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Seidel Integral Complete Split Graphs

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Abstract

In the paper we consider a generalized join operation, that is, the H-join on graphs where H is an arbitrary graph. In terms of Seidel matrix of graphs we determine the Seidel spectrum of the graphs obtained by this operation on regular graphs. Some additional consequences regarding S-integral complete split graphs are also obtained, which allows to exhibit many infinite families of Seidel integral complete split graphs.

Keywords: Seidel spectrum, Seidel integral graph, H-join of graphs, complete split graph.

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1. Introduction

All graphs considered here are simple and undirected. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. If $e \in E(G)$ has end vertices u and v, then we say that u and v are adjacent and this edge is denoted by uv. A graph G is called p-regular if every vertex has the same degree equal to p.

For a graph G, let $M = M(G)$ be a graph matrix associated with G. The M-polynomial is defined as $P_M(G, x) = |xI - M|$, where I is the identity matrix. The M-eigenvalues are the roots of the M-polynomial, and the M-spectrum of G, denoted also by $Spec_M(G)$, is the multiset consisting of the M-eigenvalues. If the eigenvalues are all integers, then the graph G is called M-integral. If we consider $A(G)$ (adjacency matrix), we say A-eigenvalues, A-polynomial, A-spectrum. A graph is called integral if its spectrum consists entirely of integers. The research

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for integral graphs started already in 1974 [9] and has continued to this day so far. A survey of integral graphs is given in [2]. It has been discovered recently (see for example [1]) that integral graphs can play a role in the quantum spin networks of quantum computing.

If we consider a matrix $L(G) = D(G) - A(G)$ instead of $A(G)$, where $D(G)$ is the diagonal matrix of vertex-degrees in G, we get the Laplacian eigenvalues and the Laplacian spectrum, while in the case of matrix $Q(G) = D(G) + A(G)$ we get the signless Laplacian eigenvalues and signless Laplacian spectrum. Similarly, if we consider a distance matrix instead of $A(G)$, we get the distance eigenvalues and distance spectrum. A graph is Laplacian (resp. signless Laplacian, distance) integral if its Laplacian (resp. signless Laplacian, distance) spectrum consists entirely of integers. For connections between adjacency, Laplacian resp. signless Laplacian spectral theories see [5]. Some interesting results on Laplacian (signless Laplacian and distance) integral graphs have been found in [5, 7, 10, 11, 14, 15, 16, 17].

Similarly, if we consider a matrix $S(G) = J_{n\times n} - I_n - 2A(G)$, where $J_{n\times n}$ is the $n \times n$ matrix with all entries equal to 1, we get the Seidel eigenvalues, the Seidel spectrum and the Seidel characteristic polynomial $P_S(G, x) = |xI_n - S(G)|$ of the Seidel matrix of G. Although the Seidel matrix has been investigated in a number of books and papers, the Seidel integral graphs had been studied in a few papers so far [12, 13, 15, 18, 20, 21], dealing mostly with complete multipartite graphs. Throughout the paper the corresponding characteristic polynomials are denoted by $P_S(G, x) = |xI_n - S(G)|$. We denote the Seidel spectrum (multiset of eigenvalues) of a square matrix S by $\{\lambda_1^{(s_1)}, ..., \lambda_m^{(s_m)}\}$, where (s_i) is the multiplicity of the eigenvalue λ_i for $1 \le i \le m$. A graph G is S-integral, if all the eigenvalues of its $S-$polynomial are integers.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. The union $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The notation nG is short for $G \cup G \cup ... \cup G$. The complete product $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. The sum $G_1 + G_2$ of graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. Further, let K_n denote the complete graph on n vertices, and let $\overline{K_n}$ denote the graph with n vertices and no edges.

In 2002, Hansen et al. [8] characterized integral graphs in the classes of split complete graphs, whose definitions are recalled below. Note that he used $G_1 \nabla G_2$ instead of $G_1 \vee G_2$.
For $a, b, n \in N$, we have the following classes of graphs:
• the complete split graph $CS_b^a = \overline{K_a} \vee K_b$;
• the multiple complete split-like graph $MCS_{b,n}^a = \overline{K_a} \vee nK_b$;

• the extended complete split-like graph $ECS_b^a = \overline{K_a} \vee (K_b + K_2)$;
• the multiple extended complete split-like graph $MECS_{b,n}^a = \overline{K_a} \vee n(K_b + K_2)$.

In 2010 Freitas et al. [7] gave the characterization of signless Laplacian integrality of those classes of graphs. From [11, 14] it follows that every graph of those classes is Laplacian integral. In [17] the characterization of distance integrality of those classes of graphs is given.

The reminder of this paper is organized as follows. In part 3 we determine the S-spectrum of the H-join of regular graphs and some results on S-integrality of these graphs. In part 4 we apply these results to S-spectrum and S-integrality of the families of complete split graphs, multiple complete split-like graphs, extended complete split-like graphs and multiple extended complete split-like graphs. These results allow us to exhibit many infinite families of Seidel-integral graphs.

2. Preliminaries

In [6] the following result, which is known as the Fiedler's lemma, was obtained.

Lemma 2.1. ([6]) Let A and B be symmetric matrices of orders m and n, respectively, with corresponding eigenpairs $(\alpha_i, \overrightarrow{u_i})$, $i = 1, 2, ..., m$ and $(\beta_i, \overrightarrow{v_i})$, $i = 1, 2, ..., n$, respectively. Suppose that $||\overrightarrow{u_1}|| = ||\overrightarrow{v_1}|| = 1$. Then, for any ρ, the matrix

$$C = \begin{pmatrix} A & \rho \overrightarrow{u_1}\overrightarrow{v_1}^T \\ \rho \overrightarrow{v_1}\overrightarrow{u_1}^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, ..., \alpha_m, \beta_2, ..., \beta_n, \gamma_1, \gamma_2$, where γ_1, γ_2 are the eigenvalues of the matrix

$$\widetilde{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

Note that the results based on Lemma 2.1 were applied for computation of graph energy.

In [19] H-join of graphs $G_1, G_2, ..., G_k$, where H is an arbitrary graph of order k, is defined as follows.

Definition 2.2. [19] Let H be a graph with $V(H) = \{1, 2, ..., k\}$, and G_i be disjoint graphs of order n_i $(i = 1, 2, ..., k)$. Then the graph $\vee_H\{G_1, G_2, ..., G_k\}$ is formed by taking the graphs $G_1, G_2, ..., G_k$ and joining every vertex of G_i to every vertex of G_j whenever i is adjacent to j in H.

Note that if $H = P_2$ then the P_2-join of graphs G_1 and G_2 is $\vee_{P_2}\{G_1, G_2\} = G_1 \vee G_2$.

In [3] the following generalization of the Fiedler's lemma was given.

Definition 2.3. ([3]) For $j \in \{1, 2, ..., k\}$, let M_j be an $m_j \times m_j$ symmetric matrix, with corresponding eigenpairs $(\alpha_{rj}, \overrightarrow{u_{rj}})$, $1 \leq r \leq m_j$. Moreover, for $q \in \{1, ..., k-1\}$ and $l \in \{q+1, ..., k\}$, let $\rho_{q,l}$ be arbitrary constants. Let $\widehat{\alpha}$ be the k-tuple

$$\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k}), \tag{1}$$

where $\alpha_{i_j,j}$ is chosen from the elements of $\{\alpha_{1,j}, ..., \alpha_{m_j,j}\}$, with $j \in \{1, ..., k\}$. Then, considering an arbitrary $\frac{k(k-1)}{2}$-tuple of reals

$$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k}),$$

we define the symmetric matrices

$$C_{\widehat{\alpha}}(\widehat{\rho}) = \begin{pmatrix} M_1 & \rho_{1,2}\overrightarrow{u_{i_1,1}}\overrightarrow{u_{i_2,2}}^T & ... & \rho_{1,k}\overrightarrow{u_{i_1,1}}\overrightarrow{u_{i_k,k}}^T \\ \rho_{1,2}\overrightarrow{u_{i_2,2}}\overrightarrow{u_{i_1,1}}^T & M_2 & ... & \rho_{2,k}\overrightarrow{u_{i_2,2}}\overrightarrow{u_{i_k,k}}^T \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{1,k}\overrightarrow{u_{i_k,k}}\overrightarrow{u_{i_1,1}}^T & \rho_{2,k}\overrightarrow{u_{i_k,k}}\overrightarrow{u_{i_2,2}}^T & ... & M_k \end{pmatrix} \tag{2}$$

and

$$\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho}) = \begin{pmatrix} \alpha_{i_1,1} & \rho_{1,2} & ... & \rho_{1,k} \\ \rho_{1,2} & \alpha_{i_2,2} & ... & \rho_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{1,k} & \rho_{2,k} & ... & \alpha_{i_k,k} \end{pmatrix}. \tag{3}$$

Theorem 2.4. [3] For $j \in \{1, 2, ..., k\}$, let M_j be an $m_j \times m_j$ symmetric matrix, with corresponding eigenpairs $(\alpha_{rj}, \overrightarrow{u_{rj}})$ $\forall r \in I_j = \{1, 2, ..., m_j\}$ and suppose that for each j the system of eigenvectors $\{\overrightarrow{u_{rj}}\}, r \in I_j$, is orthonormal. Consider a $\frac{k(k-1)}{2}$-tuple of scalars $\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k})$ and the k-tuple $\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k})$ defined in (1). Then the matrix $C_{\widehat{\alpha}}(\widehat{\rho})$ in (2) has the multiset of eigenvalues $(\bigcup_{j=1}^{k}\{\alpha_{1,j}, ..., \alpha_{m_j,j}\} - \{\alpha_{i_j,j}\}) \cup \{\gamma_1, ..., \gamma_k\}$, where $\gamma_1, ..., \gamma_k$ are eigenvalues of the matrix $\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho})$ in (3).

The following Lemma is given in [4].

Lemma 2.5. [4] Let

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

be 2×2 block symmetric matrix. Then the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

3. Seidel-Spectra of H-Join of Regular Graphs

For arbitrary graph H with $V(H) = \{1, 2, ..., k\}$, considering the H-join of regular graphs, $\vee_H\{G_1, G_2, ..., G_k\}$, where G_i is a r_i-regular graph of order n_i, we define

$$C_S(H) = \begin{cases} n_i - 2r_i - 1, & i = j; \\ -\sqrt{n_i n_j}, & ij \in E(H); \\ \sqrt{n_i n_j}, & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 3.1. Let H be a graph with $V(H) = \{1, 2, ..., k\}$, and G_i be an r_i-regular graph of order $n_i(i = 1, 2, ..., k)$. If $G = \vee_H\{G_1, G_2, ..., G_k\}$ then

$$Spec_S(G) = (\bigcup_{i=1}^{k}(Spec_S(G_i) - \{n_i - 2r_i - 1\})) \cup Spec(C_S(H)), \tag{5}$$

i.e.

$$P_S(G, x) = P(C_S(H)) \prod_{i=1}^{k} \frac{P_S(G_i, x)}{x - n_i + 2r_i + 1}. \tag{6}$$

Proof. Let $M_i = S_i$ be the Seidel matrix of the r_i-regular graph G_i of order n_i for $i \in \{1, 2, ..., k\}$. Each pair $(n_i - 2r_i - 1, \frac{1}{\sqrt{n_i}}\overrightarrow{1_{n_i}})$ is an eigenpair of $M_i = S_i$. Let

$$\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k}) = (n_1 - 2r_1 - 1, ..., n_k - 2r_k - 1)$$

and

$$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k})$$

such that for $q \in \{1, ..., k-1\}$ and $l \in \{q+1, ..., k\}$,

$$\rho_{q,l} = \rho_{l,q} = \begin{cases} -\sqrt{n_q n_l}, & ql \in E(H); \\ \sqrt{n_q n_l}, & \text{otherwise.} \end{cases} \tag{7}$$

Then using (3)

$$\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho}) = \begin{pmatrix} \alpha_{i_1,1} & \rho_{1,2} & \cdots & \rho_{1,k} \\ \rho_{1,2} & \alpha_{i_2,2} & \cdots & \rho_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{1,k} & \rho_{2,k} & \cdots & \alpha_{i_k,k} \end{pmatrix} = C_S(H). \tag{8}$$

From definition of $\rho_{q,l}$ and eigenpairs $(n_i - 2r_i - 1, \frac{1}{\sqrt{n_i}}\overrightarrow{1_{n_i}})$ of S_i follows that

$$C_{\widehat{\alpha}}(\widehat{\rho}) = (C_{ij})_{k \times k}, \text{ with} \tag{9}$$

$$C_{i,j} = \begin{cases} S_i, & i = j; \\ -J_{n_i \times n_j}, & ij \in E(H); \\ J_{n_i \times n_j}, & \text{otherwise.} \end{cases}$$

Hence $C_{\widehat{\alpha}}(\widehat{\rho}) = S(G)$. Now, using Theorem 2.4, the matrix $S(G)$ has eigenvalues consisting of those of $C_S(H)$ and those of S_i, except for $n_i - 2r_i - 1$ for $i \in \{1, 2, ..., k\}$. $\square$

The following corollaries of the Theorem 3.1 are useful for the S-spectra of complete split graphs.

Corollary 3.2. Let $H = K_{1,n}$ be a graph with $V(K_{1,n}) = \{1, 2, ..., n+1\}$, and G_i be an r_i-regular graph of order $n_i (i = 1, 2, ..., n+1)$. If $G = \vee_{K_{1,n}}\{G_1, G_2, ..., G_{n+1}\}$ then

$$Spec_S(G) = (\bigcup_{i=1}^{n+1} (Spec_S(G_i) - \{n_i - 2r_i - 1\})) \cup Spec(C_S(K_{1,n})), \tag{10}$$

i.e.

$$P_S(G, x) = P(C_S(K_{1,n})) \prod_{i=1}^{n+1} \frac{P_S(G_i, x)}{x - n_i + 2r_i + 1}, \tag{11}$$

where

$$C_S(K_{1,n}) = \begin{pmatrix} n_1 - 2r_1 - 1 & -\sqrt{n_1 n_2} & ... & -\sqrt{n_1 n_{n+1}} \\ -\sqrt{n_2 n_1} & n_2 - 2r_2 - 1 & ... & \sqrt{n_2 n_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ -\sqrt{n_{n+1} n_1} & \sqrt{n_{n+1} n_2} & ... & n_{n+1} - 2r_{n+1} - 1 \end{pmatrix}. \tag{12}$$

Corollary 3.3. Let $H = P_2$ be a graph with $V(P_2) = \{1, 2\}$, and G_i be an r_i-regular graph of order $n_i (i = 1, 2)$. If $G = \vee_{P_2}\{G_1, G_2\}$ then

$$Spec_S(G) = \cup_{i=1}^{2} (Spec_S(G_i) - \{n_i - 2r_i - 1\})) \cup Spec(C_S(P_2)), \tag{13}$$

i.e.

$$P_S(G, x) = P(C_S(P_2)) \frac{P_S(G_1, x) P_S(G_2, x)}{(x - n_1 + 2r_1 + 1)(x - n_2 + 2r_2 + 1)}, \tag{14}$$

where

$$C_S(P_2) = \begin{pmatrix} n_1 - 2r_1 - 1 & -\sqrt{n_1 n_2} \\ -\sqrt{n_2 n_1} & n_2 - 2r_2 - 1 \end{pmatrix}. \tag{15}$$

Corollary 3.4. Let G_i be a S-integral r_i-regular graph of order n_i. Let $G = \vee_{K_{1,n}}\{G_1, G_2, ..., G_{n+1}\}$. Then $G = \vee_{K_{1,n}}\{G_1, G_2, ..., G_{n+1}\}$ is S-integral if and only if $C_S(K_{1,n})$ has only integral eigenvalues.

Corollary 3.5. Let G_i be a S-integral r_i-regular graph of order $n_i (i = 1, 2)$. Let $G = \vee_{P_2}\{G_1, G_2\}$. Then $G = \vee_{P_2}\{G_1, G_2\}$ is S-integral if and only if

$$x^2+[2(r_1+r_2+1)-(n_1+n_2)]x+(r_2-n_2)(2r_1+1)+(r_1-n_1)(2r_2+1)+(r_1+r_2+1) \tag{16}$$

has only integral roots.

Proof. From S-integrality of G_i , $(i = 1, 2)$ it follows that $G = \vee_{P_2}\{G_1, G_2\}$ is S-integral if and only if (14) has integral eigenvalues. Hence

$$\begin{vmatrix} x - n_1 + 2r_1 + 1 & \sqrt{n_1 n_2} \\ \sqrt{n_2 n_1} & x - n_2 + 2r_2 + 1 \end{vmatrix} = (x-n_1+2r_1+1)(x-n_2+2r_2+1)-n_1n_2 =$$

$$x^2+[2(r_1+r_2+1)-(n_1+n_2)]x+(r_2-n_2)(2r_1+1)+(r_1-n_1)(2r_2+1)+(r_1+r_2+1).$$

□

4. Application to Seidel Integral Complete Split Graphs

Let $CS^a_b = \overline{K_a} \vee K_b$ be a complete split graph. The following theorem gives the S-polynomial of this graph.

Theorem 4.1. Let $CS^a_b = \overline{K_a} \vee K_b$ be a complete split graph. Then, the S-polynomial of CS^a_b is $P_S(CS^a_b, x) = (x+1)^{a-1}(x-1)^{b-1}(x^2+(b-a)x-2ab+a+b-1)$.

Proof. From (14) of Corollary 3.3 we have $P_S(CS^a_b, x) = \frac{P_S(\overline{K_a},x)\cdot P_S(K_b,x)}{(x-a+1)(x+b-1)} \cdot P(C_S(P_2))$, where

$$C_S(P_2) = \begin{pmatrix} a-1 & -\sqrt{ab} \\ -\sqrt{ab} & -b+1 \end{pmatrix}.$$

Because of $P_S(\overline{K_a}, x) = (x+1)^{a-1}(x-a+1)$, $P_S(K_b, x) = (x-1)^{b-1}(x+b-1)$ and $P(C_S(P_2), x) = (x^2 + x(b-a) - 2ab + a + b - 1)$, we have the proof. □

Corollary 4.2. Let $CS^a_b = \overline{K_a} \vee K_b$ be a complete split graph. Then CS^a_b is S-integral if and only if $(a+b)^2+4(a-1)(b-1)$ is a perfect square and its S-spectrum is $\{(-1)^{(a-1)}, 1^{(b-1)}, \frac{a-b\pm\sqrt{(a+b)^2+4(a-1)(b-1)}}{2}\}$.

Proof. Using Theorem 4.1, CS_b^a is S-integral if and only if $x^2+x(b-a)-2ab+a+b-1=0$ has integer zeros, from which follows that $x_{1,2} = \frac{a-b\pm\sqrt{(b-a)^2-4(-2ab+a+b-1)}}{2}$ $= \frac{a-b\pm\sqrt{(a+b)^2+4(a-1)(b-1)}}{2}$ are integers. So $(a+b)^2+4(a-1)(b-1)$ has to be a perfect square. □

The following Corollary gives sufficient conditions for a complete split graph CS_b^a to be S-integral.

Corollary 4.3.
1. Let $a=t, b=2t^2-5t+3$ or $b=t, a=2t^2-5t+3$. Then CS_b^a is S-integral for any $t \in N, t > 1$.
2. Let $a=t, b=(t-2)^2$ or $b=t, a=(t-2)^2$. Then CS_b^a is S-integral for any $t \in N, t > 2$.
3. Let $a=1+2t, b=3+3t$ or $b=1+2t, a=3+3t$. Then CS_b^a is S-integral for any $t \in N$.
4. Let $a=2+2t, b=1+3t$ or $b=2+2t, a=1+3t$. Then CS_b^a is S-integral for any $t \in N$.

Proof. For case 1, it is easy to verify that $(a+b)^2+4(a-1)(b-1)=s^2$ for $s=2t^2-2t-1$. Similarly, for case 2, $s=(t+1)(t-2)$, for case 3, $s=7t+4$, for case 4, $s=7t+3$. □

Let $ECS_b^a = \overline{K_a} \vee (K_b+K_2)$ be an extended complete split-like graph. The following theorem gives the S-polynomial of this graph.

Theorem 4.4. Let $ECS_b^a = \overline{K_a} \vee (K_b+K_2)$ be an extended complete split-like graph. Then, the S-polynomial of ECS_b^a is $P_S(ECS_b^a, x) = (x+1)^{a+b-2}(x-3)^{b-1}(x+2b-3)(x^2+x(2-a)-2ab-a+1)$.

Proof. From (14) of Corollary 3.3 we have $P_S(ECS_b^a, x) = \frac{P_S(\overline{K_a},x)\cdot P_S(K_b+K_2,x)}{(x-a+1)(x+1)} \cdot P(C_S(P_2))$, where

$$C_S(P_2) = \begin{pmatrix} a-1 & -\sqrt{2ab} \\ -\sqrt{2ab} & -1 \end{pmatrix}.$$

We know that $P_S(\overline{K_a}, x) = (x+1)^{a-1}(x-a+1)$. Using Lemma 2.5 for the matrix

$$S(K_b+K_2) = \begin{pmatrix} -K_b & K_b-I_b \\ K_b-I_b & -K_b \end{pmatrix}$$

we have $P_S(K_b+K_2, x) = (x+1)^b(x-3)^{b-1}(x+2b-3)$. Since $P(C_S(P_2), x) = (x^2+x(2-a)-2ab-a+1)$, we have the proof. □

Corollary 4.5. The graph ECS_b^a is S-integral if and only if a^2+8ab is a perfect square and its S-spectrum is $\{(-1)^{(a+b-2)}, 3^{(b-1)}, 3-2b, \frac{a-2\pm\sqrt{a^2+8ab}}{2}\}$.

Proof. From Theorem 4.4 follows that ECS_b^a is S-integral if and only if $P(C_S(P_2), x) = x^2 + x(2-a) - 2ab - a + 1$ has only integral eigenvalues, which is if and only if $x_{1,2} = \frac{a-2\pm\sqrt{a^2+8ab}}{2}$ are integers, from which follows that $a^2 + 8ab$ has to be a perfect square. □

Corollary 4.6. The graph $ECS_b^a = \overline{K_a} \vee (K_b + K_2)$ is S-integral if and only if there exist integers k, p and q such that either $(a, b) = (2kq^2, k(p^2 - q^2)/4)$ or $(a, b) = (k(p-q)^2, kpq/2)$.

Proof. From Corollary 4.5, the necessary and sufficient condition for ECS_b^a to be S-integral is that, for some integer r, $a^2 + 8ab = r^2$. This is equivalent to $(a + 4b)^2 = r^2 + (4b)^2$ showing that $r, 4b, a + 4b$ form a Pythagorean triple. By Euclid's formula there exist integers k, p, q such that either $r = 2kpq, 4b = k(p^2 - q^2), a + 4b = k(p^2 + q^2)$ or $r = k(p^2 - q^2), 4b = 2kpq, a + 4b = k(p^2 + q^2)$, from which either $(a, b) = (2kq^2, k(p^2 - q^2)/4)$ or $(a, b) = (k(p-q)^2, kpq/2)$. □

Let $MCS_{b,n}^a = \overline{K_a} \vee nK_b$ be a multiple complete split-like graph. The following theorem gives the S-polynomial of this graph.

Theorem 4.7. Let $MCS_{b,n}^a$ be a multiple complete split-like graph. Then, the S-polynomial of $MCS_{b,n}^a$ is $P_S(MCS_{b,n}^a, x) = (x+1)^{a-1}(x-1)^{n(b-1)}(x + 2b - 1)^{n-1}(x^2 - x(a + nb - 2b) + a - 2ab - nb + 2b - 1)$.

Proof. From Corollary 3.2 $P_S(MCS_{b,n}^a, x) = \frac{P_S(\overline{K_a}, x) \cdot P_S^n(K_b, x)}{(x-a+1)(x+b-1)^n} \cdot P(C_S(K_{1,n}))$, where

$$C_S(K_{1,n}) = \begin{pmatrix} a-1 & -\sqrt{ab} & \dots & -\sqrt{ab} \\ -\sqrt{ab} & -b+1 & \dots & b \\ \vdots & \vdots & \vdots & \vdots \\ -\sqrt{ab} & b & \dots & -b+1 \end{pmatrix}.$$

Hence

$$P(C_S(K_{1,n})) = \begin{vmatrix} x-a+1 & b & \dots & b \\ a & x+b-1 & \dots & -b \\ \vdots & \vdots & \vdots & \vdots \\ a & -b & \dots & x+b-1 \end{vmatrix} =$$

$$= (x + 2b - 1)^{n-1}(x^2 - x(a + nb - 2b) + a - 2ab - nb + 2b - 1).$$

Because of $P_S(\overline{K_a}, x) = (x+1)^{a-1}(x - a + 1)$ and $P_S(K_b, x) = (x-1)^{b-1}(x + b - 1)$, we have the proof. □

Corollary 4.8. The graph $MCS_{b,n}^a$ is S-integral if and only if $(a+bn)^2 + 4(ab - a - b^2n + b^2 + bn - 2b + 1)$ is a perfect square and its S-spectrum is $\{(-1)^{(a-1)}, 1^{(nb-n)}, (1-2b)^{(n-1)}, \frac{a+b(n-2)\pm\sqrt{(a+bn)^2+4(ab-a-b^2n+b^2+bn-2b+1)}}{2}\}$.

Proof. Using Theorem 4.7, $P_S(MCS^a_{b,n}, x)$ have integer zeros if and only if $x^2 - x(a + nb - 2b) + a - 2ab - nb + 2b - 1 = 0$ has integer roots. Since the roots are $x_{1,2} = \frac{a+b(n-2)\pm\sqrt{(a+bn)^2+4(ab-a-b^2n+b^2+bn-2b+1)}}{2}$, $(a + bn)^2 + 4(ab - a - b^2n + b^2 + bn - 2b + 1)$ has to be a perfect square. □

The following Corollary gives sufficient conditions for $MCS^a_{b,n}$ to be S-integral.

Corollary 4.9. 1. Let $a = (n - 1)t, b = t + 2$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N, n > 1$.

2. Let $a = (n - 1)t + 1, b = t$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N$.

3. Let $a = n + (n - 1)t, b = t + 1$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N, n > 1$.

4. Let $a = (n - 1) + (n - 1)t, b = t + 3$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N, n > 1$.

5. Let $a = 1 + (n - 1)t, b = 1 + (n + 1)t$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N$.

6. Let $a = (n-1) + (n-1)t, b = 2n+2+(n+1)t$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N, n > 1$.

7. Let $a = 2+2nt, b = 1+(n+2)t$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N$.

8. Let $a = n + 2nt, b = n + 2 + (n + 2)t$. Then $MCS^a_{b,n}$ is S-integral for any $n, t \in N$.

Proof. For case 1, it is easy to verify that $(a+bn)^2+4(ab-a-b^2n+b^2+bn-2b+1) = (2nt+2n-t-2)^2$. Similarly, in case 2, $(2nt-t+1)^2$, in case 3, $(2nt+2n-t)^2$, in case 4, $(2nt+4n-t-3)^2$, in case 5, $(n^2t+n+t+1)^2$, in case 6, $(n^2t+2n^2-n+t+1)^2$, in case 7, $(n^2t + 2nt + n + 4t + 2)^2$, in case 8, $(n^2t + n^2 + 2nt + n + 4t + 2)^2$. □

Let $MECS^a_{b,n} = \overline{K_a} \vee n(K_b + K_2)$ be a multiple extended complete split-like graph. The following theorem gives the S-polynomial of this graph.

Theorem 4.10. Let $MECS^a_{b,n}$ be a multiple extended complete split-like graph. Then, the S-polynomial of $MECS^a_{b,n}$ is $P_S(MECS^a_{b,n}, x) = (x + 1)^{a+nb-n-1}(x - 3)^{n(b-1)}(x+2b-3)^n(x+2b+1)^{n-1}(x^2-x(a+2nb-2b-2)-2ab-a-2nb+2b+1)$.

Proof. From Corollary 3.2, $P_S(MECS^a_{b,n}, x) = \frac{P_S(\overline{K_a},x)\cdot(P^n_S(K_b+K_2,x))^n}{(x-a+1)(x+1)^n}\cdot P(C_S(K_{1,n}))$, where

$$C_S(K_{1,n}) = \begin{pmatrix} a-1 & -\sqrt{2ab} & \ldots & -\sqrt{2ab} \\ -\sqrt{2ab} & -1 & \ldots & 2b \\ \vdots & \vdots & \vdots & \vdots \\ -\sqrt{2ab} & 2b & \ldots & -1 \end{pmatrix}.$$

Hence

$$P(C_S(K_{1,n})) = \begin{vmatrix} x-a+1 & 2b & ... & 2b \\ a & x+1 & ... & -2b \\ \vdots & \vdots & \vdots & \vdots \\ a & -2b & ... & x+1 \end{vmatrix} =$$

$$= (x+2b+1)^{n-1}(x^2 - x(a+2nb-2b-2) - 2ab - a - 2nb + 2b + 1).$$

Because of $P_S(\overline{K_a}, x) = (x+1)^{a-1}(x-a+1)$ and $P_S(K_b + K_2, x) = (x+1)^b(x-3)^{b-1}(x+2b-3)$, we have the proof. □

Corollary 4.11. The graph $MECS^a_{b,n}$ is S-integral if and only if $(a+2bn-2b)^2+8ab$ is a perfect square and its S-spectrum is $\{(-1)^{(a+nb-n-1)}, 3^{(nb-n)}, (3-2b)^{(n)}, (-1-2b)^{(n-1)}, \frac{a+2bn-2b-2\pm\sqrt{(a+2bn-2b)^2+8ab}}{2}\}$.

Proof. From Theorem 4.10 follows that $P_S(MECS^a_{b,n}, x)$ has integral eigenvalues if and only if $x^2 - x(a+2nb-2b-2) - 2ab - a - 2nb + 2b + 1 = 0$ has integer roots. The roots of the equation are $x_{1,2} = \frac{a+2bn-2b-2\pm\sqrt{(a+2bn-2b)^2+8ab}}{2}$. Hence $(a+2bn-2b)^2+8ab$ has to be a perfect square. □

The following Corollary gives sufficient conditions for $MECS^a_{b,n}$ to be S-integral.

Corollary 4.12.

1. Let $a = (2n-1)+(2n-1)t, b = 1+t$. Then $MECS^a_{b,n}$ is S-integral for any $n, t \in N$.
2. Let $a = (3n-2)+(3n-2)t, b = 3+3t$. Then $MECS^a_{b,n}$ is S-integral for any $n, t \in N$.
3. Let $a = (2n-1)+(2n-1)t, b = (2n+1)+(2n+1)t$. Then $MECS^a_{b,n}$ is S-integral for any $n, t \in N$.
4. Let $a = 2n+2nt, b = n+1+(n+1)t$. Then $MECS^a_{b,n}$ is S-integral for any $n, t \in N$.

Proof. For case 1, it is easy to verify that $(a+2bn-2b)^2+8ab = (t+1)^2(4n-1)^2$. Similarly, for case 2, $(a+2bn-2b)^2+8ab = (t+1)^2(9n-4)^2$, for case 3, $(a+2bn-2b)^2+8ab = (t+1)^2(4n^2+1)^2$, for case 4, $(a+2bn-2b)^2+8ab = 4(t+1)^2(n^2+n+1)^2$. □

5. Conclusions

In the paper we studied Seidel spectra of H-join of regular graphs and applied the obtained results to Seidel integrality of complete split graphs, extended complete split-like graphs, multiple complete split-like graphs and multiple extended complete split-like graphs.

In Corollary 4.6 we gave necessary and sufficient conditions for Seidel integrality of extended complete split-like graphs. However, for complete split graphs, multiple complete split-like graphs and multiple extended complete split-like graphs we found only sufficient conditions.

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