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On Eigenvalues of Permutation Graphs

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Abstract

The aim of this paper is to determine an upper bound for the number of non-co-spectral permutation graphs in terms of automorphism group of a graph G. As a corollary, we determine the eigenvalues of all permutation graphs $P_\alpha(C_n)$, where $\alpha \in Aut(C_n)$.

Keywords: Permutation graph, Petersen graph, automorphism group.

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1. Introduction

All graphs considered in this paper are simple of finite orders, namely, undirected graphs with no loops or parallel edges and with finite number of vertices. Let $G = (V, E)$ be a graph with n vertices and m edges. Let G_1 and G_2 be two graphs with vertex sets $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_n\}$, respectively. For any permutation $\alpha \in \mathbb{S}_n$, the (G_1, G_2)-permutation graph of labeled graphs G_1 and G_2 is the union of G_1 and G_2 together with the edges joining the vertex $v_i \in V(G_1)$ to the vertex $u_{\alpha(i)} \in V(G_2)$. If $G_1 = G_2 = G$, then we show it by $P_\alpha(G)$ and we call it as permutation graph. The edges $v_i u_{\alpha(i)}$ are called the permutation edges of $P_\alpha(G)$. Therefore, if α is the identity map on S, then $P_\alpha(G) \cong G \times K_2$, where $\times$ denotes the Cartesian product of two graphs, see [1]. Note that the graph $P_\alpha(G)$ depends not only on the choice of the permutation α but also on the particular labeling of G as well. For example, there are four permutation graphs which can be obtained from the cycle graph C_5 and one of them is the Petersen graph. The other graphs are depicted in Figure 1.

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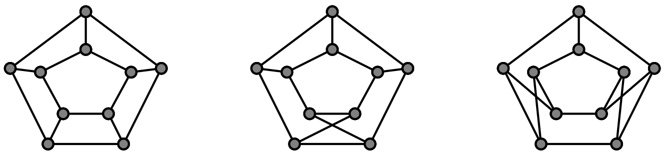

Figure 1: Three permutation graphs constructed by the cycle graph C_5.

It is natural to ask, for given graph G of order n and two permutations $\alpha, \beta \in \mathbb{S}_n$, under which conditions, two permutation graphs $P_\alpha(G)$ and $P_\beta(G)$ are co-spectral? We answer to this question by means of the concept of the automorphism group of a graph G. If $G \cong C_n$, then the permutation graph $P_\alpha(G)$ is isomorphic with the generalized Petersen graph $P(n;k)$, for some k and for some α. Some spectral properties of these graphs are studied in [3,8]. In [3], the author has been proved that the gap between two greatest eigenvalues of the generalized Petersen graphs $P(n;k)$ and $P(n;k')$, where $2 \leq k,\ k' \leq n-1$, tends to zero as n is sufficiently large. As a corollary, an explicit upper bound on the size of this gap is provided. In [8], the authors have been determined the distances between spectra of generalized Petersen graph $P(n,2)$ and all permutation graphs on $2n$ vertices constructed by the cycle graph C_n. The entire spectrum of $P(n;k)$ has been given explicitly by Gera and Stănică [5]. In fact, they provided closed form trigonometric expressions for each eigenvalue of $P(n;k)$.

The paper is organized as follows. In Section 2, we introduce some definitions and concepts that we need throughout this paper. In Section 3, we determine an upper bound for the number distinct non-co-spectral permutation graphs of a group of order n. As a result, we compute an upper bound for the number of distinct non-co-spectral permutation graphs on the cycle graph C_n. Our notation is standard and mainly taken from standard algebraic graph theory books such as [2,6].

2. Definitions and Preliminaries

Suppose $A(G)$ is the adjacency matrix of the graph G and I_n denotes the identity matrix of order n. The characteristic polynomial of G is defined as $\chi(G,\lambda) = det(\lambda I_n - A(G))$. The eigenvalues of G are the roots of $\chi(G,\lambda)$ and compose the spectrum of G. It is a well-known fact that since $A(G)$ is real symmetric matrix, the eigenvalues of $A(G)$ are real numbers. Consider the graph G on n vertices. We denote the eigenvalues of G in descending order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_s(G)$ be the distinct eigenvalues of G with multiplicities $t_1, t_2, \ldots, t_s$, respectively. The multiset $spec(G) = \{[\lambda_1(G)]^{t_1}, [\lambda_2(G)]^{t_2}, \ldots, [\lambda_s(G)]^{t_s}\}$ of eigenvalues of $A(G)$ is called the spectrum of G. For two graphs G and G', if their spectrum are the same, then G and G' are co-spectral with respect to adjacency matrix, otherwise they are non-co-spectral.

The complete and the cyclic graphs on n vertices are denoted by K_n and C_n, respectively. A bijection σ on the vertex set of a graph G is named a *graph automorphism* if it preserves the edge set of G. In the other words, $e = uv$ is an edge of G if and only if $\sigma(e) = \sigma(u)\sigma(v)$ is an edge of G. Let $Aut(G)$ be the set of all graph automorphisms of G. Then $Aut(G)$ under the composition of mappings forms a group. A graph G is called *vertex-transitive* if $Aut(G)$ acting on $V(G)$ has one orbit. We can similarly define an edge-transitive graph just by considering $Aut(G)$ acting on $E(G)$. The Petersen graph is a cubic graph on 10 vertices and 15 edges which is vertex-transitive, see Figure 2. The automorphism graph of this graph is isomorphic with the symmetric group $\mathbb{S}_5$.

Definition 2.1. The generalized Petersen graph $P(n, k)$ is a connected cubic graph with respectively vertex and edge sets:

$$V(P(n, k)) = \{u_i, v_i \mid 1 \leq i \leq n\},$$

and

$$E(P(n, k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 1 \leq i \leq n\},$$

where the subscripts are expressed as integers modulo n ($n \geq 5$), and k is an integer where $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

This class of graphs was introduced by Coxeter in 1950 and it is commonly used in interconnection networks (see [9]). $P(n, k)$ is vertex-transitive if and only if $n = 10$ and $k = 2$ or if $k^2 = \pm 1 \pmod{n}$ [4].

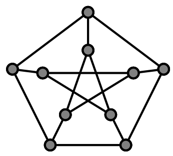

Figure 2: The Petersen graph $P(5, 2)$.

3. Main Results

Let G be a graph of order n and $\alpha \in \mathbb{S}_n$. The structure of adjacency matrix of resulted permutation graph is as follows:

$$M = \begin{pmatrix} A(G) & P_\alpha \\ P_\alpha^t & A(G) \end{pmatrix}.$$

Suppose α and β are two permutations on $\mathbb{S}_n$ which are not an automorphism of Γ. It is clear that
$$\alpha.\Gamma = \beta.\Gamma \Longleftrightarrow \alpha\beta^{-1} \in \Gamma.$$
This yields that $P^t_{\alpha\beta^{-1}}AP_{\alpha\beta^{-1}} = A$ if and only if $(P_\alpha P_{\beta^{-1}})^t AP_\alpha P_{\beta^{-1}} = A$ or equivalently, $P^t_\beta AP_\beta = P^t_\alpha AP_\alpha$.

Theorem 3.1. The number of distinct non-co-spectral permutation graphs on Γ is at most $\frac{n!}{|\Gamma|}$.

Proof. Let G be a graph with adjacency matrix A and the automorphism group $\Gamma = Aut(G)$. Also, let $\alpha, \beta \in \mathbb{S}_n$ are such that $\alpha.\Gamma = \beta.\Gamma$. By above disscusion, we yield that $P^t_\beta AP_\beta = P^t_\alpha AP_\alpha$. Let
$$M_1 = \begin{pmatrix} A & P_\alpha \\ P^t_\alpha & A \end{pmatrix}$$
and
$$M_2 = \begin{pmatrix} A & P_\beta \\ P^t_\beta & A \end{pmatrix}.$$
One can check easily that
$$|\lambda I - M_1| = \begin{vmatrix} \lambda I - A & -P_\alpha \\ -P^t_\alpha & \lambda I - A \end{vmatrix}$$
and
$$|\lambda I - M_2| = \begin{vmatrix} \lambda I - A & -P_\beta \\ -P^t_\beta & \lambda I - A \end{vmatrix}.$$
Thus,
$$\begin{aligned} |\lambda I - M_1| &= |\lambda I - A| \times |\lambda I - A - (P_\alpha(\lambda I - A)^{-1}P^t_\alpha)| \\ &= |\lambda I - A| \times |\lambda I - A + P_\alpha(\lambda I - A)^{-1}P^t_\alpha| \end{aligned}$$
and
$$|\lambda I - M_2| = |\lambda I - A| \times |\lambda I - A + P_\beta(\lambda I - A)^{-1}P^t_\beta|.$$
Since $P_\alpha AP^t_\alpha = P_\beta AP^t_\beta$, we obtain $\lambda I - P_\alpha AP^t_\alpha = \lambda I - P_\beta AP^t_\beta$ if and only if $P_\alpha(\lambda I - A)P^t_\alpha = P_\beta(\lambda I - A)P^t_\beta$. This implies that $P_\alpha(\lambda I - A)^{-1}P^t_\alpha = P_\beta(\lambda I - A)^{-1}P^t_\beta$ and so M_1 and M_2 are co-spectral. Therefore, the maximum number of non-co-spectral permutation graphs on Γ is $\frac{|\mathbb{S}_n|}{|\Gamma|}$ and we are done. □

It is a well-known fact that the automorphism group of the cycle graph C_n is isomorphic with dihedral group D_{2n} of order $2n$ with the following presentation:
$$D_{2n} = \langle x, y : \ x^n = y^2 = 1, \ y^{-1}xy = x^{-1}\rangle.$$

Corollary 3.2. The number of distinct non-co-spectral permutation graphs on the cycle graph C_n is at most $\frac{(n-1)!}{2}$.

Corollary 3.3. Suppose G is a graph with the adjacency matrix

$$M = \begin{pmatrix} A & P_\alpha \\ P_\alpha^t & A \end{pmatrix}.$$

The eigenvalues of M are the union of eigenvalues of A and $A + P_\alpha(\lambda I - A)^{-1}P_\alpha^t$.

Example 3.4. Consider the cycle graph C_4 and suppose that $\Gamma = Aut(C_4)$. It is easy to see that $\Gamma \cong D_8$. Also let $\alpha = (12)$ and $\beta = (123)$ be two permutations of the symmetric $\mathbb{S}_4$. We obtain

$$(123) \cdot (12)^{-1} = (123)(12) = (13) \in \Gamma$$

and thus $\alpha.\Gamma = \beta.\Gamma$. On the other hand,

$$A(C_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad P_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_\beta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is not difficult to see that,

$$P_\beta^t A P_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_\alpha^t A P_\alpha.$$

Set

$$M_1 = \begin{pmatrix} A(C_4) & P_\alpha \\ P_\alpha^t & A(C_4) \end{pmatrix}, \quad M_2 = \begin{pmatrix} A(C_4) & P_\beta \\ P_\beta^t & A(C_4) \end{pmatrix}.$$

Clearly, we have

$$AP_\beta = AP_\alpha = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and

$$P_\alpha^t P_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} I_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_\beta^t P_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} I_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Corollary 3.5. ([8]) Let G be the permutation graph $P_\alpha(C_n)$, where $\alpha \in Aut(C_n)$ with the adjacency matrix M. Then $\chi(M,\lambda) = \chi(C_n, \lambda+1)\chi(C_n, \lambda-1)$.

By relabeling of vertices of a permutation graph G, its adjacency matrix can be written as follows:

$$M = \begin{pmatrix} A & I \\ I & B \end{pmatrix},$$

where A is the adjacency matrix of G and B is the adjacency matrix of G obtained from relabeling of vertices of G. Applying elementary row and column operators yield that M is similar to

$$\begin{pmatrix} A & I \\ 0 & B - A^{-1} \end{pmatrix}.$$

In other words, $|\lambda I - M| = 0$ implies that $|\lambda I - A| = 0$ or $|\lambda I - B + A^{-1}| = 0$. Hence, the spectrum of M is the union of spectrum of A together with the spectrum of $B - A^{-1}$.

Example 3.6. Let $A = A(C_5)$ and $\alpha = (123) \in \mathbb{S}_5$. Then for $B = (A(C_5))_\alpha$, we obtain

$$A = A(C_5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = (A(C_5))_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

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