Some Graph Polynomials of the Power Graph and its Supergraphs

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Abstract

In this paper, exact formulas for the dependence, independence, vertex cover and clique polynomials of the power graph and its supergraphs for certain finite groups are presented.

Keywords: Dependence polynomial, independence polynomial, vertex cover polynomial, clique polynomial, power graph.

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1. Introduction

Let $\Gamma$ be an undirected simple graph with edge set $E(\Gamma)$, and vertex set $V(\Gamma)$. We use $|\Gamma|$ to denote the number of vertices of $\Gamma$. A set of vertices in a graph such that no two of them are adjacent, is called an independent set. For the graph $\Gamma$, a set $S$ of vertices is a clique, if every two distinct vertices in $S$ are adjacent. The clique number of $\Gamma$, $\omega(\Gamma)$, is the size of the largest clique in $\Gamma$. A vertex cover of a graph is a set $S$ of vertices such that each edge of the graph is incident to at least one vertex of $S$. The dependence polynomial is introduced by Fisher and Solow in [3]. For a graph $\Gamma$ this polynomial is defined as

$$f_{\Gamma}(z) = 1 - c_1 z + c_2 z^2 - c_3 z^3 + \cdots + (-1)^{\omega(\Gamma)} c_{\omega(\Gamma)} z^{\omega(\Gamma)},$$

where $c_k$ is the number of complete subgraphs of size $k$ in $\Gamma$. The clique polynomial of $\Gamma$, $D_{\Gamma}(z)$, is defined as

$$D_{\Gamma}(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_{\omega(\Gamma)} z^{\omega(\Gamma)},$$

where $c_k$
is the number of cliques with \( k \) vertices in \( \Gamma \). The relation between the dependence and clique polynomials can be described as \( D_{\Gamma}(z) = f_{\Gamma}(z) \). The independence polynomial of the graph \( \Gamma \) is defined as \( I_{\Gamma}(z) = \sum_{k=0}^n \frac{(-1)^k}{k!} \delta_k(z) \), in which \( \delta_k \) is the number of independent vertex sets of size \( k \) of \( \Gamma \). The dependence and independence polynomials are in relation \( I_{\Gamma}(z) = f_{\Gamma}(z) \). Let \( c_k \) be the number of vertex covers of size \( k \) of \( \Gamma \) and let \( |\Gamma| = n \). The vertex cover polynomial of \( \Gamma \) which is denoted by \( \Psi_{\Gamma}(z) \) is defined as \( \Psi_{\Gamma}(z) = 1 - c_1 z + c_2 z^2 - c_3 z^3 + \cdots + (-1)^n c_n z^n \). This polynomial is related to the independence polynomial by \( \Psi_{\Gamma}(z) = z^n I_{\Gamma}(z^{-1}) \).

Following Sabidussi [11, p. 396], the \( A \)-join of a set of graphs \( \{G_a\}_{a \in A} \) is defined as the graph \( H \) with the vertex and edge sets

\[
V(H) = \{ (x, y) \mid x \in V(A) \land y \in V(G_a) \},
\]

\[
E(H) = \{ (x, y)(x', y') \mid xx' \in E(A) \text{ or else } x = x' \land yy' \in E(G_a) \}.
\]

If \( A \) is labeled and has \( p \) points, then the \( A \)-join of \( H_1, H_2, \ldots, H_p \) is denoted by \( H_{[1, H_1, 2, H_2, \ldots, p]} \).

If \( \Gamma_1 \) and \( \Gamma_2 \) are two graphs with disjoint vertex sets, then the graph union \( \Gamma_1 \cup \Gamma_2 \) is a graph with \( V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2) \) and \( E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2) \). The join of two graphs \( \Gamma_1 \) and \( \Gamma_2 \), denoted by \( \Gamma_1 + \Gamma_2 \), is a graph obtained from \( \Gamma_1 \) and \( \Gamma_2 \) by joining each vertex of \( \Gamma_1 \) to all vertices of \( \Gamma_2 \). Following Došlíc [1], for given vertices \( y \in V(\Gamma_1) \) and \( z \in V(\Gamma_2) \), a splice of \( \Gamma_1 \) and \( \Gamma_2 \) by vertices \( y \) and \( z \), \( (\Gamma_1, \Gamma_2)(y, z) \), is defined by identifying the vertices \( y \) and \( z \) in the union of \( \Gamma_1 \) and \( \Gamma_2 \).

Let \( G \) be a finite group. The order of \( x \in G \) is denoted by \( o(x) \). Moreover, we use \( \pi_x(G) \) to denote the set of all element orders of \( G \) and \( \Omega_i(G) \) stands for the number of all elements of order \( i \) of \( G \). The notation \( \phi \) is used for the Euler’s totient function. The power graph is introduced by Kelarev and Quinn in [7]. Two vertices \( x \) and \( y \) are adjacent in the power graph if and only if one is a power of the other. Following Feng et al. [2], let \( C(G) = \{ C_1, \ldots, C_k \} \) be the set of all cyclic subgroups of \( G \) and define \( L_G \) to be the graph with vertex set \( C(G) \) in which two cyclic subgroups are adjacent if one is contained in the other. For complete graph \( K_{b_i} \), where \( b_i = \phi(|C_i|) \) and \( C_i \in C(G) \), the power graph \( P(G) \) is isomorphic to \( L_G[K_{b_1}, K_{b_2}, \ldots, K_{b_k}] \).

Choose a finite group \( G \). The cyclic graph \( \Gamma_G \) is a simple graph with vertex set \( G \). Two elements \( x, y \in G \) are adjacent in the cyclic graph if and only if \( (x, y) \) is cyclic [8]. For \( C(G) = \{ C_1, \ldots, C_k \} \), define \( W_G \) to be the graph with vertex set \( C(G) \) in which two cyclic subgroups \( C_i \) and \( C_j \) are adjacent if one is contained in the other or there exists a cyclic subgroup \( C_k \) such that \( C_i \subseteq C_k \) and \( C_j \subseteq C_k \). As a result, \( \Gamma_G = W_G[K_{b_1}, K_{b_2}, \ldots, K_{b_k}] \) with \( b_i = \phi(|C_i|) \). Set \( \pi_x(G) = \{ a_1, \ldots, a_k \} \) and assume that \( \Delta_G \) is a graph with vertex set \( \pi_x(G) \) and edge set \( E(\Delta_G) = \{ xy \mid x, y \in \pi_x(G), x \neq y \} \). As defined in [4, 5], the main supergraph \( S(G) \) is a graph with vertex set \( G \) in which two vertices \( x \) and \( y \) are adjacent if and only if \( o(x) | o(y) \) or \( o(y) | o(x) \). In [5], the authors have proved that \( S(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}] \). Note that the graphs \( S(G) \) and \( \Gamma_G \) are
supergraphs of the power graph. We refer the reader to [10] for group theory and to [13] for graph theoretical concepts and notations.

2. Results

In this section, we first state some results that will be kept throughout this paper.

**Theorem 2.1.** [3] Assume $H$ is a graph with $k$ vertices and $G_1, \ldots, G_k$ are $k$ given graphs. Then the dependence polynomial of the graph $H[G_1, \ldots, G_k]$ is

$$f_{H[G_1, \ldots, G_k]}(z) = \sum_{A \in C_H} (-1)^{|A|} \prod_{i \in A} (1 - f_{G_i}(z)),$$

where $C_H$ is the set of all subsets of vertices of $H$ that corresponds to complete subgraphs of $H$.

**Theorem 2.2.** [3] Let $\Gamma_1$ and $\Gamma_2$ be two graphs. Then

$$f_{\Gamma_1 \cup \Gamma_2}(z) = f_{\Gamma_1}(z) + f_{\Gamma_2}(z) - 1,$$

$$f_{\Gamma_1 + \Gamma_2}(z) = f_{\Gamma_1}(z)f_{\Gamma_2}(z).$$

**Theorem 2.3.** [12] If $\Gamma_1$ and $\Gamma_2$ are two graphs, then

$$f_{(\Gamma_1 \Gamma_2)(y,z)}(x) = f_{\Gamma_1}(x) + f_{\Gamma_2}(x) - (1 - x).$$

By using Theorem 2.1 and this fact that $f_{K_n}(z) = (1 - z)^n$, the following result holds:

**Corollary 2.4.** The dependence polynomials of graphs $P(G) = L_G[K_{b_1}, \ldots, K_{b_k}]$, $S(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}]$ and $\Gamma_G = W_G[K_{b_1}, \ldots, K_{b_k}]$ are as follows:

$$f_{P(G)}(z) = \sum_{A \in C_{L_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{b_i}),$$

$$f_{S(G)}(z) = \sum_{A \in C_{\Delta_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{\Omega_{a_i}(G)}),$$

$$f_{\Gamma_G}(z) = \sum_{A \in C_{W_G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - z)^{b_i}),$$

where $C_{L_G}, C_{\Delta_G}$ and $C_{W_G}$ are the set of all subsets of vertices of $L_G$, $\Delta_G$ and $W_G$ corresponding to complete subgraphs of $L_G$, $\Delta_G$ and $W_G$, respectively.

By using the relationship between the dependence and independence, the vertex cover and the clique polynomials and also this fact that $f_{K_n}(z) = 1 - nz$, we have the following result for the graph $S(G)$. 

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Corollary 2.5. The independence, the vertex cover and the clique polynomials of the graph $S(G)$ are:

\[
D_{S(G)}(z) = \sum_{A \in C_{\Delta G}} (-1)^{|A|} \prod_{i \in A} (1 - (1 + z)\Omega_{a_i}(G)),
\]

\[
I_{S(G)}(z) = \sum_{A \in C_{\Xi G}} (-1)^{|A|} \prod_{i \in A} \Omega_{a_i}(G)z,
\]

\[
\Psi_{S(G)}(z) = z^{|G|} \sum_{A \in C_{\Xi G}} (-1)^{|A|} \prod_{i \in A} \Omega_{a_i}(G)z^{-1},
\]

where $C_{\Delta G}$ and $C_{\Xi G}$ are defined similar to Theorem 2.1.

In the following results, we apply Theorems 2.1, 2.2 and 2.3 in order to compute the polynomials of the dihedral, semi-dihedral and dicyclic groups which can be presented as follows:

\[
D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle,
\]

\[
SD_{8n} = \langle a, b | a^4n = b^2 = 1, bab = a^{2n-1} \rangle,
\]

\[
T_{4n} = \langle a, b | a^2n = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.
\]

Theorem 2.6. For any $n \geq 0$,

\[
f_{\Gamma D_{2n}}(z) = (1 - z)((1 - z)^{n-1} - nz) - 1.
\]

Proof. By the definition of a cyclic graph and also the structure of dihedral groups, we have $\Gamma D_{2n} = P_3[K_{n-1}, K_1, \overline{K_n}]$. Now, applying Theorem 2.1 for the path $P_3$ with vertex set $V(P_3) = \{1, 2, 3\}$, we deduce that $C_{P_3} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. Therefore,

\[
f_{\Gamma D_{2n}}(z) = -(1 - f_{K_{n-1}}(z)) - (1 - f_{K_1}(z)) - (1 - f_{\overline{K_n}}(z))
+ (1 - f_{K_{n-1}}(z))(1 - f_{K_1}(z)) + (1 - f_{K_1}(z))(1 - f_{\overline{K_n}}(z))
- (1 - (1 - z)^{n-1}) - (1 - (1 - z)) - (1 - (1 - nz))
+ (1 - (1 - z)^{n-1})(1 - (1 - z)) + (1 - (1 - z))(1 - (1 - nz))
= (1 - z)((1 - z)^{n-1} - nz) - 1.
\]

Hence the result follows.

The following result is an immediate consequence of the previous theorem.

Corollary 2.7. For any $n \geq 0$,

\[
D_{\Gamma D_{2n}}(z) = (1 + z)((1 + z)^{n-1} + nz) - 1.
\]
Theorem 2.8. For any \( n \geq 0 \),
\[
I_{\Gamma_{2n}}(z) = (1 - z)^n(1 - nz + z) - z - 1.
\]

Proof. It is easy to see that \( \Gamma_{D_{2n}} = \overline{P_3[K_{n-1}, K_1, K_n]} \). Applying Theorem 2.1 for the path \( P_3 \) with vertex set \( V(P_3) = \{1, 2, 3\} \), we have \( C_{\overline{P_3}} = \{\{1\}, \{2\}, \{3\}, \{1, 3\}\} \).

Thus,
\[
I_{\Gamma_{2n}}(z) = -(1 - f_{K_{n-1}}(z)) - (1 - f_{K_1}(z)) - (1 - f_{K_n}(z)) + (1 - f_{K_n-1}(z))(1 - f_{K_1}(z)) + (1 - f_{K_1}(z))(1 - f_{K_n}(z))
\]
\[
= (1 - z)^n(1 - nz + z) - z - 1.
\]

Now the result follows from \( I_{\Gamma_{2n}}(z) = f_{\Gamma_{2n}}(z) \).

By the relationship between the independence polynomial and the vertex cover polynomial, the following result holds.

Corollary 2.9. \( \Psi_{\Gamma_{2n}}(z) = z^{2n}(1 - z^{-1})n(1 - nz^{-1} + z^{-1}) - z^{2n-1} - z^{2n} \).

We now take the dicyclic group \( T_{4n} \) into account.

Theorem 2.10. For any \( n \geq 0 \),
\[
f_{\Gamma_{T_{4n}}}(z) = (1 - z)^n + nz(z - 1)^2(z - 2) - 1.
\]

Proof. Assume that \( W \) is the graph depicted in Figure 1. Then, we can write \( \Gamma_{T_n} = W[K_{2n-2}, K_2, K_2, K_2, \ldots, K_2] \), where there are \( n+1 \) copies of the complete graph \( K_2 \). Therefore, by Theorem 2.1,

Figure 1: The graph \( W \) related to the cyclic graph of \( T_{4n} \).

\( C_W = \{\{1\}, \{2\}, \{3\}, \ldots, \{n + 2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \ldots, \{2, n + 2\}\} \),

and so
\[
f_{\Gamma_{T_{4n}}}(z) = -(1 - f_{K_{2n-2}}(z)) - (1 - f_{K_2}(z)) \underbrace{-(1 - f_{K_2}(z)) - \cdots - (1 - f_{K_2}(z))}_{n}
\]
\[
+ (1 - f_{K_2}(z))(1 - f_{K_2}(z)) + \cdots + (1 - f_{K_2}(z))(1 - f_{K_2}(z))
\]
\[
+ (1 - f_{K_{2n-2}}(z))(1 - f_{K_2}(z))
\]
\[
= (1 - z)^n + nz(z - 1)^2(z - 2) - 1.
\]
This completes the proof. \qed

Corollary 2.11. \( D_{\Gamma_{T_{4n}}} (z) = (1 + z)^{2n} - nz(-z - 1)^2(-z - 2) - 1. \)

Theorem 2.12. Let \( n \geq 0. \) Then

\[
I_{\Gamma_{T_{4n}}} (z) = -4nz + (-1)^{n+1}(2nz - 2z)(2z)^n \\
+ \sum_{i=2}^{n} (-1)^i \frac{n(n-1) \ldots (n-(i-2))}{(i-1)!} (2nz - 2z)(2z)^{i-1} \\
+ \sum_{i=2}^{n} (-1)^i \frac{n(n-1) \ldots (n-(i-1))}{i!} (2z)^i.
\]

Proof. According to the structure of \( W, \ \overline{W} \) is the graph union of a single vertex at node 2 and the graph \( K_{n+1}. \) Therefore, the set \( C_{\overline{W}} \) can be decomposed into singleton subsets, two-element subsets, ..., \( (n+1) \)-element subsets. We have

\[
\Gamma_{T_{4n}} = W[K_{2n-2}, K_2, K_2, \ldots, K_2].
\]

By applying Theorem 2.1 for singleton subsets and also for \( (n+1) \)-element subsets, the first and the second terms of the formula are obtained. Since the graph corresponding to the vertex 1 is different from those corresponding to the other vertices, we consider two different categories of subsets: subsets containing vertex 1, and those which do not contain vertex 1. We know that the number of subsets with \( i \) elements, \( 1 \leq i \leq n+1, \) is \( \binom{n+1}{i}. \) Moreover, the number of subsets containing vertex 1 is \( \frac{n(n-1) \ldots (n-(i-2))}{(i-1)!} \) and the number of subsets which do not contain vertex 1 is \( \frac{n(n-1) \ldots (n-(i-1))}{i!} \). Now, the result follows from Theorem 2.1 and so \( I_{\Gamma_{T_{4n}}} (z) = f_{\Gamma_{T_{4n}}} (z). \)

The following result is an immediate consequence of the previous theorem.

Corollary 2.13. Let \( n \geq 0. \) Then

\[
\Psi_{\Gamma_{T_{4n}}} (z) = z^{2n} [-4nz^{-1} + (-1)^{n+1}(2nz^{-1} - 2z^{-1})(2z^{-1})^n \\
+ \sum_{i=2}^{n} (-1)^i \frac{n(n-1) \ldots (n-(i-2))}{(i-1)!} (2nz^{-1} - 2z^{-1})(2z^{-1})^{i-1} \\
+ \sum_{i=2}^{n} (-1)^i \frac{n(n-1) \ldots (n-(i-1))}{i!} (2z^{-1})^i].
\]

We now consider cyclic groups. Suppose \( d_i, 1 \leq i \leq t, \) are all divisors of \( n \) different from \( n. \) Then \( P(Z_n) = K_{\phi(n)+1} + \Delta_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \ldots, K_{\phi(d_t)}], \) where \( \Delta_n \) is the graph with vertex and edge sets \( V(\Delta_n) = \{d_i \mid 1, n \neq d_i, 1 \leq i \leq t\} \) and \( E(\Delta_n) = \{d_id_j \mid d_id_j, 1 \leq i < j \leq t\}, \) respectively [9].
**Theorem 2.14.** Let \( n \geq 0 \). Then

\[
 f_{P(Z_n)}(x) = (1 - x)^{\phi(n)+1} \sum_{A \in C_{\Delta n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)}),
\]

where \( C_{\Delta n} \) is defined similar to Theorem 2.1.

**Proof.** The proof follows from Theorem 2.1 and Theorem 2.2. \( \square \)

In what follows, we compute all polynomials for the power graph of groups \( D_{2n}, T_{4n} \) and \( SD_{8n} \).

**Theorem 2.15.** Let \( n \geq 0 \). Then

\[
 f_{P(D_{2n})}(x) = (1 - x)[-x(n - 1) + (1 - x)^{\phi(n)} \sum_{A \in C_{\Delta n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})],
\]

where \( C_{\Delta n} \) is defined similar to Theorem 2.1.

**Proof.** Note that \( P(Z_n) \) can be written as \( P(Z_n) \) and \( S_n \), where \( S_n \) is the star graph with root vertex of degree \( n - 1 \) and \( P(Z_n) \) is an induced subgraph of \( P(D_{2n}) \) obtained from \( \langle a \rangle \). Hence, by Theorems 2.3 and 2.14,

\[
 f_{P(D_{2n})}(x) = f_{S_n}(x) + f_{P(Z_n)}(x) - (1 - x)
 = (1 - x)(1 + (n - 1)(-x)) - (1 - x)
 + (1 - x)^{\phi(n) + 1} \sum_{A \in C_{\Delta n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})
 = (1 - x)[-x(n - 1) + (1 - x)^{\phi(n)} \sum_{A \in C_{\Delta n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})],
\]

which completes the proof. \( \square \)

The dependence polynomial of \( P^*(T_{4n}) \) is the subject of our next result.

**Theorem 2.16.** For any \( n \geq 0 \),

\[
 f_{P^*(T_{4n})}(x) = (1 - x)[nx^2 - 2nx + (1 - x)^{\phi(2n) - 1} \sum_{A \in C_{\Delta 2n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})].
\]
Proof. Following Hamzeh and Ashrafi [6], we define the rooted graph $B$ to be $B = K_1 + (\cup_{i=1}^n K_2)$ with root vertex at node $r$, where $V(K_1) = \{r\}$. We consider $P^*(Z_{2n})$ as a rooted graph with root vertex at node $a$ such that $a$ is adjacent to all vertices of this graph. Moreover, we construct $P^*(T_{4n})$ by identifying the vertex $a$ in $P^*(Z_{2n})$ and the vertex $r$ in $B$, i.e. $P^*(T_{4n}) = P^*(Z_{2n})B$. By the graph structure of $B$, $\omega(B) = 3$ and so $f_B(z) = 1 - (2n + 1)z + 3nz^2 - nz^3$. Now by Theorems 2.3 and 2.14 and the dependence polynomial of the graph $B$,

$$f_{P^*(T_{4n})}(x) = f_{P^*(Z_{2n})}(x) + f_{B}(x) - (1 - x)$$

$$= (1 - x)[nx^2 - 2nx$$

$$+ (1 - x)^{\phi(2n)-1} \sum_{A \in C_{2n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})].$$

Consequently, the proof is completed. \qed

We now compute the dependence polynomial of $P^*(SD_{8n})$.

**Theorem 2.17.** Let $n \geq 0$. Then

$$f_{P^*(SD_{8n})}(x) = -nx^3 + 3nx^2 - 4nx$$

$$+ (1 - x)^{\phi(4n)} \sum_{A \in C_{2n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)}).$$

Proof. Similar to the proof of Theorem 2.16, we define the rooted graph $B$ to be $B = K_1 + (\cup_{i=1}^n K_2)$ with root vertex at node $r$, where $V(K_1) = \{r\}$. We also consider $P^*(Z_{4n})$ as a rooted graph with root vertex at node $a$ such that $a$ is connected to all other vertices of $P^*(Z_{4n})$. Moreover, we construct another graph $A$ by identifying the vertex $a$ in $P^*(Z_{4n})$ and the vertex $r$ in $B$, i.e. $A = P^*(Z_{4n})B$. By the graph structure of $P^*(SD_{8n})$, it can be seen that $P^*(SD_{8n}) = A \cup K_{2n}$. Thus by Theorem 2.2,

$$f_{P^*(SD_{8n})}(x) = f_{A}(x) + f_{K_{2n}}(x) - 1$$

$$= f_{A}(x) + 1 - (2n)x - 1$$

$$= f_{A}(x) - 2n.$$ 

Next, we compute the dependence polynomial of the graph $A$. By Theorem 2.3 and the dependence polynomial of $B$,

$$f_{A}(x) = f_{P^*(Z_{4n})}(x) + f_{B}(x) - (1 - x)$$

$$= (1 - x)^{\phi(4n)} \sum_{A \in C_{2n}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)})$$

$$+ 1 - (2n + 1)x + 3nx^2 - nx^3 - (1 - x).$$
As a consequence,
\[
f_{P^*(SD_{16})}(x) = -nx^3 + 3nx^2 - 4nx \\
+ (1 - x)^{\phi(4n)} \sum_{A \in C_{2^k}} (-1)^{|A|} \prod_{i \in A} (1 - (1 - x)^{\phi(d_i)}).
\]

The proof is completed. \[\square\]

**Conflicts of Interest.** The author declares that there are no conflicts of interest regarding the publication of this article.

**References**


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