other $NMl^2$ required equations, we use the relationship between

$U_j$ and $\Psi_{ij}$, which can be obtained from Equation (11). Actually, substituting the collocation points

$$x_s = \frac{2s-1}{2NM}, \qquad s = 1, 2, \ldots, NM,$$

into equation

$$\varphi_{ij}(x_s, X_j^T B(x_s)) = \Psi_{ij}^T B(x_s), \quad i, j = 1, 2, \ldots, l, \tag{18}$$

another $NMl^2$ required equations can be obtained. Combining Equations (17) and (18), we get a system of nonlinear algebraic equations with $NM(l^2 + l)$ equations an the same number of unknowns. The unknown vectors $U_j$ can be obtained by solving this system and consequently the solution of the system of integral Equation (10) will be obtained by $u_j(x) = U_j^T B(x)$, for $j = 1, 2, \ldots, l$.

# 4. Numerical Examples

In this section, three examples are presented to illustrate the accuracy of the proposed method. In each case, the absolute error of method with different values of $N$ and $M$, which have been computed by Maple 16, are tabulated and the absolute error functions are plotted.

**Example 4.1.** Consider the following system of integro-differential equations:

$$u_1(x) + u_2'(x) = \frac{93}{35}x + \int_0^1 xt^2 u_1^2(t)\mathrm{d}t + \int_0^1 xt^2 u_2^2(t)\mathrm{d}t,$$
$$u_2(x) + u_1'(x) = \frac{11}{12}x^2 + 1 + \int_0^1 x^2 t u_1^2(t)\mathrm{d}t - \int_0^1 x^2 t u_2^2(t)\mathrm{d}t,$$

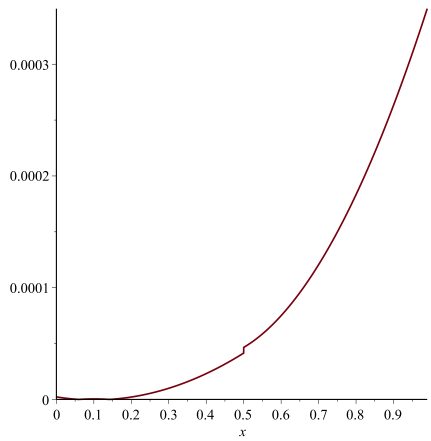
with the initial conditions $u_1(0) = u_2(0) = 0$. The exact solutions of this system are $u_1(x) = x$ and $u_2(x) = x^2$. The corresponding values of absolute errors with $N = 2$ and $M = 3$ as well as $N = 3$ and $M = 4$ are given in Table 1. Also, the absolute error functions with $N = 2$ and $M = 3$ are plotted in Figure 1, and with $N = 3$ and $M = 4$ are plotted in Figure 2.

**Example 4.2.** Consider the following system of integro-differential equations:

$$u_1(x) + 3u_2(x) - u_2'(x) = -\frac{1}{9}x(1 + 2e^3) + \int_0^1 e^{x-2t} u_1^2(t)\mathrm{d}t + \int_0^1 xtu_2(t)\mathrm{d}t$$
$$u_2(x) + 2u_1'(x) + u_2'(x) = 4e^{3x} + \frac{15}{7}e^x - x - \frac{1}{7}e^{x+7} + \int_0^1 xe^{-3t} u_1^3(t)\mathrm{d}t + \int_0^1 e^{x+t} u_2^2(t)\mathrm{d}t$$

Table 1: The corresponding values of absolute errors for Example 4.1.

| x | Absolute error of $(u_1(x), u_2(x))$ with $N = 2$ and $M = 3$ | Absolute error of $(u_1(x), u_2(x))$ with $N = 3$ and $M = 4$ |
|---|---|---|
| 0.0 | $(2.239\times10^{-6}, 5.486\times10^{-7})$ | $(7.481\times10^{-11}, 2.568\times10^{-10})$ |
| 0.1 | $(4.477\times10^{-7}, 2.763\times10^{-6})$ | $(5.700\times10^{-9}, 3.491\times10^{-8})$ |
| 0.2 | $(2.131\times10^{-6}, 1.159\times10^{-5})$ | $(4.630\times10^{-8}, 1.412\times10^{-7})$ |
| 0.3 | $(9.977\times10^{-6}, 2.703\times10^{-5})$ | $(1.567\times10^{-7}, 3.243\times10^{-7})$ |
| 0.4 | $(2.309\times10^{-5}, 4.909\times10^{-5})$ | $(3.732\times10^{-7}, 5.930\times10^{-7})$ |
| 0.5 | $(4.652\times10^{-5}, 8.009\times10^{-5})$ | $(7.319\times10^{-7}, 9.591\times10^{-7})$ |
| 0.6 | $(7.496\times10^{-5}, 1.156\times10^{-4})$ | $(1.272\times10^{-6}, 1.439\times10^{-6})$ |
| 0.7 | $(1.205\times10^{-4}, 1.633\times10^{-4})$ | $(2.032\times10^{-6}, 2.054\times10^{-6})$ |
| 0.8 | $(1.832\times10^{-4}, 2.231\times10^{-4})$ | $(3.057\times10^{-6}, 2.827\times10^{-6})$ |
| 0.9 | $(2.631\times10^{-4}, 2.950\times10^{-4})$ | $(4.389\times10^{-6}, 3.787\times10^{-6})$ |



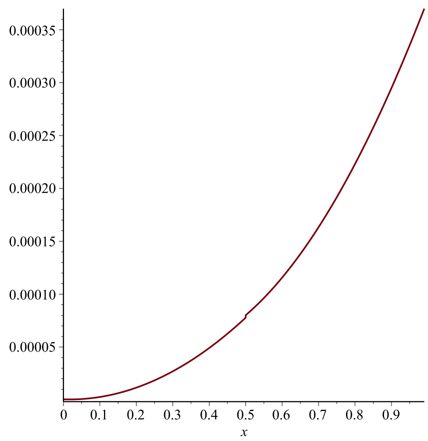




Figure 1: The absolute error functions of $u_1(x)$ (left figure) and $u_2(x)$ (right figure) for Example 4.1 with $N = 2$ and $M = 3$.

with the initial conditions $u_1(0) = u_2(0) = 1$. The exact solutions of this system are $u_1(x) = e^x$ and $u_2(x) = e^{3x}$. The corresponding values of absolute errors with $N = M = 3$ as well as $N = 3$ and $M = 4$ are given in Table 2. Also, the absolute error functions with $N = M = 3$ are plotted in Figure 3, and with $N = 3$ and $M = 4$ are plotted in Figure 4.

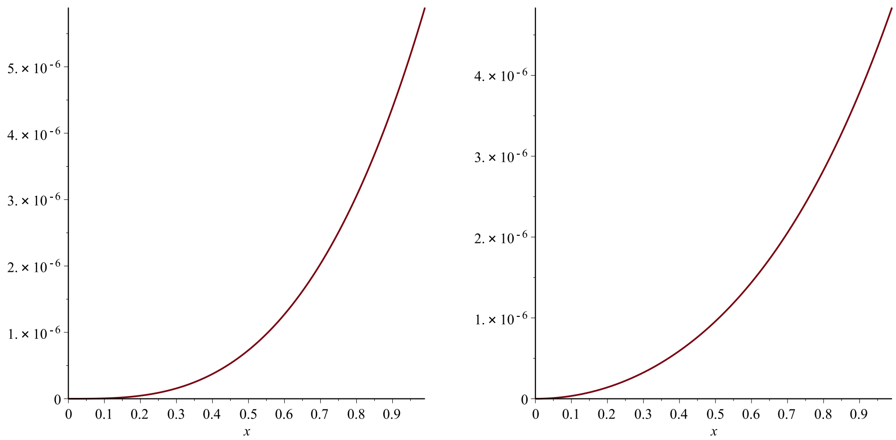



Figure 2: The absolute error functions of $u_1(x)$ (left figure) and $u_2(x)$ (right figure) for Example 4.1 with $N = 3$ and $M = 4$.

Table 2: The corresponding values of absolute errors for Example 4.2.

| x | Absolute error of $(u_1(x), u_2(x))$ with $N = M = 3$ | Absolute error of $(u_1(x), u_2(x))$ with $N = 3$ and $M = 4$ |
|---|---|---|
| 0.0 | $(1.802\times10^{-4}, 1.410\times10^{-2})$ | $(5.860\times10^{-6}, 9.993\times10^{-4})$ |
| 0.1 | $(5.282\times10^{-4}, 5.794\times10^{-3})$ | $(1.438\times10^{-4}, 3.770\times10^{-4})$ |
| 0.2 | $(5.612\times10^{-4}, 4.171\times10^{-3})$ | $(3.528\times10^{-4}, 2.522\times10^{-4})$ |
| 0.3 | $(1.083\times10^{-3}, 1.169\times10^{-3})$ | $(6.243\times10^{-4}, 1.135\times10^{-3})$ |
| 0.4 | $(1.967\times10^{-3}, 9.716\times10^{-3})$ | $(9.874\times10^{-4}, 2.473\times10^{-3})$ |
| 0.5 | $(2.821\times10^{-3}, 1.370\times10^{-3})$ | $(1.474\times10^{-3}, 9.290\times10^{-4})$ |
| 0.6 | $(3.873\times10^{-3}, 1.984\times10^{-2})$ | $(2.066\times10^{-3}, 3.753\times10^{-3})$ |
| 0.7 | $(6.119\times10^{-3}, 2.069\times10^{-2})$ | $(2.823\times10^{-3}, 5.144\times10^{-3})$ |
| 0.8 | $(7.734\times10^{-3}, 2.421\times10^{-2})$ | $(3.816\times10^{-3}, 3.399\times10^{-3})$ |
| 0.9 | $(8.969\times10^{-3}, 5.602\times10^{-2})$ | $(5.028\times10^{-3}, 6.929\times10^{-3})$ |

**Example 4.3.** Consider the following system of integral equations:

$$\begin{aligned} u_1(x) &= -\sin(5x) + x\left(\frac{23}{125}\cos(5) - \frac{2}{25}\sin(5) + \frac{2}{125}\right) \\ &\quad + x\left(\frac{4}{9}e^{-3} - \frac{1}{9}\right)\int_0^1 \left(-xt^2u_1(t) + xtu_2(t)\right)\mathrm{d}t \\ u_2(x) &= e^{-3x} + x\left(-\frac{2}{5}\cos(5) + \frac{1}{25}\sin(5) + \frac{1}{5}\right) \\ &\quad + x^2\left(\frac{4}{9}e^{-3} - \frac{1}{9}\right) + \int_0^1 \left(x(t+1)u_1(t) + x^2tu_2(t)\right)\mathrm{d}t \end{aligned}$$

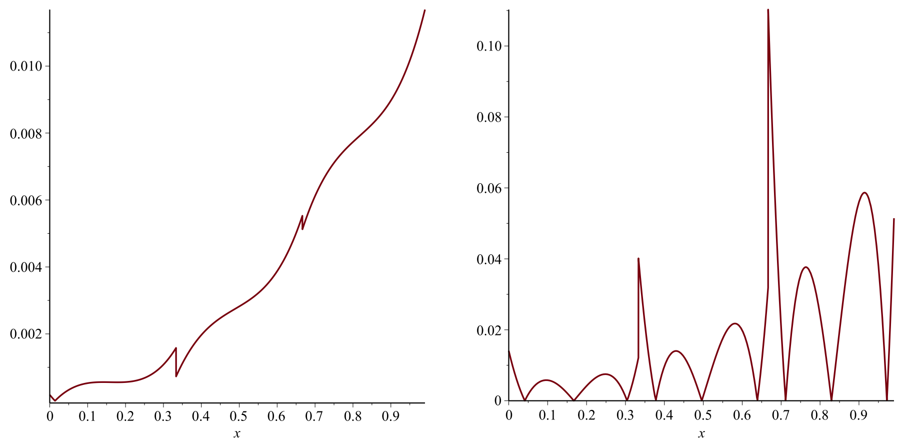



Figure 3: The absolute error functions of $u_1(x)$ (left figure) and $u_2(x)$ (right figure) for Example 4.2 with $N = M = 3$.

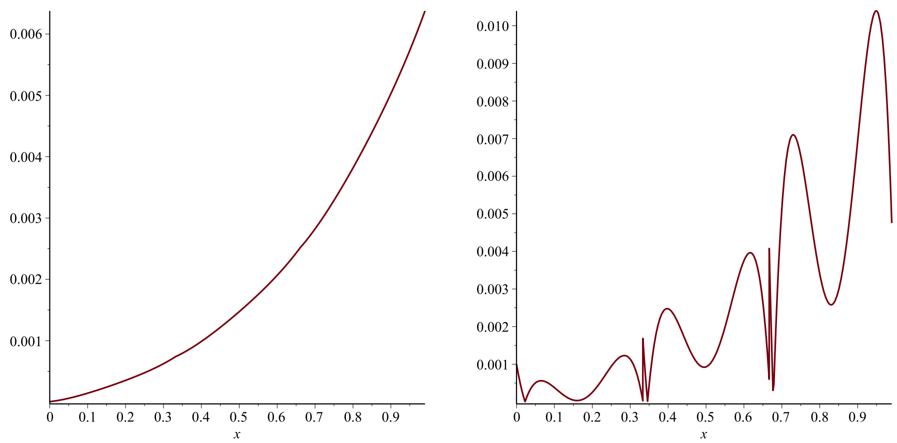



Figure 4: The absolute error functions of $u_1(x)$ (left figure) and $u_2(x)$ (right figure) for Example 4.2 with $N = 3$ and $M = 4$.

with the exact solution $u_1(x) = \sin(-5x)$ and $u_2(x) = e^{-3x}$. The maximum absolute errors are reported in Table 3 for different values of $M$ and $N$. As one observes, the approximate solutions converge to exact solution rapidly when $M$ and $N$ increase.

Also, the comparison among the Bessel polynomials method (BPM) for $N = 15$ [18], modified homotopy perturbation method (MHPM) for $N = 10$ [9], and the presented method for $N = 2$ and $M = 15$ is shown in Table 4. Moreover, the absolute error functions with $N = 2$ and $M = 15$ are plotted in Figure 5.

Table 3: Maximum absolute errors for Example 4.3.

| N | M | Maximum absolute error of $u_1(x)$ | Maximum absolute error of $u_2(x)$ |
|---|---|---|---|
| 2 | 3 | $4.70\times10^{-2}$ | $5.85\times10^{-3}$ |
| 2 | 5 | $1.13\times10^{-3}$ | $4.28\times10^{-5}$ |
| 2 | 10 | $3.58\times10^{-9}$ | $1.09\times10^{-11}$ |
| 2 | 15 | $1.56\times10^{-15}$ | $3.21\times10^{-14}$ |
| 3 | 3 | $1.39\times10^{-2}$ | $1.76\times10^{-3}$ |
| 3 | 5 | $9.49\times10^{-5}$ | $6.93\times10^{-6}$ |
| 3 | 10 | $8.64\times10^{-11}$ | $8.94\times10^{-11}$ |

Table 4: The corresponding values of absolute errors of $(u_1(x), u_2(x))$ for Example 4.3.

| x | BPM [18] for $N = 15$ | MHPM [9] for $N = 10$ | Presented method for $N = 2$ and $M = 15$ |
|---|---|---|---|
| 0.1 | $(6.85\times10^{-12}, 9.41\times10^{-13})$ | $(8.78\times10^{-7}, 6.35\times10^{-6})$ | $(1.63\times10^{-16}, 1.86\times10^{-15})$ |
| 0.4 | $(5.84\times10^{-13}, 4.23\times10^{-12})$ | $(3.51\times10^{-6}, 2.86\times10^{-5})$ | $(1.75\times10^{-16}, 1.84\times10^{-15})$ |
| 0.7 | $(1.15\times10^{-12}, 8.24\times10^{-12})$ | $(6.14\times10^{-6}, 5.58\times10^{-5})$ | $(6.75\times10^{-17}, 2.05\times10^{-14})$ |
| 1 | $(1.49\times10^{-12}, 1.30\times10^{-11})$ | $(8.78\times10^{-6}, 8.78\times10^{-5})$ | $(2.16\times10^{-15}, 6.30\times10^{-14})$ |

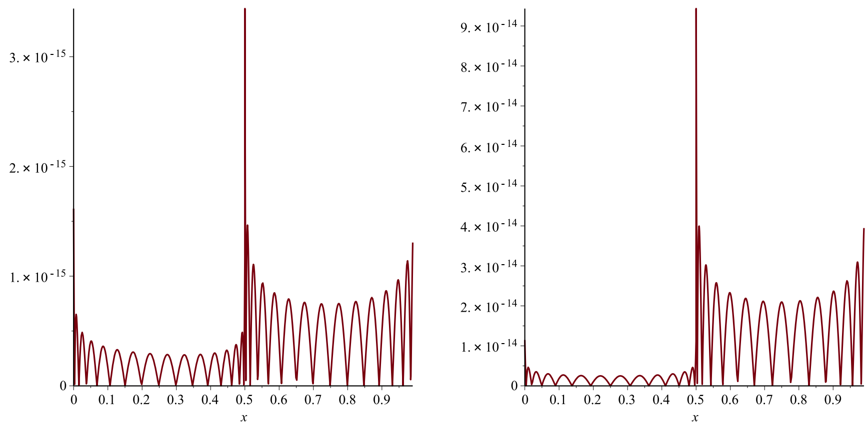



Figure 5: The absolute error functions of $u_1(x)$ (left figure) and $u_2(x)$ (right figure) for Example 4.3 with $N = 2$ and $M = 15$.

**Confilcts of Interest.** The authors declare that there are no conflicts of interest regarding the publication of this article.

# References

[1] J. Abdul, *Introduction to Integral Equations with Applications*, Wiley, New York, 1999.

[2] M. Alipour, D. Baleanu and F. Babaei, Hybrid Bernstein Block-Pulse functions method for second kind integral equations with convergence analysis, *Abstr. Appl. Anal.* **2014** (2014) 623763.

[3] R. Y. Chang and M. L. Wang, Shifted Legendre direct method for variational problems, *J. Optim. Theory Appl.* **39** (1983) 299 − 307.

[4] K. B. Datta and B. M. Mohan, *Orthogonal Functions in Systems and Control*, World Scientific, Singapore, 1995.

[5] S. M. Hashemiparast, M. Sabzevari and H. Fallahgoul, Improving the solution of nonlinear Volterra integral equations using rationalized Haar s-functions, *Vietnam J. Math.* **39** (2) (2011) 145 − 157.

[6] S. M. Hashemiparast, M. Sabzevari and H. Fallahgoul, Using crooked lines for the higher accuracy in system of integral equations, *J. Appl. Math. Inform.* **29** (2011) 145 − 159.

[7] S. M. Hashemiparast, H. Fallahgoul and A. Hosseyni, Fourier series approximation for periodic solution of system of integral equations using Szego-Bernstein weights, *Int. J. Comput. Math.* **87** (2010) 1485 − 1496.

[8] C. H. Hsiao, Hybrid function method for solving Fredholm and Volterra integral equations of the second kind, *J. Comput. Appl. Math.* **230** (2009) 59 − 68.

[9] M. Javidi, Modified homotopy perturbation method for solving system of linear Fredholm integral equations, *Math. Comput. Modelling* **50** (2009) 159 − 165.

[10] Z. Jiang and W. Schanfelberger, *Block-Pulse Functions and their Applications in Control Systems*, Springer-Verlag, Berlin, 1992.

[11] K. Maleknejad and M. Tavassoli Kajani, Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions, *Appl. Math. Comput.* **145** (2003) 623 − 629.

[12] K. Maleknejad, B. Basirat and E. Hashemzadeh, Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations, *Comput. Math. Appl.* **61** (2011) 2821 − 2828.

[13] A. Saadatmandi and M. Dehghan, A Legendre collocation method for fractional integro-differential equations, *J. Vib. Control* **17** (2011) 2050 – 2058.

[14] M. Sabzevari, Erratum to "Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions", *Appl. Math. Comput.* **339** (2018) 302 – 307.

[15] H. Sadeghi Goghari and M. Sadeghi Goghari, Two computational methods for solving linear Fredholm fuzzy integral equation of the second kind, *Appl. Math. Comput.* **182** (2006) 791 – 796.

[16] P. K. Sahu and S. S. Ray, Hybrid Legendre Block-Pulse functions for the numerical solutions of system of nonlinear Fredholm-Hammerstein integral equations, *Appl. Math. Comput.* **270** (2015) 871 – 878.

[17] M. Tavassoli Kajani and A. Hadi Vencheh, Solving second kind integral equations with Hybrid Chebyshev and Block-Pulse functions, *Appl. Math. Comput.* **163** (2005) 71 – 77.

[18] S. Yüzbaşi, N. Şahin and M. Sezer, Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases, *Comput. Math. Appl.* **61** (2011) 3079 – 3096.

Mehdi Sabzevari
Department of Applied Mathematics,
University of Kashan,
Kashan, I. R. Iran
e-mail: sabzevari@kashanu.ac.ir

Fatemeh Molaei
Department of Applied Mathematics,
University of Kashan,
Kashan, I. R. Iran
e-mail: fatemehmolaie@gmail.com