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A Study of PageRank in Undirected Graphs

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Abstract

The PageRank (PR) algorithm is the base of Google search engine. In this paper, we study the PageRank sequence for undirected graphs of order six by computing their PR vectors. In continuing, we provide an ordering of graphs by variance of PR vector whose variation is proportional with variance of degree sequence. Finally, we introduce a relation between domination number and PR-variance of given graphs.

Keywords: PageRank algorithm, Google matrix, domination number, isomorphism.

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1. Introduction

All graphs considerd in this paper are undirected connected graphs without loops and parallel edges. Graph theory has been increasingly applied to model phenomena by mathematical formulas. One of these usages is to model the web pages as a directed graph. Sergey Brin and Larry Page in 1998, after discovery of Google, introduced PageRank to make web pages accessible to web surfers [3, 9]. Since then, many researches have been conducted on different aspects of this algorithm. Such studies can be generally classified into PR vector calculations [15–17] and beyond the web applications of this algorithm, see [5].

In PageRank algorithm, each page of the web considered as a vertex of graph, and two vertices are adjacent if and only if there is a link from one page to another.

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By defining a Markov process on this graph, we form Google matrix. It has been proved that the maximum eigenvalue of Google matrix is 1 and the dimension of corresponding eigenvector is also one [2]. An eigenvector for which each entry is corresponded to the PageRank of a page is called PR vector.

Today, diverse applications of PR algorithm are in chemistry [12], social network analysis [8], traffic modeling [14], sports [4, 10], biology [18], etc. Recently, different versions of PR have also been introduced, see [1, 7, 13].

Grolmusz in [6] proved that if the initial vector for power iteration method is proportional with vertices degree and the power method will converges to initial vector. Another interesting application of PR is Iso-Rank used for vertex similarity between two graphs. Let G and H be two graphs. By computing PR on tensor product of Markov transition matrices P and Q (for G and H), we achieve the similarity matrix X in which X_{ij} is similarity between vertices i in G and j in H, for more details see [11].

A subset $D \subseteq V$ of vertices of graph G is called a dominating set if every vertex not in D is adjacent to at least one vertex in D. The dominating number of graph G is defined as the minimum cardinality of all dominating sets. The aim of this paper is to determine the PR vector of simple graphs of order 6. This is the first attempt to determine the PR vector of connected graphs of this order. Since, the number of graphs of order n is very huge, here we report our results only on graphs of order six. The number of such graphs is 112. It is easy to see that the number of graphs of order 7 is $\sum_{i=6}^{21} m_i \binom{21}{i}$, where m_i is the number of connected graphs of order 7 with i edges. We compare our results with dominating number of given graphs.

2. PageRank Algorithm

The PageRank algorithm is based on link between pages of the web. In this graph, each node of V corresponds to a web page, and we have $(i, j) \in E$, if there is an outlink from node i to node j. The adjacency matrix of directed graph G is an $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $(i, j) \in E$, $a_{ij} = -1$ if $(j, i) \in E$ and zero otherwise. Suppose that l_i is the out degree of node i, namely the number of outlinks of node i. If page i has no outlink to other pages, then it is called a dangling node. We characterize dangling nodes by a vector $d \in R^{n\times 1}$ defined as

$$d = (d_i), \tag{1}$$

where $d_i = 1$ if i is a dangling node and zero otherwise. Now we present transition matrix $P = (p_{ij}) \in R^{n\times n}$ by the following way: if i is dangling node, $p_{ij} = 0$ for all $j = 1, ..., n$ and otherwise,

$$p_{ij} = a_{ij}/l_i.$$

Each entry p_{ij} is the probability of jumping from the node i to node j. In personalization PageRank algorithm, we need an extra probability of jumping from one node to other nodes. For this purpose, we use a personalization vector v which is a probability distribution vector. If the web graph has dangling nodes, then $P + du^T$ is the transition matrix where u is a probability distribution vector, that assigns probability of jumping from these dangling nodes. After these modifications, we are ready to introduce the Google matrix which is

$$G = \alpha(P + du^T) + (1 - \alpha)ev^T, \tag{2}$$

where u, v are positive vectors in $R^{n\times 1}$ such that $u^T e = v^T e = 1$. There exists a unique positive eigenvector of G^T called PageRank vector defined as follows:

$$\begin{cases} \pi^T = \pi^T G \\ \pi^T e = 1 \end{cases}. \tag{3}$$

In this paper, we consider only simple graphs which means that in Equation (1) we have $d = 0$. Here, by substituting $d = 0$ in Equations (2, 3) and applying Equation (4), one can see that for the simple graph G with PageRank vector π, we have

$$\pi^T = \alpha\pi^T P + (1 - \alpha)v^T. \tag{4}$$

A list of non-decreasing PageRank of vertices in a graph is called PageRank sequence (PR-sequence).

3. Main Results

Let us to explain our method for computing the PageRank vector of a given graph. Consider the simple graph H depicted in Figure 1. The adjacency matrix of this graph is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The degree sequence vector of vertices of H is $(4, 2, 4, 3, 2, 3)$ and thus $l_1 = l_3 = 4, l_2 = l_5 = 2$ and $l_4 = l_6 = 3$. This means that by using Equation (3) the transition matrix is

$$P = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}.$$

Hence, the Google matrix of graph H is $G = 0.85P + \frac{0.15}{6}J$, where $J = (1)_{6\times 6}$ is a matrix with entries 1. Now from Equation (4), the PageRank vector of graph H is

$$\pi^T = [0.217374, 0.116870, 0.214955, 0.167705, 0.118709, 0.164387].$$

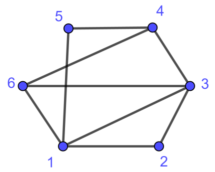

Figure 1: The graph H on 6 vertices and 9 edges.

In general, there are 112 graphs of order 6, see Figure 2. Here, we compute variance of PR vector for all graphs as we shown in the Table 1.

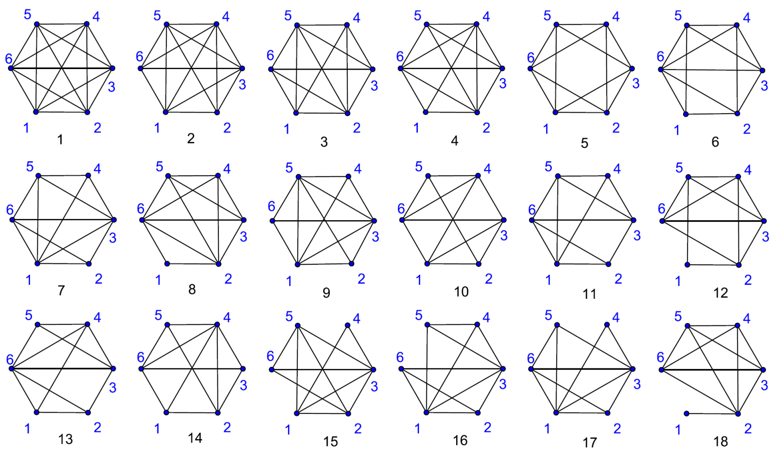

Figure 2: All connected graphs on six vertices.

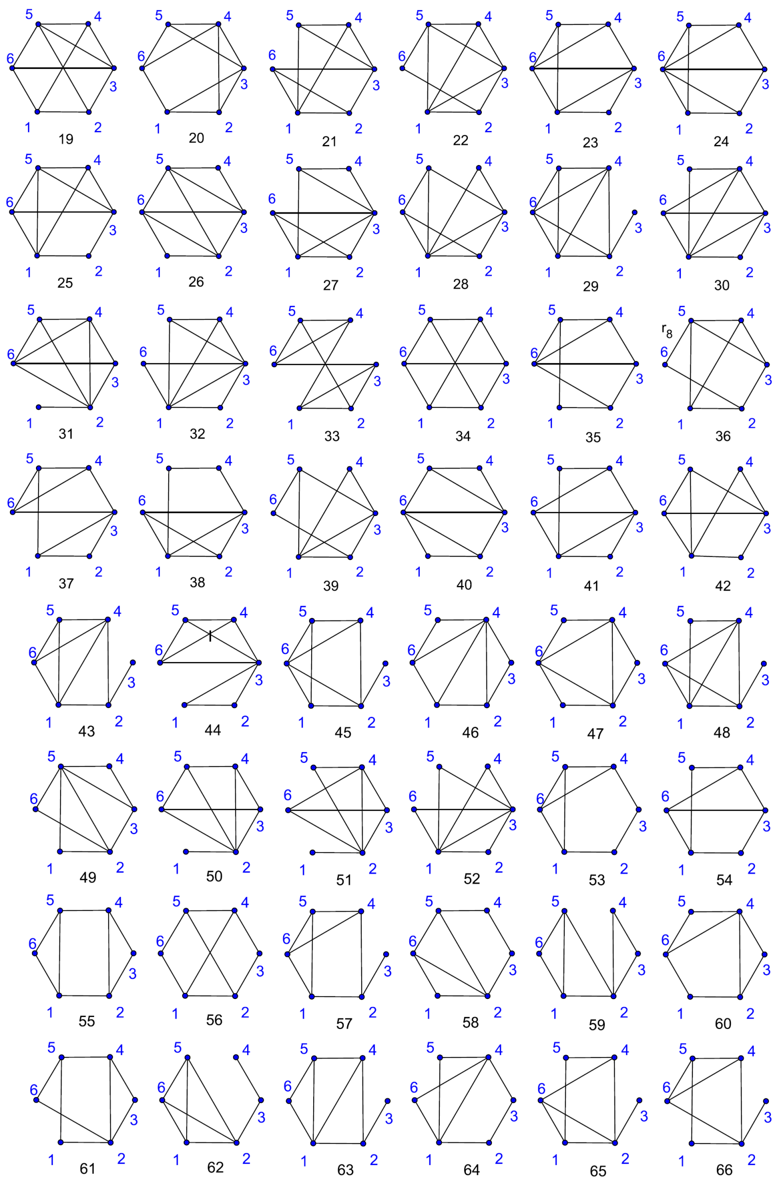

Figure 2: Continued.

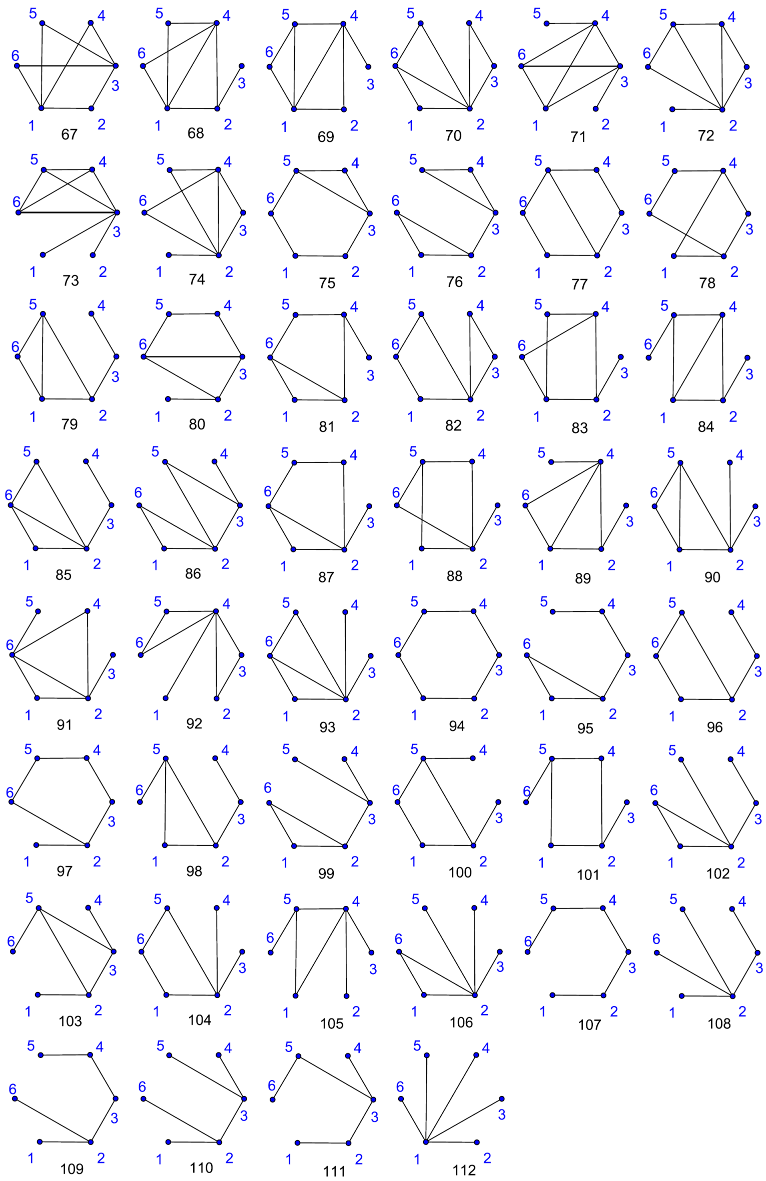

Figure 2: Continued.

Table 1: The PageRank of vertices of connected graphs of order six.

Num	**PageRank**	$\sigma^2(pr)$	γ
1	(0.166667,0.166667,0.166667,0.166667,0.166667,0.166667)	0.000000	1
2	(0.177239,0.145523,0.177239,0.177239,0.177239,0.145523)	0.000268	1
3	(0.155379,0.189243,0.155379,0.155379,0.189243,0.155379)	0.000306	1
4	(0.121743,0.189690,0.154593,0.189690,0.154593,0.189690)	0.000780	1
5	(0.166667,0.166667,0.166667,0.166667,0.166667,0.166667)	0.000000	2
6	(0.130609,0.167090,0.165853,0.165853,0.167090,0.203506)	0.000532	1
7	(0.203882,0.129687,0.203882,0.129687,0.166431,0.166431)	0.001101	1
8	(0.094801,0.205295,0.164870,0.164870,0.164870,0.205295)	0.001632	1
9	(0.204195,0.129139,0.204195,0.129139,0.204195,0.129139)	0.001690	1
10	(0.179372,0.141256,0.179372,0.179372,0.141256,0.179372)	0.000387	2
11	(0.180574,0.139895,0.180574,0.139895,0.179531,0.179531)	0.000430	2
12	(0.102318,0.181924,0.177945,0.177945,0.181924,0.177945)	0.000998	2
13	(0.140494,0.140494,0.179851,0.179851,0.138905,0.220406)	0.001075	1
14	(0.139321,0.180872,0.139321,0.220293,0.139321,0.180872)	0.001105	1
15	(0.181575,0.178448,0.221719,0.101277,0.178448,0.138532)	0.001720	1
16	(0.220764,0.139618,0.220764,0.139618,0.139618,0.139618)	0.001756	1
17	(0.221879,0.138454,0.221879,0.100438,0.138454,0.178896)	0.002445	1
18	(0.064021,0.229538,0.176610,0.176610,0.176610,0.176610)	0.002977	1
19	(0.153942,0.151698,0.195482,0.151698,0.195482,0.151698)	0.000499	2
20	(0.152803,0.151476,0.195721,0.195721,0.151476,0.152803)	0.000507	2
21	(0.196809,0.151595,0.196809,0.151595,0.151595,0.151595)	0.000545	2
22	(0.194244,0.154574,0.194244,0.149396,0.196903,0.110639)	0.001204	2
23	(0.197307,0.108855,0.197307,0.151043,0.151043,0.194446)	0.001296	2
24	(0.151948,0.151948,0.151948,0.151948,0.151948,0.240261)	0.001300	1
25	(0.197889,0.109102,0.197889,0.150396,0.194327,0.150396)	0.001312	2
26	(0.108248,0.195876,0.195876,0.108248,0.195876,0.195876)	0.002048	2
27	(0.194695,0.149929,0.241627,0.109735,0.154085,0.149929)	0.002073	1
28	(0.241282,0.150903,0.197415,0.107969,0.150903,0.151529)	0.002137	1
29	(0.192444,0.206174,0.068812,0.192444,0.147683,0.192444)	0.002698	2
30	(0.242429,0.107888,0.196116,0.196116,0.107888,0.149563)	0.002935	1
31	(0.067276,0.248683,0.149217,0.192803,0.149217,0.192803)	0.003716	1
32	(0.242567,0.107472,0.242567,0.149961,0.149961,0.107472)	0.003818	1
33	(0.166667,0.166667,0.166667,0.166667,0.166667,0.166667)	0.000000	2
34	(0.166667,0.166667,0.166667,0.166667,0.166667,0.166667)	0.000000	2
35	(0.120330,0.168232,0.164756,0.164756,0.168232,0.213692)	0.000874	2
36	(0.165543,0.169332,0.165543,0.164708,0.215999,0.118876)	0.000946	2
37	(0.168632,0.118825,0.216687,0.164835,0.166186,0.164835)	0.000960	2
38	(0.214838,0.162287,0.214838,0.122876,0.122876,0.162287)	0.001703	2
39	(0.214807,0.167179,0.214807,0.116292,0.167179,0.119736)	0.001875	2
40	(0.117984,0.166667,0.215349,0.117984,0.166667,0.215349)	0.001896	2
41	(0.217374,0.116870,0.214955,0.167705,0.118709,0.164387)	0.001938	2
42	(0.218094,0.117689,0.218094,0.117689,0.164217,0.164217)	0.002020	2

Table 1: Continued.

Num	**PageRank**	$\sigma^2(pr)$	γ
43	(0.211821,0.179503,0.075860,0.211821,0.160498,0.160498)	0.002513	2
44	(0.122510,0.122510,0.267308,0.162557,0.162557,0.162557)	0.002816	1
45	(0.163914,0.224732,0.072756,0.163914,0.162890,0.211793)	0.002851	2
46	(0.164845,0.166920,0.117551,0.266212,0.117551,0.166920)	0.002947	1
47	(0.116959,0.216374,0.116959,0.216374,0.116959,0.216374)	0.002965	2
48	(0.212942,0.223238,0.072439,0.212942,0.115500,0.162939)	0.003799	2
49	(0.116379,0.216431,0.166321,0.117511,0.266979,0.116379)	0.004003	1
50	(0.071188,0.271692,0.164280,0.164280,0.164280,0.164280)	0.004034	1
51	(0.071392,0.272895,0.162858,0.213282,0.116715,0.162858)	0.005012	1
52	(0.267856,0.116072,0.267856,0.116072,0.116072,0.116072)	0.006144	1
53	(0.184649,0.134465,0.134465,0.184649,0.180886,0.180886)	0.000625	2
54	(0.187611,0.130415,0.184443,0.184443,0.130415,0.182673)	0.000791	2
55	(0.185065,0.185065,0.129870,0.185065,0.185065,0.129870)	0.000812	2
56	(0.185065,0.185065,0.129870,0.185065,0.185065,0.129870)	0.000812	2
57	(0.182196,0.196921,0.080793,0.182196,0.178947,0.178947)	0.001815	2
58	(0.127550,0.239487,0.132678,0.133618,0.184337,0.182329)	0.001928	2
59	(0.182573,0.241177,0.132610,0.132610,0.182573,0.128458)	0.001968	2
60	(0.130470,0.186122,0.128707,0.239871,0.128707,0.186122)	0.002062	2
61	(0.129370,0.241701,0.128488,0.183978,0.187093,0.129370)	0.002117	2
62	(0.174034,0.237245,0.151335,0.089316,0.174034,0.174034)	0.002272	2
63	(0.236499,0.194896,0.080220,0.181594,0.180417,0.126374)	0.003036	2
64	(0.236499,0.194896,0.080220,0.181594,0.180417,0.126374)	0.003036	2
65	(0.240461,0.132344,0.132344,0.240461,0.127195,0.127195)	0.003273	2
66	(0.183374,0.245912,0.077256,0.183374,0.128913,0.181169)	0.003294	2
67	(0.129538,0.248642,0.077836,0.180752,0.182481,0.180752)	0.003327	2
68	(0.243244,0.128378,0.243244,0.128378,0.128378,0.128378)	0.003518	2
69	(0.237306,0.193773,0.079902,0.237306,0.125856,0.125856)	0.004315	2
70	(0.127750,0.297482,0.131429,0.131429,0.127750,0.184160)	0.004586	1
71	(0.179439,0.076799,0.243761,0.243761,0.076799,0.179439)	0.005673	2
72	(0.076386,0.302275,0.128111,0.182559,0.182559,0.128111)	0.006004	1
73	(0.077475,0.077475,0.308681,0.178789,0.178789,0.178789)	0.007304	1
74	(0.076511,0.303007,0.127245,0.238749,0.127245,0.127245)	0.007315	1
75	(0.150262,0.147369,0.206493,0.142013,0.206493,0.147369)	0.000959	2
76	(0.145985,0.208029,0.208029,0.145985,0.145985,0.145985)	0.001027	2
77	(0.145985,0.208029,0.145985,0.145985,0.208029,0.145985)	0.001027	2
78	(0.143736,0.209536,0.143736,0.209536,0.146727,0.146727)	0.001104	2
79	(0.161121,0.237500,0.177757,0.100546,0.161121,0.161954)	0.001919	2
80	(0.085695,0.214216,0.205370,0.144675,0.144675,0.205370)	0.002556	2
81	(0.140905,0.204669,0.086869,0.218363,0.144784,0.204409)	0.002610	2
82	(0.146033,0.271314,0.143746,0.143746,0.146033,0.149129)	0.002632	2
83	(0.206982,0.215421,0.086036,0.206982,0.142289,0.142289)	0.002664	2
84	(0.138533,0.268000,0.161567,0.093665,0.138533,0.199703)	0.003660	2

Table 1: Continued.

Num	PageRank	$\sigma^2(pr)$	γ
85	(0.202673,0.212203,0.085125,0.202673,0.212203,0.085125)	0.004008	2
86	(0.142722,0.268543,0.216348,0.086300,0.143365,0.142722)	0.004199	2
87	(0.142242,0.276992,0.083861,0.145597,0.145258,0.206049)	0.004416	2
88	(0.143070,0.279073,0.084302,0.143070,0.207415,0.143070)	0.004550	2
89	(0.203469,0.213456,0.085480,0.273647,0.083149,0.140799)	0.005845	2
90	(0.202152,0.285012,0.085566,0.085566,0.202152,0.139552)	0.006083	2
91	(0.141794,0.274808,0.083397,0.141794,0.083397,0.274808)	0.007699	2
92	(0.082887,0.144151,0.144151,0.340509,0.144151,0.144151)	0.007854	1
93	(0.141534,0.345449,0.083726,0.083726,0.141534,0.204031)	0.009676	1
94	(0.166667,0.166667,0.166667,0.166667,0.166667,0.166667)	0.000000	2
95	(0.156757,0.229889,0.168580,0.184572,0.103444,0.156757)	0.001703	2
96	(0.161121,0.237500,0.177757,0.100546,0.161121,0.161954)	0.001919	2
97	(0.094532,0.245403,0.164769,0.165263,0.165263,0.164769)	0.002283	2
98	(0.158816,0.234820,0.176569,0.100041,0.237471,0.092283)	0.003958	2
99	(0.157746,0.231898,0.256984,0.097813,0.097813,0.157746)	0.004411	2
100	(0.163199,0.242961,0.093840,0.093840,0.242961,0.163199)	0.004455	2
101	(0.162837,0.243244,0.093919,0.162837,0.243244,0.093919)	0.004468	2
102	(0.159247,0.313254,0.176622,0.100063,0.091567,0.159247)	0.006355	2
103	(0.093074,0.240259,0.240259,0.093074,0.240259,0.093074)	0.006499	3
104	(0.163161,0.322800,0.093594,0.093594,0.163161,0.163688)	0.007018	2
105	(0.161041,0.092896,0.092896,0.319512,0.240511,0.093145)	0.009027	2
106	(0.161390,0.398822,0.092799,0.092799,0.092799,0.161390)	0.014064	1
107	(0.109743,0.199398,0.190859,0.190859,0.199398,0.109743)	0.001959	2
108	(0.108953,0.197537,0.282093,0.104927,0.197537,0.108953)	0.005141	3
109	(0.106294,0.286924,0.191109,0.199565,0.109814,0.106294)	0.005333	2
110	(0.106383,0.287235,0.287235,0.106383,0.106383,0.106383)	0.008722	2
111	(0.105188,0.377361,0.197946,0.109128,0.105188,0.105188)	0.012004	2
112	(0.472975,0.105405,0.105405,0.105405,0.105405,0.105405)	0.022518	1

Then we sort all graphs with 6 vertices and m edges by $var(PR)$. We have applied this technique for ordering all graphs on six vertices. These graphs can be classified in 11 classes based on the number of edges as reported in Table 2.

Table 2: Graphs on 6 vertices together with edge classes.

Graphs	1	2	3,4	5-9	10-18	19-32	33-52	53-74	75-93	94-106	107-112
Edges	15	14	13	12	11	10	9	8	7	6	5

In Figures 3 and 4, we have drawn the diagrams of $Var(PR)$ and $Var(deg)$ of each class in Table 2.

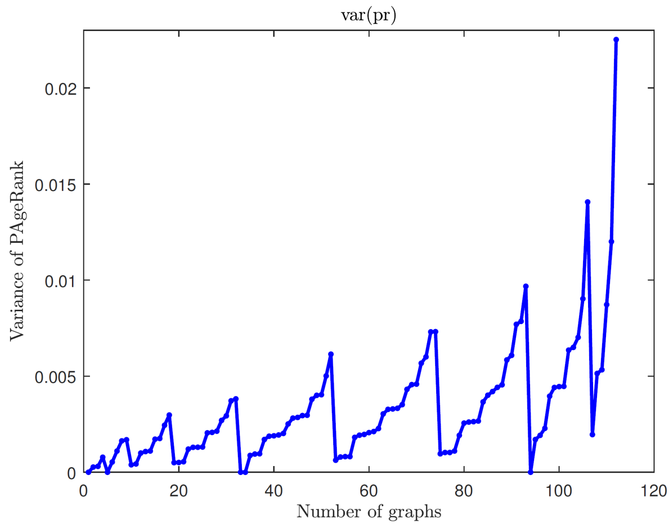

Figure 3: A diagram for Var(pr).

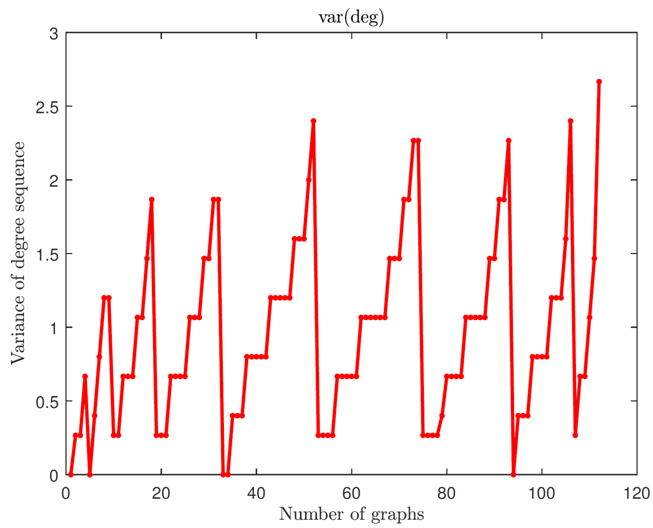

Figure 4: A diagram for Var(degree).

By comparing these diagrams, the following results can be derived:

a) $Var(PR)$ of r-regular graphs ($r = 5, 4, 3, 2$) is zero.

b) As we can see in the diagrams, the trend of changes in $Var(deg)$ is proportional to the $Var(PR)$ of the graphs.

c) There are non-isomorphic graphs with the same PR sequence. We call them as Co-PR graphs, depicted in Figure 5.

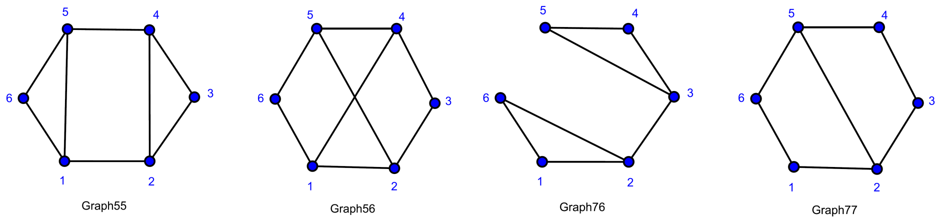

Figure 5: Co-PR graphs on 6-vertices.

In general, there are four Co-PR graphs G_{55}, G_{56}, G_{76} and G_{77} of order 6 as given in Figure 5. In other words, we have

$$\begin{aligned} PR_{55} &= PR_{56} = (0.208029, 0.208029, 0.145985, 0.145985, 0.145985, 0.145985), \\ PR_{76} &= PR_{77} = (0.185065, 0.185065, 0.185065, 0.185065, 0.129870, 0.129870). \end{aligned}$$

This yields the following theorem.

Theorem 3.1. There are no Co-PR graphs on 3, 4 and 5 vertices.

Note that the graphs G_{76} and G_{77} can be obtained from G_{55} and G_{56} after removing an edge, respectively. This yields the following conjecture.

Conjecture 3.2. *Let G and H be two Co-PR graphs on $n \geq 6$ vertices. There are two edges $e \in E(G)$ and $f \in E(H)$ such that G/e and H/f are Co-PR.*

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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