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Planarity of Inclusion Graph of Cyclic Subgroups of Finite Group

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Abstract

Let G be a finite group. The inclusion graph of cyclic subgroups of G, $\mathcal{I}_c(G)$, is the (undirected) graph with vertices of all cyclic subgroups of G, and two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$, are adjacent if and only if $\langle a\rangle \subset \langle b\rangle$ or $\langle b\rangle \subset \langle a\rangle$. In this paper, we classify all finite abelian groups, whose inclusion graph is planar. Also, we study planarity of this graph for finite group G, where $|\pi(Z(G))| \geq 2$.

Keywords: Inclusion graph, power graph, planarity, abelian group.

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1. Introduction

There is a large literature which is devoted to study the ways of associating a graph to a group for the purpose of investigating the algebraic structure using properties of the associated graph (see for example [1, 2, 9, 10, 11, 12]). The inclusion graph is an interesting graph associated with a group. In [8], P. Devi and R. Rajkumar assigned a graph to subgroups of a group as follows: For a finite group G, the inclusion graph of subgroups of G is a graph whose vertices are all proper subgroups of G and two distinct vertices H and K are adjacent if and only if $H \subset K$ or $K \subset H$.

The directed power graph of a semigroup S was defined by Kelarev and Quinn [13] as the digraph $\Gamma(S)$ with vertex set S, in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Motivated by this,

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Chakrabarty et al. [7] defined the (undirected) power graph $\Gamma(S)$ of a semigroup S as a graph with vertex set S and two distinct vertices x and y joined if one is a power of the other. They proved that for a finite group G, the power graph $\Gamma(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and some positive integer m. In [5, 6], Cameron and Ghosh obtained interesting results about power graphs of finite groups. Mirzargar et al. [15], considered some graph theoretical properties of the power graph $\Gamma(G)$ that can be related to the group theoretical properties of G, such as clique number, independence number and chromatic number. Recently, Bubboloni et al. [3], considered *quotient power graph*, $\tilde{\Gamma}(G)$, obtained from $\Gamma(G)$, where defined a quotient graph with vertex set $[G] = \{[x] : x \in G\}$, $[x] = \{x^m : 1 \leq m \leq \circ(x), (m, \circ(x)) = 1\}$, and $\{[x], [y]\} \in [E]$ is an edge in $\tilde{\Gamma}(G)$ if there exists $x' \in [x]$ and $y' \in [y]$ such that $\{x', y'\} \in E$. Shaker and Iranmanesh in [17], focused on the investigation of (proper) quotient power graphs of finite groups, and studied some graph theoretical properties of the (proper) quotient power graph of a finite group G.

Inspired by ideas from Shaker and Iranmanesh in [17], we study *inclusion graph of cyclic subgroups* of a group G. We see that inclusion graph of cyclic subgroups of a group is exactly quotient power graph of a group. We denote $\mathcal{I}_c^*(G) = \mathcal{I}_c(G) - \langle e \rangle$.

Let Γ be a graph. $V(\Gamma)$ and $E(\Gamma)$ are vertices and edges of Γ, respectively. Also, we denote by $\pi(n)$ the set of all prime divisors of a positive integer n. Given a group G, we shall write $\pi(G)$ instead of $\pi(|G|)$. Also, all groups and graphs in this paper are assumed to be finite.

In [17, Corollary 3.14], Shaker and Iranmanesh proved that, if G be a finite group, where $\pi_e(G) \subseteq \{1, p, p^2, p^3, pq, p^2q\}$, then $\mathcal{I}_c(G)$ is planar, but it's not true for all groups to satisfy in this condition.
In this paper, we study on the planarity of inclusion graph of cyclic subgroups of finite groups (or equivalently quotient power graph of a finite group) and classify all finite abelian groups, whose inclusion graphs are planar . In addition, we present a necessary condition for finite groups, whose inclusion graphs are planar. Also, we study planarity of this graph for finite group G, where $|\pi(Z(G))| \geq 2$.

By a similar argument to [8, Theorem 2.4], we immediately conclude the following theorem.

Theorem 1.1. *Let G be a finite group. Then $\mathcal{I}_c(G)$ is complete if and only if $G \cong \mathbb{Z}_{p^m}$, for some prime p and positive integer m. Moreover, $\mathcal{I}_c(\mathbb{Z}_{p^m}) \cong K_{m+1}$.*

Corollary 1.2. *$\mathcal{I}_c^*(G)$ is complete if and only if $G \cong \mathbb{Z}_{p^m}$ and $\mathcal{I}_c^*(\mathbb{Z}_{p^m}) \cong K_m$, for some prime p and positive integer m.*

2. Planarity of (Proper) Inclusion Graphs

In this section, we discuss the planarity of (proper) inclusion graph of cyclic subgroups of finite groups and we classify all abelian groups with the planar graph.

At the end of, we introduce a necessary condition for finite groups, with a planar graph. We begin with the following theorem from [18](Kuratowski's Theorem), that will be needed in our proofs.

Theorem 2.1. *[18, Theorem 9.10] A finite graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 2.2. *Let G be a finite group and $x \in G$ such that $\circ(x) = pqrs$, where p, q, r and s are distinct prime numbers. Then $\mathcal{I}_c^*(G)$ is not planar.*

Proof. Let $\circ(x) = pqrs$. Then the vertices $\langle x\rangle, \langle x^p\rangle, \langle x^q\rangle$ are all adjacent to the vertices $\langle x^{pq}\rangle, \langle x^{pqr}\rangle, \langle x^{pqs}\rangle$. Thus $K_{3,3}$ is a subgraph of $\mathcal{I}_c^*(G)$ and by Theorem 2.1, $\mathcal{I}_c^*(G)$ is not planar. □

Lemma 2.3. *Let G be a finite group and $x \in G$ such that $\circ(x) = p^2qr$, where p, q and r are distinct prime numbers. Then $\mathcal{I}_c^*(G)$ is not planar.*

Proof. Let $\circ(x) = p^2qr$. Then the vertices $\langle x\rangle, \langle x^p\rangle, \langle x^q\rangle$ are all adjacent to the vertices $\langle x^{pq}\rangle, \langle x^{pqr}\rangle, \langle x^{p^2q}\rangle$. Thus $\mathcal{I}_c^*(G)$ has a subgraph isomorphic to $K_{3,3}$. So $\mathcal{I}_c^*(G)$ is not planar, by Theorem 2.1. □

Lemma 2.4. *Let G be a finite abelian group and $|\pi(G)| = 3$. Then $\mathcal{I}_c^*(G)$ is planar if and only if $G \cong \mathbb{Z}_{pqr}$, where p, q, r are distinct primes.*

Proof. Let $\mathcal{I}_c^*(G)$ is planar. By Lemma 2.3, G has no element of order p^2qr. Let G has a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{qr}$. There are elements a_i, $i \in \{1, 2, 3\}$ of $\mathbb{Z}_p \times \mathbb{Z}_p$ such that $\langle a_i\rangle \neq \langle a_j\rangle$ for $1 \leq i, j \leq 3$. Let b, c are elements of $\mathbb{Z}_{qr}$ of orders q, r, respectively. Then the vertices $\langle b\rangle, \langle c\rangle, \langle bc\rangle$ are all adjacent to $\langle a_ibc\rangle$, $i = \{1, 2, 3\}$, and so $K_{3,3}$ is a subgraph of $\mathcal{I}_c^*(G)$, a contradiction. Thus $|G| = pqr$. Now, let $G \cong \mathbb{Z}_{pqr}$ and $x, y, z \in G$ of order p, q, r, respectively. Then all non-identity cyclic subgroups of G are $\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle xy\rangle, \langle xz\rangle, \langle yz\rangle$ and $\langle xyz\rangle$ and by the Figure 1, $\mathcal{I}_c^*(G)$ is planar. □

Theorem 2.5. *Let G be a nilpotent group and $\mathcal{I}_c^*(G)$ be a planar graph. Then $|\pi(G)| \leq 3$. And, if $|\pi(G)| = 3$, then $G \cong \mathbb{Z}_{pqr}$, where p, q, r are distinct primes .*

Proof. Since G is nilpotent, $G = P_1 \times P_2 \times \cdots \times P_t$, where $P_1, P_2, \cdots, P_t$ are distinct Sylow subgroups of G. By Lemma 2.2, $t \leq 3$, where $t = |\pi(G)|$.

If $|\pi(G)| = 3$ and P_1 has a subgroup of order p_1^2, then G has an abelian subgroup of order $p_1^2p_2p_3$, which contract Lemma 2.4. So the proof is completed. □

Lemma 2.6. *Let G be a finite abelian p-group and $exp(G) \leq p^3$. Then $\mathcal{I}_c^*(G)$ is planar.*

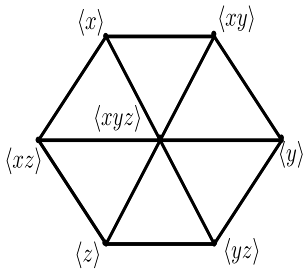

Figure 1: $\mathcal{I}_c^*(\mathbb{Z}_{pqr})$.

Proof. We can consider that $exp(G) = p^3$. Let $G \cong \underbrace{\mathbb{Z}_{p^3} \times \cdots \times \mathbb{Z}_{p^3}}_{n-times}$, and $C_{p^\nu}(G)$ be the number of cyclic subgroups of order p^ν. Then we have

$$C_p(G) = \frac{p^n - 1}{p-1} \quad , \quad C_{p^2}(G) = \frac{p^{n-1}(p^n-1)}{p-1} \quad , \quad C_{p^3}(G) = \frac{p^{2(n-1)}(p^n-1)}{p-1}.$$

Since any connected components of $\mathcal{I}_c^*(G)$ contains exactly one subgroup of order p, so the number of connected component of $\mathcal{I}_c^*(G)$ is equal to $\frac{p^n-1}{p-1}$. Also, every vertex of order p is adjacent to p^{n-1} vertices of order p^2 and every vertex of order p^2 is adjacent to p^{n-1} vertices of order p^3. Suppose that $H_i = \langle a_i\rangle$, $1 \le i \le \frac{p^n-1}{p-1}$ are the vertices of order p, $T_{ij} = \langle b_{ij}\rangle$, $1 \le j \le p^{n-1}$ are the vertices of order p^2 and $N_{ijk} = \langle c_{ijk}\rangle$, $1 \le k \le p^{n-1}$ are the vertices of order p^3 where are adjacent to T_{ij}. Then any connected component of $\mathcal{I}_c^*(G)$ is as Figure 2, which shows $\mathcal{I}_c^*(G)$ is planar. $\square$

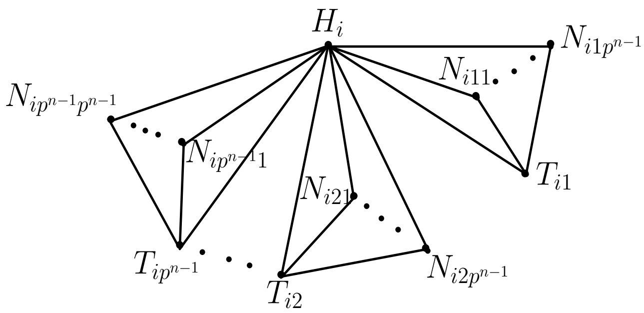

Figure 2: Every connected component of $\mathcal{I}_c^*(\mathbb{Z}_{p^3} \times \cdots \times \mathbb{Z}_{p^3})$.

Lemma 2.7. *Let G be a finite abelian p-group of exponent p^4. Then $\mathcal{I}_c^*(G)$ is planar if and only if $G \cong \mathbb{Z}_{p^4}$ or $G \cong \mathbb{Z}_{2^4} \times \mathbb{Z}_{2^\alpha}$, $\alpha \leq 4$.*

Proof. Suppose that $\mathcal{I}_c^*(G)$ is planar and $G \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_p$, $p \geq 3$. Also, suppose that x and y are elements of G of orders p^4 and p, respectively and $G = \langle x\rangle\langle y\rangle$. Then the vertices $\langle x\rangle, \langle xy\rangle, \langle xy^2\rangle$ are all adjacent to $\langle x^p\rangle, \langle x^{p^2}\rangle, \langle x^{p^3}\rangle$, and so $K_{3,3}$ is a subgraph of $\mathcal{I}_c^*(G)$, a contradiction by Theorem 2.1.

Assume that $p = 2$, and $G \cong \mathbb{Z}_{2^4} \times H$, where H is an abelian 2-group of exponent 2^α, $\alpha \leq 4$. Firstly, let H be not cyclic. We can assume that $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose that, $a_i = (x_i, 0, 0)$ are the elements of orders $2^i, 1 \leq i \leq 4$ and $b = (0, y, 0), c = (0, 0, z)$ are the elements of order 2. Then $\langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle$ are all adjacent to the vertices $\langle a_4\rangle, \langle a_4 b\rangle, \langle a_4 c\rangle$. Thus $K_{3,3}$ is the subgraph of $\mathcal{I}_c^*(G)$, which is a contradiction. So H is cyclic, which completes the proof.
For planarity of $G \cong \mathbb{Z}_{2^4} \times \mathbb{Z}_{2^\alpha}$, $\alpha \leq 4$, we can assume $\alpha = 4$. The number of cyclic subgroups of G is as follows:

$$C_2(G) = 3, C_{2^2}(G) = 6, C_{2^3}(G) = 12, C_{2^4}(G) = 24.$$

Then $\mathcal{I}_c^*(G)$ has three connected components and the inclusion graph of cyclic subgroups of G is as Figure 3. Hence $\mathcal{I}_c^*(G)$ is planar. □

Lemma 2.8. *Let $G \cong \mathbb{Z}_{p^2} \times H$ and H be an elementary abelian group, where $p \nmid |H|$. Then $\mathcal{I}_c^*(G)$ is planar.*

Proof. Suppose that $H = \underbrace{\mathbb{Z}_q \times \cdots \times \mathbb{Z}_q}_{n-times}$, where q is prime. The number of cyclic subgroups of H is equal to $k = \frac{q^n-1}{q-1}$, also G has one subgroup of each order p and p^2. Assume that $H_1, H_2, \cdots, H_k$ are all subgroups of order q and T_1 and T_2 are subgroups of order p and p^2, respectively. Then by Figure 4, $\mathcal{I}_c^*(G)$ is planar. □

Lemma 2.9. *Let $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_q$, where $q \geq 3$ is a prime. Then $\mathcal{I}_c^*(G)$ is planar.*

Proof. Let $a = (x, 0, 0), b = (0, y, 0), c = (0, 0, z)$ be the elements of G of orders $4, 4, q$, respectively. Then all cyclic subgroups of G are as follows:

$$\langle a\rangle, \langle a^2\rangle, \langle b\rangle, \langle b^2\rangle, \langle c\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle ab^2\rangle, \langle a^2b^2\rangle, \langle ac\rangle, \langle a^2c\rangle$$

$$\langle bc\rangle, \langle b^2c\rangle, \langle abc\rangle, \langle a^2bc\rangle, \langle ab^2c\rangle, \langle a^2b^2c\rangle, \langle ab^3\rangle, \langle ab^3c\rangle.$$

Hence, by Figure 5, $\mathcal{I}_c^*(G)$ is planar. □

In Theorem 2.10, and Corollary 2.11, we characterize all finite abelian groups, whose (proper) inclusion graphs of cyclic subgroups are planar, where shows that the [17, Corollary 3.14], is not true.

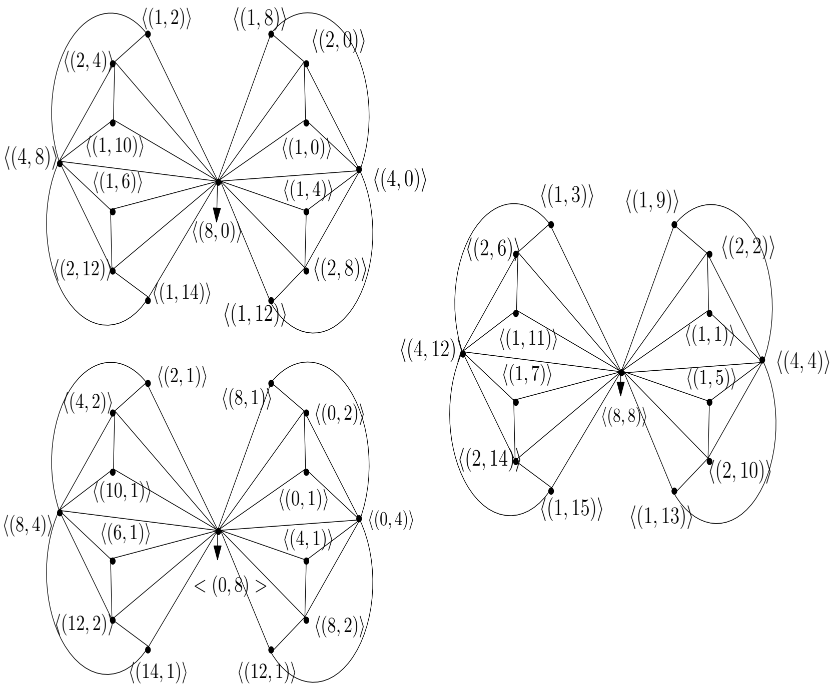

Figure 3: $\mathcal{I}_c^*(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^4})$.

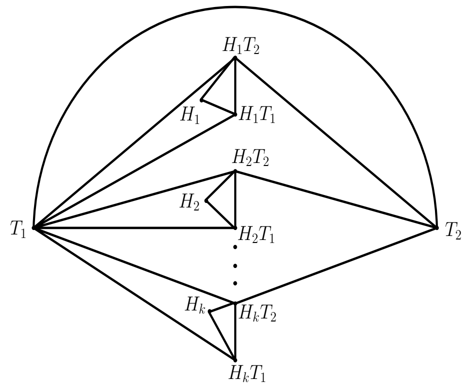

Figure 4: $\mathcal{I}_c^*(\mathbb{Z}_{p^2} \times H)$, H is an elementary abelian q-group.

Theorem 2.10. *Let G be a finite abelian group and p, q, r be distinct prime numbers. Then $\mathcal{I}_c^*(G)$ is planar if and only if G isomorphic to one of the following*

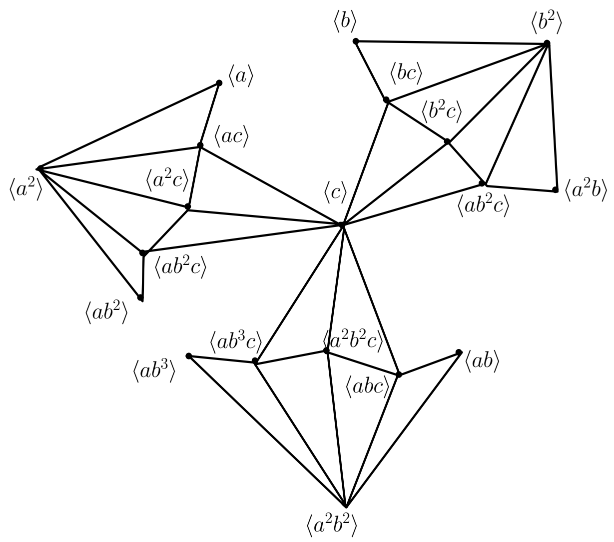

Figure 5: $\mathcal{I}_c^*(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_q)$.

groups:

(1) $\mathbb{Z}_{p^\alpha}$*, where* $\alpha \leq 4$.

(2) G *is a p-group and* $exp(G) \leq p^3$.

(3) $\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_q$*,* $\alpha \leq 3$.

(4) $\mathbb{Z}_{p^\alpha} \times H$*, where* H *is an elementary abelian q-group and* $\alpha \leq 2$.

(5) $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta}$*,* $\alpha, \beta \leq 4$.

(6) $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_q$*,* $\alpha, \beta \leq 2$.

(7) $\mathbb{Z}_{pqr}$.

Proof. Let $G \cong P_1 \times P_2 \times \cdots \times P_t$, where $P_1, \ldots, P_t$ are p_i-sylow subgroups of G, respectively. By Theorem 2.5, $t \leq 3$. We consider the following cases:
Case 1. G be a cyclic group.
Subcase 1.a. $t = 1$. Then $G \cong \mathbb{Z}_{p^\alpha}$ and by Corollary 1.2, $\mathcal{I}_c^*(G)$ is isomorphic to K_α. So $\mathcal{I}_c^*(G)$ is planar if and only if $\alpha \leq 4$.
Subcase 1.b. $t = 2$ and $G \cong \mathbb{Z}_{p^\alpha q^\beta}$.

- **i.** Suppose $\alpha \geq 4$ and $a, b \in G$ such that $\circ(a) = p^4, \circ(b) = q$. Then the vertices $\langle b \rangle, \langle a^{p^2} \rangle, \langle a^{p^3} \rangle$ are all adjacent to $\langle ab \rangle, \langle a^p b \rangle, \langle a^{p^2} b \rangle$. Thus $\mathcal{I}_c^*(G)$ has a subgraph isomorphic to $K_{3,3}$.

ii. Let $\alpha \geq 2, \beta \geq 2$ and $a, b \in G$ of orders p^2, q^2, respectively. Then the vertices $\langle a^p \rangle, \langle b^q \rangle, \langle a^p b^q \rangle$ are all adjacent to the vertices $\langle a^p b \rangle, \langle ab^q \rangle, \langle ab \rangle$. Thus $K_{3,3}$ is the subgraph of $\mathcal{I}_c^*(G)$.

iii. Let $\alpha \leq 3$, $\beta = 1$. It is enough to consider $|G| = p^3 q$. Let $a, b \in G$ such that $\circ(a) = p^3, \circ(b) = q$. Then all the cyclic subgroups of G are $\langle a \rangle, \langle a^p \rangle, \langle a^{p^2} \rangle, \langle b \rangle, \langle ab \rangle$, $\langle a^p b \rangle, \langle a^{p^2} b \rangle$. By Figure 6, $\mathcal{I}_c^*(G)$ is planar. Hence $\mathcal{I}_c^*(\mathbb{Z}_{p^\alpha q})$ is planar for all $\alpha \leq 3$.

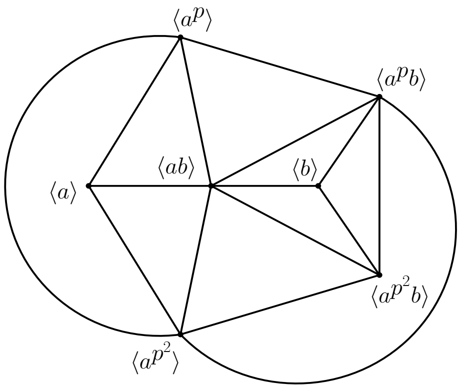

Figure 6: $\mathcal{I}_c^*(\mathbb{Z}_{p^3 q})$

In this subcase, $\mathcal{I}_c^*(G)$ is planar if and only if G satisfies in (3) of theorem.
Subcase 1.c. Let $t = 3$. Then by Theorem 2.5, $\mathcal{I}_c^*(G)$ is planar if and only if $G \cong \mathbb{Z}_{pqr}$.
Case 2. G is not cyclic. By Theorem 2.5, $t \leq 2$ and we consider the following cases.
Subcase 2.a. $t = 1$. Then $exp(G) \leq p^4$ for some prime p.

i. If $exp(G) = p^4$, then by Lemma 2.7, $\mathcal{I}_c^*(G)$ is planar if and only if $G \cong \mathbb{Z}_{2^4} \times \mathbb{Z}_{2^\alpha}$, $\alpha \leq 4$.

ii. If $exp(G) \leq p^3$, by Lemma 2.6, $\mathcal{I}_c^*(G)$ is planar.

Subcase 2.b. $t = 2$. By subcase 1.b (i), $exp(P_i) \leq p_i^3$.

i. Let $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_q$ and suppose that $a_i = (x_i, 0, 0), b = (0, y, 0), c = (0, 0, z)$ be the elements of G of orders $p^i, 1 \leq i \leq 3$, p, q, respectively. Then the vertices $\langle a_1 \rangle, \langle a_2 \rangle, \langle c \rangle$ are all adjacent to the vertices $\langle a_2 c \rangle, \langle a_3 c \rangle, \langle a_3 bc \rangle$. Hence $K_{3,3}$ is a subgraph of $\mathcal{I}_c^*(G)$ and so it's not planar.

ii. Let $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_q$. By a similar argument of the last case, one can show that $\mathcal{I}_c^*(G)$ is non-planar.

iii. Let $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_q$ and $p \geq 3$. Suppose that $a = (x, 0, 0), b = (0, y, 0), c = (0, 0, z)$ are the elements of G of orders p^2, p, q, respectively. Then the vertices $\langle a^p \rangle, \langle a^p c \rangle, \langle c \rangle$ are all adjacent to the vertices $\langle ac \rangle, \langle abc \rangle, \langle ab^2 c \rangle$. Thus $K_{3,3}$ is a subgraph of $\mathcal{I}_c^*(G)$, and so it's not planar.

iv. Let $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ and $a = (x, 0, 0, 0), b = (0, y, 0, 0), c = (0, 0, z, 0), d = (0, 0, 0, w)$ be the elements of G, such that $\circ(a) = \circ(b) = p$ and $\circ(c) = \circ(d) = q$. Then $\mathcal{I}_c^*(G)$ contains a subdivision of $K_{3,3}$ (see Figure 7). Thus by Theorem 2.1 , $\mathcal{I}_c^*(G)$ is non-planar.

v. If $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$, Then by Lemma 2.8, $\mathcal{I}_c^*(G)$ is planar.

vi. Let $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_q$. By a similar argument, we conclude that $\mathcal{I}_c^*(G)$ is non-planar.

vii. If $G \cong \mathbb{Z}_4 \times \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_q$, $\alpha = 1, 2$, then by Lemma 2.9, $\mathcal{I}_c^*(G)$ is planar.

□

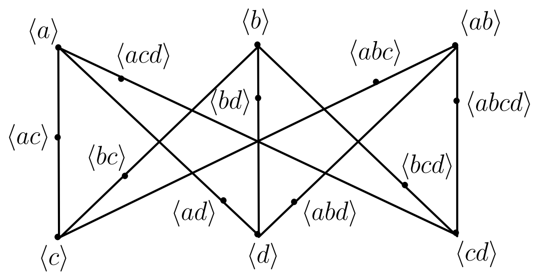

Figure 7: A subgraph of $\mathcal{I}_c^*(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q)$.

Similar to above theorem, we have the following results.

Theorem 2.11. *Let G be a finite abelian group and p, q are distinct prime numbers. Then $\mathcal{I}_c(G)$ is a planar graph if and only if G isomorphic to one of the following groups:*

(1) $\mathbb{Z}_{p^\alpha}$, *where* $\alpha \leq 3$.

(2) *G is a p-group and $exp(G) \leq p^2$.*

(3) $\mathbb{Z}_{p^\alpha q}$, $\alpha \leq 2$.

(4) $\mathbb{Z}_p \times H$, *where H is an elementary abelian q-group.*

(5) $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta}$, $\alpha, \beta \leq 3$.

Theorem 2.12. *Let G be a non-abelian finite group such that $|\pi(Z(G))| = 2$. If $\mathcal{I}_c^*(G)$ is planar, then $|\pi(G)| \leq 3$ and $Z(G)$ is isomorphic to $\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$, $\alpha \leq 2$.*

Proof. Assume that $\pi(G) \geq 3$, $\pi(Z(G)) = \{p, q\}$ and $r \in \pi(G) - \pi(Z(G))$. Also we let P be an r-sylow subgroup of G. We claim that $|P| = r$ and $P \trianglelefteq G$.
If $|P| \geq r^2$, then P has an abelian subgroup H of order r^2. Hence $HZ(G)$ has an abelian subgroup of order pqr^2, which shows the graph is not planar, a contradiction. If P is not a normal subgroup, then G has at least three r-sylow subgroups $\langle c_1\rangle, \langle c_2\rangle, \langle c_3\rangle$. Suppose that a, b are elements of $Z(G)$ of orders p, q, respectively. Then $\{\langle abc_1\rangle, \langle abc_2\rangle, \langle abc_3\rangle\}$ and $\{\langle ab\rangle, \langle a\rangle, \langle b\rangle\}$ are the vertices of a $K_{3,3}$, a contradiction. Thus $P \trianglelefteq G$.
If $|\pi(G)| \geq 4$, $s \in \pi(G) - \{p, q, r\}$ and Q be an s-sylow subgroup of G, then by claim, Q is a normal subgroup of order s. Consequently $PQZ(G) \cong Z(G) \times \mathbb{Z}_{rs}$, where has a non-planar graph, a contradiction.
Now, we assume that $Z(G) \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_q$ and $b \notin Z(G)$. Since $H = \langle Z(G), \{b\}\rangle$ is an abelian subgroup, which contains $Z(G)$, by Theorem 2.10, $|H| = 16q$ and $o(b) = 4$. Thus the order of any elements of $G - Z(G)$ is 4 and also G has exactly 3 elements of order 2. But $|G - Z(G)| \geq |Z(G)| = 8q$ and therefore G has at least $8q/6$ cyclic subgroups of order 4, which have unique subgroup of order 2. Let $\langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle$ be distinct subgroups of order 4, which contain $\langle x\rangle$ of order 2 and y be an element of order q in $Z(G)$. One can see that $\{\langle a_1y\rangle, \langle a_2y\rangle, \langle a_3y\rangle\}$ and $\{\langle xy\rangle, \langle y\rangle, \langle x\rangle\}$ is a $K_{3,3}$, a contradiction.
On the other hand if $x \in G - Z(G)$, then $\langle Z(G) \cup \{x\}\rangle$ is an abelian subgroup strictly included the subgroup $Z(G)$. Now the result follows by Theorem 2.10. □

Similarly, we can conclude the following result.

Theorem 2.13. *Let G be a non-abelian finite group such that $|\pi(Z(G))| = 2$. If $\mathcal{I}_c(G)$ is planar, then $|\pi(G)| = 2$ and $Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$.*

Theorem 2.14. *Let G be a finite group and $|\pi(Z(G))| \geq 3$. Then $\mathcal{I}_c^*(G)$ is planar if and only if $G \cong \mathbb{Z}_{pqr}$.*

Proof. Suppose that $\mathcal{I}_c^*(G)$ is a planar graph. By Theorem 2.10, $Z(G) \cong \mathbb{Z}_{pqr}$. We consider the following three cases:
Case 1. Let $y \in G \setminus Z(G)$, and $\circ(y) = s$, $s \notin \pi(Z(G))$. Then G has an element of order $pqrs$, which contract Lemma 2.2.
Case 2. Suppose that $y \in G\setminus Z(G)$, and $\circ(y) = p$. Then $\langle y, Z(G)\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{qr}$. Hence by Lemma 2.4, $\mathcal{I}_c^*(G)$ is not planar, a contradiction.
Case 3. $G\setminus Z(G)$ has no elements of prime order. Let $y \in G\setminus Z(G)$, and $\circ(y) = p^2$. Then $y^p \in Z(G)$ and $\langle y, Z(G)\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{qr}$, which contract Lemma 2.3. □

Similarly, we can conclude the following result.

Corollary 2.15. *Let G be a finite group and $|\pi(Z(G))| \geq 3$. Then $\mathcal{I}_c(G)$ is not planar.*

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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