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Some Graph Polynomials of the Power Graph and its Supergraphs

Asma Hamzeh ⋆

Abstract

In this paper, exact formulas for the dependence, independence, vertex cover and clique polynomials of the power graph and its supergraphs for certain finite groups are presented.

Keywords: Dependence polynomial, independence polynomial, vertex cover polynomial, clique polynomial, power graph.

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1. Introduction

Let Γ be an undirected simple graph with edge set $E(\Gamma)$, and vertex set $V(\Gamma)$. We use $|\Gamma|$ to denote the number of vertices of Γ. A set of vertices in a graph such that no two of them are adjacent, is called an independent set. For the graph Γ, a set S of vertices is a clique, if every two distinct vertices in S are adjacent. The clique number of Γ, $\omega(\Gamma)$, is the size of the largest clique in Γ. A vertex cover of a graph is a set S of vertices such that each edge of the graph is incident to at least one vertex of S. The dependence polynomial is introduced by Fisher and Solow in [3]. For a graph Γ this polynomial is defined as

$$f_\Gamma(z) = 1 - c_1 z + c_2 z^2 - c_3 z^3 + \cdots + (-1)^{\omega(\Gamma)} c_{\omega(\Gamma)} z^{\omega(\Gamma)},$$

where c_k is the number of complete subgraphs of size k in Γ. The clique polynomial of Γ, $D_\Gamma(z)$, is defined as $D_\Gamma(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_{\omega(\Gamma)} z^{\omega(\Gamma)}$, where c_k

⋆Corresponding author (E-mail: hamze2006@yahoo.com, hamzeh@irc.ac.ir)
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is the number of cliques with k vertices in Γ. The relation between the dependence and clique polynomials can be described as $D_\Gamma(-z) = f_\Gamma(z)$. The independence polynomial of the graph Γ is defined as $I_\Gamma(z) = \sum_{k=0}^{n}(-1)^k i_k z^k$, in which i_k is the number of independent vertex sets of size k of Γ. The dependence and independence polynomials are in relation $I_{\overline{\Gamma}}(z) = f_\Gamma(z)$. Let c_k be the number of vertex covers of size k of Γ and let $|\Gamma| = n$. The vertex cover polynomial of Γ which is denoted by $\Psi_\Gamma(z)$ is defined as $\Psi_\Gamma(z) = 1 - c_1 z + c_2 z^2 - c_3 z^3 + \cdots + (-1)^n c_n z^n$. This polynomial is related to the independence polynomial by $\Psi_\Gamma(z) = z^n I_\Gamma(z^{-1})$.

Following Sabidussi [11, p. 396], the $A-$join of a set of graphs $\{G_a\}_{a\in A}$ is defined as the graph H with the vertex and edge sets

$$\begin{aligned} V(H) &= \{(x, y) \mid x \in V(A) \ \& \ y \in V(G_x)\}, \\ E(H) &= \{(x, y)(x', y') \mid xx' \in E(A) \ \textit{or else} \ x = x' \ \& \ yy' \in E(G_x)\}. \end{aligned}$$

If A is labeled and has p points, then the $A-$join of $H_1, H_2, \ldots, H_p$ is denoted by $A[H_1, H_2, \ldots, H_p]$.

If Γ_1 and Γ_2 are two graphs with disjoint vertex sets, then the graph union $\Gamma_1 \cup \Gamma_2$ is a graph with $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$. The join of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 + \Gamma_2$, is a graph obtained from Γ_1 and Γ_2 by joining each vertex of Γ_1 to all vertices of Γ_2. Following Došlić [1], for given vertices $y \in V(\Gamma_1)$ and $z \in V(\Gamma_2)$, a splice of Γ_1 and Γ_2 by vertices y and z, $(\Gamma_1.\Gamma_2)(y.z)$, is defined by identifying the vertices y and z in the union of Γ_1 and Γ_2.

Let G be a finite group. The order of $x \in G$ is denoted by $o(x)$. Moreover, we use $\pi_e(G)$ to denote the set of all element orders of G and $\Omega_i(G)$ stands for the number of all elements of order i of G. The notation ϕ is used for the Euler's totient function. The power graph is introduced by Kelarev and Quinn in [7]. Two vertices x and y are adjacent in the power graph if and only if one is a power of the other. Following Feng et al. [2], let $C(G) = \{C_1, \ldots, C_k\}$ be the set of all cyclic subgroups of G and define L_G to be the graph with vertex set $C(G)$ in which two cyclic subgroups are adjacent if one is contained in the other. For complete graph K_{b_i}, where $b_i = \phi(|C_i|)$ and $C_i \in C(G)$, the power graph $\mathcal{P}(G)$ is isomorphic to $L_G[K_{b_1}, K_{b_2}, \ldots, K_{b_k}]$.

Choose a finite group G. The cyclic graph Γ_G is a simple graph with vertex set G. Two elements $x, y \in G$ are adjacent in the cyclic graph if and only if $\langle x, y\rangle$ is cyclic [8]. For $C(G) = \{C_1, \ldots, C_k\}$, define W_G to be the graph with vertex set $C(G)$ in which two cyclic subgroups C_i and C_j are adjacent if one is contained in the other or there exists a cyclic subgroup C_k such that $C_i \subseteq C_k$ and $C_j \subseteq C_k$. As a result, $\Gamma_G = W_G[K_{b_1}, K_{b_2}, \ldots, K_{b_k}]$ with $b_i = \phi(|C_i|)$. Set $\pi_e(G) = \{a_1, \ldots, a_k\}$ and assume that Δ_G is a graph with vertex set $\pi_e(G)$ and edge set $E(\Delta_G) = \{xy \mid x, y \in \pi_e(G), x|y \ \textit{or} \ y|x\}$. As defined in [4, 5], the main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$. In [5], the authors have proved that $\mathcal{S}(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}]$. Note that the graphs $\mathcal{S}(G)$ and Γ_G are supergraphs of the power graph. We refer the reader to [10] for group theory and to [13] for graph theoretical concepts and notations.

2. Results

In this section, we first state some results that will be kept throughout this paper.

Theorem 2.1. [3] *Assume H is a graph with k vertices and $G_1, \ldots, G_k$ are k given graphs. Then the dependence polynomial of the graph $H[G_1, \ldots, G_k]$ is*

$$f_{H[G_1,\ldots,G_k]}(z) = \sum_{A \in C_H} (-1)^{|A|} \prod_{i \in A} (1 - f_{G_i}(z)),$$

where C_H is the set of all subsets of vertices of H that corresponds to complete subgraphs of H.

Theorem 2.2. [3] *Let Γ_1 and Γ_2 be two graphs. Then*

$$\begin{aligned} f_{\Gamma_1 \bigcup \Gamma_2}(z) &= f_{\Gamma_1}(z) + f_{\Gamma_2}(z) - 1, \\ f_{\Gamma_1 + \Gamma_2}(z) &= f_{\Gamma_1}(z) f_{\Gamma_2}(z). \end{aligned}$$

Theorem 2.3. [12] *If Γ_1 and Γ_2 are two graphs, then*

$$f_{(\Gamma_1.\Gamma_2)(y,z)}(x) = f_{\Gamma_1}(x) + f_{\Gamma_2}(x) - (1 - x).$$

By using Theorem 2.1 and this fact that $f_{K_n}(z) = (1-z)^n$, the following result holds:

Corollary 2.4. *The dependence polynomials of graphs $\mathcal{P}(G) = L_G[K_{b_1}, \ldots, K_{b_k}]$, $\mathcal{S}(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}]$ and $\Gamma_G = W_G[K_{b_1}, \ldots, K_{b_k}]$ are as follows:*

$$\begin{aligned} f_{\mathcal{P}(G)}(z) &= \sum_{A \in C_{L_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{b_i}), \\ f_{\mathcal{S}(G)}(z) &= \sum_{A \in C_{\Delta_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{\Omega_{a_i}(G)}), \\ f_{\Gamma_G}(z) &= \sum_{A \in C_{W_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{b_i}), \end{aligned}$$

where C_{L_G}, C_{Δ_G} and C_{W_G} are the set of all subsets of vertices of L_G, Δ_G and W_G corresponding to complete subgraphs of L_G, Δ_G and W_G, respectively.

By using the relationship between the dependence and independence, the vertex cover and the clique polynomials and also this fact that $f_{\overline{K_n}}(z) = 1 - nz$, we have the following result for the graph $\mathcal{S}(G)$.

Corollary 2.5. *The independence, the vertex cover and the clique polynomials of the graph $\mathcal{S}(G)$ are:*

$$\begin{aligned}
D_{\mathcal{S}(G)}(z) &= \sum_{A\in C_{\Delta_G}} (-1)^{|A|} \prod_{i\in A} (1-(1+z)^{\Omega_{a_i}(G)}),\\
I_{\mathcal{S}(G)}(z) &= \sum_{A\in C_{\overline{\Delta_G}}} (-1)^{|A|} \prod_{i\in A} \Omega_{a_i}(G)z,\\
\Psi_{\mathcal{S}(G)}(z) &= z^{|G|} \sum_{A\in C_{\overline{\Delta_G}}} (-1)^{|A|} \prod_{i\in A} \Omega_{a_i}(G)z^{-1},
\end{aligned}$$

where C_{Δ_G} and $C_{\overline{\Delta_G}}$ are defined similar to Theorem 2.1.

In the following results, we apply Theorems 2.1, 2.2 and 2.3 in order to compute the polynomials of the dihedral, semi-dihedral and dicyclic groups which can be presented as follows:

$$\begin{aligned}
D_{2n} &= < a, b | a^n = b^2 = 1, bab = a^{-1} >,\\
SD_{8n} &= < a, b | a^{4n} = b^2 = 1, bab = a^{2n-1} >,\\
T_{4n} &= < a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} > .
\end{aligned}$$

Theorem 2.6. *For any $n \geq 0$,*

$$f_{\Gamma_{D_{2n}}}(z) = (1-z)((1-z)^{n-1} - nz) - 1.$$

Proof. By the definition of a cyclic graph and also the structure of dihedral groups, we have $\Gamma_{D_{2n}} = P_3[K_{n-1}, K_1, \overline{K_n}]$. Now, applying Theorem 2.1 for the path P_3 with vertex set $V(P_3) = \{1, 2, 3\}$, we deduce that $C_{P_3} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. Therefore,

$$\begin{aligned}
f_{\Gamma_{D_{2n}}}(z) &= -(1 - f_{K_{n-1}}(z)) - (1 - f_{K_1}(z)) - (1 - f_{\overline{K_n}}(z))\\
&+ (1 - f_{K_{n-1}}(z))(1 - f_{K_1}(z)) + (1 - f_{K_1}(z))(1 - f_{\overline{K_n}}(z))\\
&= -(1-(1-z)^{n-1}) - (1-(1-z)) - (1-(1-nz))\\
&+ (1-(1-z)^{n-1})(1-(1-z)) + (1-(1-z))(1-(1-nz))\\
&= (1-z)((1-z)^{n-1} - nz) - 1.
\end{aligned}$$

Hence the result follows. □

The following result is an immediate consequence of the previous theorem.

Corollary 2.7. *For any $n \geq 0$,*

$$D_{\Gamma_{D_{2n}}}(z) = (1+z)((1+z)^{n-1} + nz) - 1.$$

Theorem 2.8. *For any $n \geq 0$,*

$$I_{\Gamma_{D_{2n}}}(z) = (1-z)^n(1-nz+z) - z - 1.$$

Proof. It is easy to see that $\overline{\Gamma_{D_{2n}}} = \overline{P_3}[\overline{K_{n-1}}, K_1, K_n]$. Applying Theorem 2.1 for the path $\overline{P_3}$ with vertex set $V(P_3) = \{1,2,3\}$, we have $C_{\overline{P_3}} = \{\{1\},\{2\},\{3\},\{1,3\}\}$. Thus,

$$\begin{aligned} f_{\overline{\Gamma_{D_{2n}}}}(z) &= -(1-f_{\overline{K_{n-1}}}(z)) - (1-f_{K_1}(z)) - (1-f_{K_n}(z)) \\ &+ (1-f_{\overline{K_{n-1}}}(z))(1-f_{K_1}(z)) + (1-f_{K_1}(z))(1-f_{K_n}(z)) \\ &= (1-z)^n(1-nz+z) - z - 1. \end{aligned}$$

Now the result follows from $I_{\Gamma_{D_{2n}}}(z) = f_{\overline{\Gamma_{D_{2n}}}}(z)$. □

By the relationship between the independence polynomial and the vertex cover polynomial, the following result holds.

Corollary 2.9. $\Psi_{\Gamma_{D_{2n}}}(z) = z^{2n}(1-z^{-1})^n(1-nz^{-1}+z^{-1}) - z^{2n-1} - z^{2n}$.

We now take the dicyclic group T_{4n} into account.

Theorem 2.10. *For any $n \geq 0$,*

$$f_{\Gamma_{T_{4n}}}(z) = (1-z)^{2n} + nz(z-1)^2(z-2) - 1.$$

Proof. Assume that W is the graph depicted in Figure 1. Then, we can write $\Gamma_{T_{4n}} = W[K_{2n-2}, K_2, K_2, K_2, \cdots, K_2]$, where there are $n+1$ copies of the complete graph K_2. Therefore, by Theorem 2.1,

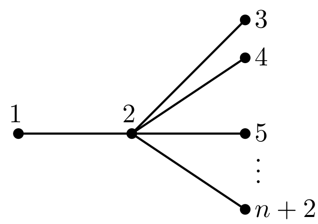

Figure 1: The graph W related to the cyclic graph of T_{4n}.

$$C_W = \{\{1\},\{2\},\{3\},\cdots,\{n+2\},\{1,2\},\{2,3\},\{2,4\},\cdots,\{2,n+2\}\},$$

and so

$$\begin{aligned} f_{\Gamma_{T_{4n}}}(z) &= -(1-f_{K_{2n-2}}(z)) - (1-f_{K_2}(z)) \underbrace{-(1-f_{K_2}(z)) - \cdots - (1-f_{K_2}(z))}_{n} \\ &+ \underbrace{(1-f_{K_2}(z))(1-f_{K_2}(z)) + \cdots + (1-f_{K_2}(z))(1-f_{K_2}(z))}_{n} \\ &+ (1-f_{K_{2n-2}}(z))(1-f_{K_2}(z)) \\ &= (1-z)^{2n} + nz(z-1)^2(z-2) - 1. \end{aligned}$$

This completes the proof. □

Corollary 2.11. $D_{\Gamma_{T_{4n}}}(z) = (1+z)^{2n} - nz(-z-1)^2(-z-2) - 1$.

Theorem 2.12. *Let $n \geq 0$. Then*

$$\begin{aligned} I_{\Gamma_{T_{4n}}}(z) &= -4nz + (-1)^{n+1}(2nz-2z)(2z)^n \\ &+ \sum_{i=2}^{n}(-1)^i \frac{n(n-1)\cdots(n-(i-2))}{(i-1)!}(2nz-2z)(2z)^{i-1} \\ &+ \sum_{i=2}^{n}(-1)^i \frac{n(n-1)\cdots(n-(i-1))}{i!}(2z)^i. \end{aligned}$$

Proof. According to the structure of W, $\overline{W}$ is the graph union of a single vertex at node 2 and the graph K_{n+1}. Therefore, the set $C_{\overline{W}}$ can be decomposed into singleton subsets, two-element subsets, ..., $(n+1)-$element subsets. We have

$$\overline{\Gamma_{T_{4n}}} = \overline{W}[\overline{K_{2n-2}}, \overline{K_2}, \overline{K_2}, \overline{K_2}, \cdots, \overline{K_2}].$$

By applying Theorem 2.1 for singleton subsets and also for $(n+1)-$element subsets, the first and the second terms of the formula are obtained. Since the graph corresponding to the vertex 1 is different from those corresponding to the other vertices, we consider two different categories of subsets: subsets containing vertex 1, and those which do not contain vertex 1. We know that the number of subsets with i elements, $1 \leq i \leq n+1$, is $\binom{n+1}{i}$. Moreover, the number of subsets containing vertex 1 is $\frac{n(n-1)\cdots(n-(i-2))}{(i-1)!}$ and the number of subsets which do not contain vertex 1 is $\frac{n(n-1)\cdots(n-(i-1))}{i!}$. Now, the result follows from Theorem 2.1 and so $I_{\Gamma_{T_{4n}}}(z) = f_{\overline{\Gamma_{T_{4n}}}}(z)$. □

The following result is an immediate consequence of the previous theorem.

Corollary 2.13. *Let $n \geq 0$. Then,*

$$\begin{aligned} \Psi_{\Gamma_{T_{4n}}}(z) &= z^{4n}[-4nz^{-1} + (-1)^{n+1}(2nz^{-1}-2z^{-1})(2z^{-1})^n \\ &+ \sum_{i=2}^{n}(-1)^i \frac{n(n-1)\cdots(n-(i-2))}{(i-1)!}(2nz^{-1}-2z^{-1})(2z^{-1})^{i-1} \\ &+ \sum_{i=2}^{n}(-1)^i \frac{n(n-1)\cdots(n-(i-1))}{i!}(2z^{-1})^i]. \end{aligned}$$

We now consider cyclic groups. Suppose d_i, $1 \leq i \leq t$, are all divisors of n different from n. Then $\mathcal{P}(Z_n) = K_{\phi(n)+1} + \Delta_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \cdots, K_{\phi(d_t)}]$, where Δ_n is the graph with vertex and edge sets $V(\Delta_n) = \{d_i \mid 1, n \neq d_i | n, 1 \leq i \leq t\}$ and $E(\Delta_n) = \{d_i d_j \mid d_i | d_j, 1 \leq i < j \leq t\}$, respectively [9].

Theorem 2.14. *Let $n \geq 0$. Then*

$$f_{\mathcal{P}(Z_n)}(x) = (1-x)^{\phi(n)+1} \sum_{A \in C_{\Delta_n}} (-1)^{|A|} \prod_{i \in A} (1-(1-x)^{\phi(d_i)}),$$

where C_{Δ_n} is defined similar to Theorem 2.1.

Proof. The proof follows from Theorem 2.1 and Theorem 2.2. □

In what follows, we compute all polynomials for the power graph of groups D_{2n}, T_{4n} and SD_{8n}.

Theorem 2.15. *Let $n \geq 0$. Then*

$$\begin{aligned} f_{\mathcal{P}(D_{2n})}(x) &= (1-x)[-x(n-1) \\ &+ (1-x)^{\phi(n)} \sum_{A \in C_{\Delta_n}} (-1)^{|A|} \prod_{i \in A} (1-(1-x)^{\phi(d_i)})], \end{aligned}$$

where C_{Δ_n} is defined similar to Theorem 2.1.

Proof. Note that $\mathcal{P}(D_{2n})$ can be written as $\mathcal{P}(D_{2n}) = S_n.\mathcal{P}(Z_n)$, where S_n is the star graph with root vertex of degree $n-1$ and $\mathcal{P}(Z_n)$ is an induced subgraph of $\mathcal{P}(D_{2n})$ obtained from $\langle a \rangle$. Hence, by Theorems 2.3 and 2.14,

$$\begin{aligned} f_{\mathcal{P}(D_{2n})}(x) &= f_{S_n}(x) + f_{\mathcal{P}(Z_n)}(x) - (1-x) \\ &= (1-x)(1+(n-1)(-x)) - (1-x) \\ &+ (1-x)^{\phi(n)+1} \sum_{A \in C_{\Delta_n}} (-1)^{|A|} \prod_{i \in A} (1-(1-x)^{\phi(d_i)}) \\ &= (1-x)[-x(n-1) \\ &+ (1-x)^{\phi(n)} \sum_{A \in C_{\Delta_n}} (-1)^{|A|} \prod_{i \in A} (1-(1-x)^{\phi(d_i)})], \end{aligned}$$

which completes the proof. □

The dependence polynomial of $\mathcal{P}^*(T_{4n})$ is the subject of our next result.

Theorem 2.16. *For any $n \geq 0$,*

$$\begin{aligned} f_{\mathcal{P}^*(T_{4n})}(x) &= (1-x)[nx^2 - 2nx \\ &+ (1-x)^{\phi(2n)-1} \sum_{A \in C_{\Delta_{2n}}} (-1)^{|A|} \prod_{i \in A} (1-(1-x)^{\phi(d_i)})]. \end{aligned}$$

Proof. Following Hamzeh and Ashrafi [6], we define the rooted graph B to be $B = K_1 + (\cup_{i=1}^{n} K_2)$ with root vertex at node r, where $V(K_1) = \{r\}$. We consider $\mathcal{P}^*(Z_{2n})$ as a rooted graph with root vertex at node a such that a is adjacent to all vertices of this graph. Moreover, we construct $\mathcal{P}^*(T_{4n})$ by identifying the vertex a in $\mathcal{P}^*(Z_{2n})$ and the vertex r in B, i.e. $\mathcal{P}^*(T_{4n}) = \mathcal{P}^*(Z_{2n}).B$. By the graph structure of B, $\omega(B) = 3$ and so $f_B(z) = 1 - (2n+1)z + 3nz^2 - nz^3$. Now by Theorems 2.3 and 2.14 and the dependence polynomial of the graph B,

$$\begin{aligned} f_{\mathcal{P}^*(T_{4n})}(x) &= f_{\mathcal{P}^*(Z_{2n})}(x) + f_B(x) - (1-x) \\ &= (1-x)[nx^2 - 2nx \\ &+ (1-x)^{\phi(2n)-1} \sum_{A \in C_{\Delta_{2n}}} (-1)^{|A|} \prod_{i \in A} (1 - (1-x)^{\phi(d_i)})]. \end{aligned}$$

Consequently, the proof is completed. □

We now compute the dependence polynomial of $\mathcal{P}^*(SD_{8n})$.

Theorem 2.17. *Let $n \geq 0$. Then*

$$\begin{aligned} f_{\mathcal{P}^*(SD_{8n})}(x) &= -nx^3 + 3nx^2 - 4nx \\ &+ (1-x)^{\phi(4n)} \sum_{A \in C_{\Delta_{4n}}} (-1)^{|A|} \prod_{i \in A} (1 - (1-x)^{\phi(d_i)}). \end{aligned}$$

Proof. Similar to the proof of Theorem 2.16, we define the rooted graph B to be $B = K_1 + (\bigcup_{i=1}^{n} K_2)$ with root vertex at node r, where $V(K_1) = \{r\}$. We also consider $\mathcal{P}^*(Z_{4n})$ as a rooted graph with root vertex at node a such that a is connected to all other vertices of $\mathcal{P}^*(Z_{4n})$. Moreover, we construct another graph A by identifying the vertex a in $\mathcal{P}^*(Z_{4n})$ and the vertex r in B, i.e. $A = \mathcal{P}^*(Z_{4n}).B$. By the graph structure of $\mathcal{P}^*(SD_{8n})$, it can be seen that $\mathcal{P}^*(SD_{8n}) = A \cup \overline{K_{2n}}$. Thus by Theorem 2.2,

$$\begin{aligned} f_{\mathcal{P}^*(SD_{8n})}(x) &= f_A(x) + f_{\overline{K_{2n}}}(x) - 1 \\ &= f_A(x) + 1 - (2n)x - 1 \\ &= f_A(x) - 2n. \end{aligned}$$

Next, we compute the dependence polynomial of the graph A. By Theorem 2.3 and the dependence polynomial of B,

$$\begin{aligned} f_A(x) &= f_{\mathcal{P}^*(Z_{4n})}(x) + f_B(x) - (1-x) \\ &= (1-x)^{\phi(4n)} \sum_{A \in C_{\Delta_{4n}}} (-1)^{|A|} \prod_{i \in A} (1 - (1-x)^{\phi(d_i)}) \\ &+ 1 - (2n+1)x + 3nx^2 - nx^3 - (1-x). \end{aligned}$$

As a consequence,

$$\begin{aligned} f_{\mathcal{P}^*(SD_{8n})}(x) &= -nx^3+3nx^2-4nx \\ &+ (1-x)^{\phi(4n)}\sum_{A\in C_{\Delta_{4n}}}(-1)^{|A|}\prod_{i\in A}(1-(1-x)^{\phi(d_i)}). \end{aligned}$$

The proof is completed. □

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Asma Hamzeh
Property and Casualty (Non-life) Insurance Research Group,
Insurance Research Center,
Tehran, Iran
E-mail: hamze2006@yahoo.com, hamzeh@irc.ac.ir