Best Proximity Point Theorems for Ćirić Type *G*-Contractions in Metric Spaces with a Graph

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Abstract

In this paper, we aim to introduce Ćirić type *G*-contractions using directed graphs in metric spaces and then to investigate the existence and uniqueness of best proximity points for them. We also discuss the main theorem and list some consequences of it.

Keywords: *G*-proximal mapping, Ćirić type *G*-contraction, best proximity point.

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1. Introduction and Preliminaries

Let (X, d) be a metric space. In [11, 12], Ćirić investigated mappings $T : X \to X$ which satisfy

$$d(Tx, Ty) \leq h \cdot \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\},$$

for all $x, y \in X$, where $h \in (0, 1)$ (known as Ćirić contractions) and proved that such mappings have a unique fixed point in complete metric spaces. He then constructed an example to show that his new contraction is a real generalization of some well-known linear contractions.

In the past decade, Jachymski [14] entered graphs in metric fixed point theory and generalized the Banach contraction principle in both metric and partially ordered metric spaces. For further works and results in metric spaces endowed with a graph, see e.g., [1, 7, 8, 10].

The main goal of the best proximity point theory is to provide sufficient conditions assuring the existence of such points. Numerous works on best proximity point theory were done and several authors have studied different contractions for having the best proximity point in metric and partially ordered metric spaces as well as metric spaces endowed with a graph (see e.g., [2–6, 13, 16, 17]).

In this paper, we introduce and investigate the notion of a Ćirić type G-contraction in metric spaces endowed with a graph and establish some results on the existence and uniqueness of best proximity points for it.

We start by reviewing a few basic notions in graph and best proximity point theory which are frequently used in this paper. For more details on graphs, the reader is referred to [9].

In an arbitrary (not necessarily simple) graph G, a link is an edge of G with distinct ends and a loop is an edge of G with identical ends. Two or more links of G with the same pairs of ends are called parallel edges of G.

Let (X, d) be a metric space and G be a directed graph with vertex set $V(G) = X$ such that the edge set $E(G)$ contains all loops, that is, $(x, x) \in E(G)$ for all $x \in X$. Assume further that G has no parallel edges. Under these hypotheses, the graph G can be easily denoted by the ordered pair $(V(G), E(G))$ and it is said that the metric space (X, d) is endowed with the graph G.

Considering a pair (A, B) of nonempty subsets of (X, d), we will use the following notations in this paper:

$$
\begin{aligned}
d(A, B) &= \inf \big\{ d(x, y) : x \in A, \ y \in B \big\} \\
A_0 &= \big\{ x \in A : \ d(x, y) = d(A, B) \text{ for some } y \in B \big\}, \\
B_0 &= \big\{ y \in B : \ d(x, y) = d(A, B) \text{ for some } x \in A \big\}.
\end{aligned}
$$

Definition 1.1. ([5]) Let (X, d) be a metric space, (A, B) be a pair of nonempty subsets of X and $T : A \to B$ be a non-self mapping. An element $x \in A$ is called a best proximity point for T if $d(x, Tx) = d(A, B)$.

By the definition of $d(A, B)$, it is apparent that each best proximity point of a non-self mapping $T : A \to B$ is a minimizer of the function $x \mapsto d(x, Tx)$ [5]. Moreover, by the above notations, it is clear that A_0 and B_0 contain all best proximity points of T and the images of them under T, respectively.

Definition 1.2. ([15]) A pair (A, B) of nonempty subsets of a metric space (X, d)

is said to have the P-property if

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B)$$

implies $d(x_1, x_2) = d(y_1, y_2)$ for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

2. Main Results

In this section, (X, d) is a metric space endowed with a graph G and (A, B) is a pair of nonempty closed subsets of X unless otherwise stated.

First, motivated from the idea of S. Basha [5], we introduce the concept of a G-proximal mapping in metric spaces endowed with a graph.

Definition 2.1. We say that a non-self mapping $T : A \to B$ is G-proximal if T satisfies

$$\left. \begin{array}{r} (y_1, y_2) \in E(G) \\ d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{array} \right\} \implies (x_1, x_2) \in E(G)$$

for all $x_1, x_2, y_1, y_2 \in A$.

Now, we are ready to give the definition of Ćirić type G-contractions in metric spaces endowed with a graph. This definition is motivated from [11].

Definition 2.2. We say that a non-self mapping $T : A \to B$ is a Ćirić type G-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot Q_T(x, y) \tag{1}$$

for all $x, y \in A$ with $(x, y) \in E(G)$ where

$$Q_T(x, y) = \max \Big\{ d(x, y), d(x, Tx) - d(A, B),$$
$$d(y, Ty) - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B) \Big\}.$$

We call the number α in (1) the Ćirić G-contractive constant of T.

Example 2.3. Consider the metric space (X, d) endowed with the complete graph G_0 whose vertex set coincides with X and $E(G_0) = X \times X$. If we set $A = B = X$ in Definition 2.2, then Ćirić type G_0-contractions are precisely the Ćirić type contractions introduced in [11].

The main result of this paper is as follows:

Theorem 2.4. Let (X, d) be complete and $T : A \to B$ be a Ćirić type G-contraction satisfying the following conditions:

(i) T is G-proximal with $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;

(ii) There exist elements $x_0, x_1 \in A_0$ such that $(x_0, x_1) \in E(G)$ and $d(x_1, Tx_0) = d(A, B)$;

(iii) T is continuous on A.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u, v \in A$ we have $(u, v) \in E(G)$, then T has a unique best proximity point in A.

Proof. From $x_1 \in A_0$ and $T(A_0) \subseteq B_0$, there exists $x_2 \in A$ such that $d(x_2, Tx_1) = d(A, B)$. In particular, $x_2 \in A_0$. Since $d(x_1, Tx_0) = d(A, B)$ and $(x_0, x_1) \in E(G)$, it follows from the G-proximality of T that $(x_1, x_2) \in E(G)$. Continuing this process, we obtain a sequence $\{x_n\}$ in A_0 such that

$$(x_n, x_{n+1}) \in E(G) \quad \text{and} \quad d(x_{n+1}, Tx_n) = d(A, B), \qquad n = 0, 1, \dots . \qquad (2)$$

Since the pair (A, B) satisfies the P-property, it follows for all $n \in \mathbb{N}$ that

$$\left. \begin{array}{l} d(x_n, Tx_{n-1}) = d(A, B) \\ d(x_{n+1}, Tx_n) = d(A, B) \end{array} \right\} \implies d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n). \qquad (3)$$

On the other hand, if $n \in \mathbb{N}$, because $(x_{n-1}, x_n) \in E(G)$, by (1) we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot Q_T(x_{n-1}, x_n), \qquad (4)$$

where

$$Q_T(x_{n-1}, x_n) = \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}) - d(A, B),$$
$$d(x_n, Tx_n) - d(A, B), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} - d(A, B) \Big\}.$$

Furthermore, using (2) and (3) as well as the triangle inequality, we get

$$Q_T(x_{n-1}, x_n) = \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}) - d(A, B),$$
$$d(x_n, Tx_n) - d(A, B), \frac{d(x_{n-1}, Tx_n) - d(A, B)}{2}$$
$$+ \underbrace{\frac{d(x_n, Tx_{n-1}) - d(A, B)}{2}}_{=0} \Big\}$$
$$= \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}) - d(A, B),$$
$$d(x_n, Tx_n) - d(A, B), \frac{d(x_n, Tx_{n-1}) - d(A, B)}{2} \Big\}$$
$$\leq \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}),$$
$$\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \Big\}$$
$$= \max \Big\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \Big\}.$$

Moreover, since

$$d(x_{n-1}, Tx_{n-1}) - d(A, B) \leq d(x_{n-1}, x_n) + \underbrace{d(x_n, Tx_{n-1}) - d(A, B)}_{=0} = d(x_{n-1}, x_n)$$

and

$$d(x_{n-1}, Tx_n) - d(A, B) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \underbrace{d(x_{n+1}, Tx_n) - d(A, B)}_{=0}$$
$$= d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

it follows from (4) that

$$d(x_n, x_{n+1}) \leq \alpha \cdot \max \Big\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \Big\}$$
$$= \max \Big\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \Big\}.$$

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then we get $d(x_n, x_{n+1}) \leq \alpha d(x_n, x_{n+1}) < d(x_n, x_{n+1})$ which is a contradiction. Therefore, $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$.

Now by induction, we find

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1), \qquad n = 0, 1, \ldots,$$

and so for all $m \geq n \geq 1$,

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \cdots + \alpha^{m-1} d(x_0, x_1) \\
&\leq [\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}] d(x_0, x_1) \\
&\leq \left(\frac{\alpha^n}{1 - \alpha}\right) d(x_0, x_1).
\end{aligned}
$$

Hence $\{x_n\}$ is a Cauchy sequence in $A_0 \subseteq A$ and since (X, d) is complete, there exists $x^* \in X$ (depending on x_0 and x_1) such that $x_n \to x^*$. Moreover, since A is closed, it follows that $x^* \in A$.

We next show that x^* is a best proximity point for T. By the continuity of T on A, we get $Tx_n \to Tx^*$. Also the joint continuity of the metric d implies that $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$. On the other hand, (2) shows that the sequence $\{d(x_{n+1}, Tx_n)\}$ is a constant sequence converging to $d(A, B)$. Therefore, from the uniqueness of the limits of converging sequences in metric spaces, we get $d(x^*, Tx^*) = d(A, B)$, that is, x^* is a best proximity point for T. Moreover, we have $x^* \in A_0$ and $Tx^* \in B_0$.

To show uniqueness, suppose that x^{**} is a best proximity point of T such that $(x^*, x^{**}) \in E(G)$. Since the pair (A, B) satisfies the P-property, $x^*, x^{**} \in A_0$ and $Tx^*, Tx^{**} \in B_0$, it follows that

$$
\left.
\begin{aligned}
d(x^*, Tx^*) &= d(A, B) \\
d(x^{**}, Tx^{**}) &= d(A, B)
\end{aligned}
\right\} \implies d(x^*, x^{**}) = d(Tx^*, Tx^{**}).
$$

Hence by (1),

$$
\begin{aligned}
d(x^*, x^{**}) &= d(Tx^*, Tx^{**}) \\
&\leq \alpha \cdot \max \Big\{ d(x^*, x^{**}), \underbrace{d(x^*, Tx^*) - d(A, B)}_{=0}, \underbrace{d(x^{**}, Tx^{**}) - d(A, B)}_{=0}, \\
&\qquad \frac{d(x^*, Tx^{**}) + d(x^{**}, Tx^*)}{2} - d(A, B) \Big\} \\
&= \alpha \cdot \max \Big\{ d(x^*, x^{**}), \frac{d(x^*, Tx^{**}) + d(x^{**}, Tx^*)}{2} - d(A, B) \Big\} \\
&\leq \alpha \cdot \max \Big\{ d(x^*, x^{**}), \frac{1}{2} \big[d(x^*, x^{**}) + \underbrace{d(Tx^{**}, x^{**}) - d(A, B)}_{=0} \\
&\qquad + d(x^{**}, x^*) + \underbrace{d(x^*, Tx^*) - d(A, B)}_{=0} \big] \Big\} \\
&= \alpha d(x^*, x^{**}).
\end{aligned}
$$

Thus, we find $d(x^*, x^{**}) = 0$ and so $x^* = x^{**}$. $\square$

Example 2.5. Let $X = \mathbb{R}^2$ be equipped with the usual metric

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad \big((x_2, y_1), (x_2, y_2) \in \mathbb{R}^2\big),$$

and put

$$A = \big\{(x, 1): \ x \in [0, 1]\big\} \quad \text{and} \quad B = \big\{(y, 0) : y \in [0, 1]\big\}.$$

Let $T : A \to B$ be defined by

$$T(x, 1) = \begin{cases} (0, 0), & 0 \le x < 1, \\[2mm] (\frac{2}{3}, 0), & x = 1, \end{cases} \qquad \big(x \in [0, 1]\big).$$

Observe that for elements $(1, 1)$ and $(\frac{1}{2}, 1)$ we have

$$Q_T\big((1, 1), (\tfrac{1}{2}, 1)\big) = \max\Big\{\tfrac{1}{2}, \sqrt{\tfrac{1}{9} + 1} - 1, \sqrt{\tfrac{1}{4} + 1} - 1, \frac{\sqrt{2} + \sqrt{\tfrac{1}{36} + 1}}{2} - 1\Big\}$$
$$= \frac{1}{2},$$

and given any $\alpha \in [0, 1)$ we obtain

$$d\big(T(1, 1), T(\tfrac{1}{2}, 1)\big) = d\big((1, 0), (\tfrac{1}{2}, 1)\big) > \frac{\alpha}{2} = \alpha \cdot Q_T\big((1, 1), (\tfrac{1}{2}, 1)\big).$$

So T is not a Ćirić type contraction. Now, define a graph G_4 by $V(G_4) = \mathbb{R}^2$ and

$$E(G_4) = \big\{\big((x_1, x_2), (x_1, x_2)\big) : (x_1, x_2) \in \mathbb{R}^2\big\} \cup \big\{\big((0, 1), (1, 1)\big), \big((1, 1), (0, 1)\big)\big\},$$

and suppose that $\mathbb{R}^2$ is endowed with G_4. Clearly, $d(A, B) = 1$, $A = A_0$ and $B = B_0$. Moreover, one can be simply show that the pair (A, B) satisfies the P-property, T is G_4-proximal and $T(A_0) \subseteq B_0$.

To show that T is a Ćirić type G_4-contraction, assume that $x \in [0, 1]$. Then we have

$$d\big(T(x, 1), T(x, 1)\big) = 0 \le \alpha \cdot Q_T\big((x, 1), (x, 1)\big),$$

and also

$$Q_T\big((0, 1), (1, 1)\big) = \max\Big\{1, 0, \sqrt{\tfrac{1}{9} + 1} - 1, \frac{\sqrt{\tfrac{4}{9} + 1} + \sqrt{2}}{2} - 1\Big\} = 1,$$

which yields

$$d\big(T(0, 1), T(1, 1)\big) = d\big((0, 0), (\tfrac{2}{3}, 0)\big) = \frac{2}{3} \le \alpha \cdot Q_T\big((0, 1), (1, 1)\big),$$

for all $\alpha \in [\frac{2}{3}, 1)$. Thus, T is a Ćirić type G_4-contraction with a Ćirić G-contractive constant $\alpha \in [\frac{2}{3}, 1)$. Moreover, all hypotheses of Theorem 2.4 are satisfied and therefore, T has a best proximity point $x^* = (0, 1)$.

Now, let $x^{**} = (x, 1) \in A$ with $x \in [0, 1]$ be another best proximity point of T. If $x \in [0, 1)$, then

$$d\big((x, 1), T(x, 1)\big) = d\big((x, 1), (0, 0)\big) = \sqrt{x^2 + 1} > d(A, B).$$

Otherwise, if $x = 1$, then

$$d\big((1, 1), T(1, 1)\big) = d\big((1, 1), (\frac{2}{3}, 0)\big) = \sqrt{\frac{1}{9} + 1} > d(A, B),$$

which is a contradiction. Hence $(0, 1)$ is the unique best proximity point of T.

Several consequences of Theorem 2.4 follow now for particular choices of the graph G. First, consider the metric space (X, d) endowed with the complete graph G_0. If we set $G = G_0$ in Theorem 2.4, then it is clear that $T : A \to B$ is G_0-proximal. Thus, we get the following result:

Corollary 2.6. Let (X, d) be a complete metric space, (A, B) be a pair of nonempty closed subsets of (X, d) and $T : A \to B$ be a Ćirić type G_0-contraction satisfying the following conditions:

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;

(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$;

(iii) T is continuous on A.

Then T has a unique best proximity point in A.

Now, suppose that $(X, \preceq)$ is a poset and consider the graph G_1 given by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. If we set $G = G_1$ in Theorem 2.4, then we obtain the following best proximity point result in complete metric spaces endowed with a partial order:

Corollary 2.7. Let $(X, \preceq)$ be a poset and (X, d) be a complete metric space. Suppose that (A, B) is a pair of nonempty closed subsets of (X, d) and $T : A \to B$ is a Ćirić type G_1-contraction satisfying the following conditions:

(i) T is G_1-proximal with $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;

(ii) There exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

(iii) T is continuous on A.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u, v \in A$ we have $u \preceq v$, then T has a unique best proximity point in A.

For the next consequence, suppose again that $(X, \preceq)$ is a poset and consider the graph G_2 given by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \preceq y \vee y \preceq x\}$. If we set $G = G_2$ in Theorem 2.4, then we obtain another best proximity point theorem in complete metric spaces endowed with a partial order.

Corollary 2.8. Let $(X, \preceq)$ be a poset and (X, d) be a complete metric space. Suppose that (A, B) is a pair of nonempty closed subsets of (X, d) and $T : A \to B$ is a Ćirić type G_2-contraction satisfying the following conditions:

(i) T is G_2-proximal with $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;

(ii) There exist comparable elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$;

(iii) T is continuous on A.

Then T has a best proximity point in A. Furthermore, if each two best proximity points are comparable, then T has a unique best proximity point in A.

Finally, let a number $\varepsilon > 0$ be fixed and consider the graph G_ε given by $V(G_\varepsilon) = X$ and $E(G_\varepsilon) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$. If we set $G = G_\varepsilon$ in Theorem 2.4, then we get the following consequence of our main theorem in complete metric spaces:

Corollary 2.9. Let $\varepsilon > 0$ be fixed and (X, d) be complete. Suppose that (A, B) is a pair of nonempty closed subsets of (X, d) and $T : A \to B$ is a Ćirić type G_ε-contraction satisfying the following conditions:

(i) T is G_ε-proximal with $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;

(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x, y) < \varepsilon$ and $d(x_1, Tx_0) = d(A, B)$;

(iii) T is continuous on A.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u, v \in A$ are ε-close, then T has a unique best proximity point in A.

Setting $A = B = X$ in Theorem 2.4, we will easily see that $d(A, B) = 0$ and obtain the following corollary in graph metric fixed point theory:

Corollary 2.10. Let (X, d) be a complete metric space endowed with a graph G and a mapping $T : X \to X$ satisfy the following conditions:

(i) T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(ii) There exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;

(iii) T is continuous on X;

(iv) There exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot \max \Big\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \Big\}$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

Then T has a fixed point in X. Furthermore, if for any two fixed points $u, v \in X$ we have $(u, v) \in E(G)$, then T has a unique fixed point in X.

Remark 1. Ćirić type G-contractions generalize a large number of other contractions introduced so far. So one can apply Theorem 2.4 for various contractions and get new results of best proximity points in complete metric spaces endowed with a graph. We list some of the contractions obtained from Ćirić type G-contractions below:

- There exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in A$ with $(x, y) \in E(G)$ (in this contraction, the mapping T is automatically continuous on A);

- There exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha \big(d(x, Tx) + d(y, Ty) - 2d(A, B) \big)$$

for all $x, y \in A$ with $(x, y) \in E(G)$;

- There exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha \big(d(x, Tx) + d(y, Ty) - 2d(A, B) \big)$$

for all $x, y \in A$ with $(x, y) \in E(G)$;

- There exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \big((d(x, Ty) - d(A, B) \big) + \gamma \big(d(y, Tx) - d(A, B) \big)$$

for all $x, y \in A$ with $(x, y) \in E(G)$;

- There exist functions $\alpha, \beta, \gamma, \zeta : X \times X \to [0, +\infty)$ with

$$\sup \big\{ \alpha(x,y) + \beta(x,y) + \gamma(x,y) + 2\zeta(x,y) : (x,y) \in X \times X \big\} = \lambda < 1$$

such that

$$d(Tx, Ty) \leq \alpha(x,y)d(x,y) + \beta(x,y)\big(d(x,Tx) - d(A,B)\big) + \gamma(x,y)\big(d(y,Ty) \\ - d(A,B)\big) + 2\zeta(x,y)\big(d(x,Ty) + d(y,Tx) - 2d(A,B)\big)$$

for $x, y \in A$ with $(x,y) \in E(G)$.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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