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On n-Nilpotent Groups and n-Nilpotency of n-Abelian Groups

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Abstract

The concept of n-nilpotent groups was introduced by Moghaddam and Mashayekhy in 1991 which is in a way a generalized version of the notion of nilpotent groups. Using the n-center subgroup, a new series was constructed, which is a generalization of the upper central series of a group. In this article some properties of such groups will be studied. Finally more results for an n-nilpotent group G are given based on the assumption that G is n-abelian.

Keywords: nilpotent group, n-abelian group, n-center subgroup, n-potent subgroup

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1. Introduction

Let G be a group and n a positive integer. In 1979 Fay and Waals [3] introduced the notions of the n-potent and the n-center subgroups of a group G, respectively as follows:

$$G_n = \langle [x, y^n] | x, y \in G \rangle,$$

$$Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\},$$

where $[x, y^n] = x^{-1}y^{-n}xy^n$. It is easy to see that G_n is a fully invariant and $Z^n(G)$ is a characteristic subgroup of the group G. In the case $n = 1$, these subgroups will be G' and $Z(G)$, the drive and center subgroups of G, respectively. Moghaddam

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et al. in [6], introduced a general version of central series and derived series. Also they defined n-nilpotent groups and explore some of their properties.
The following definition is vital in our investigation:

Definition 1.1 (6, Definition 1.2)**.** A normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_t = G$ of a group G is termed an n-central series of length t if for any $0 \leq i \leq t-1$,

$$\frac{G_{i+1}}{G_i} \leq Z^n(\frac{G}{G_i}).$$

By using the notion of n-potent subgroup, a sequence of subgroups of a group G, is defined, which is called the lower n-central series of G by the rules

$$\gamma_1^n(G) = G, \quad \gamma_2^n(G) = [G, G^n] = G_n, \quad \gamma_i^n(G) = [\gamma_{i-1}^n(G), G^n].$$

Let us mentioned that $\gamma_i^n(G)/\gamma_{i+1}^n(G)$ is a subgroup of the n-center of $G/\gamma_{i+1}^n(G)$. In view of this, it is natural to consider a second sequence of subgroups of G, called the upper n-central series of G, defined by the rules $Z_0^n(G) = 1, Z_1^n(G) = Z^n(G)$ and for $i > 1$, $Z_i^n(G)$ is the inverse image of $Z^n(G/Z_{i-1}^n(G))$ in G.(See [6]).

In order to set up the contents of this survey, we begin by introducing n-nilpotent groups, which is a generalization of nilpotent groups. It is noteworthy that Baer [1] introduced another concept termed n-nilpotent group which is totally different from the following.

Definition 1.2 (6, Definition 1.2)**.** A group G is called n-nilpotent if it has at least one n-central series of the length c such that c is the least of the lengths of its n-central series. For brevity we write $cl_n(G) = c$.

Of course, if G is a nilpotent group of class c, then it is n-nilpotent for all positive integer n. But its converse is not valid. For example consider S_3. Also the n-nilpotency of G implies that G^n is nilpotent.

A group G in which $(xy)^n = x^n y^n$ holds for all $x, y \in G$ and some fixed integer n has been called n-abelian. Furthermore, $[x^n, y] = [x, y^n] = [x, y]^n$ for any $x, y \in G$. The origins of this concept may be traced back to 1944 and are associated with the name of F. W. Levi [5]. Other self-evident fact about n-abelian groups are that every n-abelian group is $(1-n)$-abelian, and conversely. Also G/G_n is n-abelian and G_n is the smallest normal subgroup J of G such that G/J is n-abelian. A detailed introduction to n-abelian groups can be found in Baer's paper [1].

This article is a study of n-nilpotency of two disparate kinds, firstly general remarks on n-nilpotent groups and secondly n-nilpotency of n-abelian groups. In this section, we met n-nilpotent groups, defined in terms of n-central series. Section 2 begins with some elementary properties of n-nilpotent groups in general. Also, we introduce a great source of finite n-nilpotent groups. Moreover, we find a criterion for a group to be residually n-nilpotent. It frequently happens that we have some information about a subgroup or a factor group of a group and we wish to transfer this to other factor groups or subgroups. In this direction, we will show that residually n-nilpotency of a group G, is equivalent to residually n-nilpotency of G/G_n.

In Section 3, we will concentrate on the structure of n-abelian n-nilpotent groups. It is a trivial observation that an extension of one n-nilpotent group by another need not be n-nilpotent. One of the most outstanding results of this section is finding an important criterion for such an extension to be n-nilpotent. In addition, we prove that an n-abelian n-nilpotent group G is torsion-free if and only if $Z^n(G)$ is torsion-free. Also, we discuss the special role played by the Frattini subgroup in an n-abelian group. Finally, we obtain the structures of verbal and marginal subgroups of n-abelian groups, for the variety of n-nilpotent groups.

2. General Remarks on n-Nilpotent Groups

In the following lemma we list explicitly some crucial results of upper and lower n-central series of a group. We omit the proof as the reader may not find any difficulty in it.

Lemma 2.1. *Let G be any group and let i and j be positive integers. Then:*
(i) $\gamma_i^n(G) \lhd^f G$, $Z_i^n(G) \lhd^c G$;
(ii) $\gamma_i^n(G) = 1 \Longleftrightarrow Z_{i-1}^n(G) = G$;
(iii) $\gamma_i^n(G/N) = (\gamma_i^n(G)N)/N$, $Z_i^n(G/N) \geq (Z_i^n(G)N/N)$;
(iv) $\gamma_i^n(G) \leq \gamma_i(G)$, $Z_i(G) \leq Z_i^n(G)$;
(v) $\gamma_i^n(G \times H) = \gamma_i^n(G) \times \gamma_i^n(H)$;
(vi) $Z_i^n(G/Z_j^n(G)) = Z_{i+j}^n(G)/Z_j^n(G)$.

Note that the n-nilpotency of a group G is equivalent to $Z_c^n(G) = G$ also by the previous lemma, $\gamma_{c+1}^n(G) = 1$.
It can readily seen that the class of n-nilpotent groups is closed under subgroups and direct product.

We exhibit here a number of basic results that we shall need.

Theorem 2.2. [6, Theorem 1.7] *Let G be an n-nilpotent group and $1 \neq H \lhd G$. Then $H \cap Z^n(G) \neq 1$.*

This is a fairly immediate consequence of Theorem 2.2.

Corollary 2.3. *Let G be a n-nilpotent group and M be a normal minimal subgroup of G. Then $M \leq Z^n(G)$.*

Another basic property of n-nilpotent groups is the following:

Proposition 2.4. *Let G be an n-nilpotent group and let $\exp(Z^n(G)) = e$. Then G has exponent dividing ne^c where c is the n-nilpotency class of G.*

Proof. First G^n is nilpotent that follows at once G is nilpotent and the class of G^n is at most c. Furthermore $\exp(Z(G^n))|e$; and we conclude $\exp(G^n)|e^c$. Finally let $y \in G$, therefore $(y^n)^{e^c} = 1$. So one can readily see that $\exp(G)|ne^c$. □

Theorem 2.5. *Let G be an infinite n-nilpotent group and let G^n be finitely generated. Then $Z^n(G)$ contains an element of infinite order.*

Proof. Since G is an n-nilpotent group thus G^n is nilpotent. Also by the fact G^n is finitely generated and infinite we have $Z(G^n)$ contains an element of infinite order. Since $Z(G^n) \leq Z^n(G)$ the proof is ended. □

Lemma 2.6. *For a nontrivial finite group G if G^n is a p-group, then $|Z^n(G)| > 1$.*

Proof. Let G be a finite group such that G^n is a p-group. If $G^n = 1$, then $[x, y^n] = [x, 1] = 1$ for any $x, y \in G$. Thus $Z^n(G) = G$ and so $|Z(G^n)| > 1$. If $|G^n| > 1$, since $Z(G^n) \leq Z^n(G)$ and $|Z(G^n)| > 1$ then $|Z^n(G)| > 1$. □

Now the following theorem gives us a great source of finite n-nilpotent groups.

Theorem 2.7. *If G is a finite group such that G^n is a p-group, then G is n-nilpotent.*

Proof. By Lemma 2.6, $|Z^n(G)| > 1$. Suppose that $Z^n(G) = G$, then G is an n-nilpotent group of class $c = 2$. Now let $Z^n(G) < G$. Then since $|G/Z^n(G)| > 1$ another application of Lemma 2.6 yields $|Z^n(G/Z^n(G)| > 1$. It follows therefore that $|Z_2^n(G)| > 1$. Since G is finite this process is limited, so $Z_c^n(G) = G$ for some c. □

Our next result gives a simple, but important, criterion for a group to be residually n-nilpotent.

Theorem 2.8. *Let G be a group. Then G is residually n-nilpotent if and only if $\bigcap_{t=1}^{\infty} \gamma_t^n(G) = 1$.*

Proof. Suppose that G is residually n-nilpotent and $\bigcap_{t=1}^{\infty} \gamma_t^n(G) \neq 1$. Therefore there exists $x \neq 1$ such that $x \in \bigcap_{t=1}^{\infty} \gamma_t^n(G)$. Now let $N_x \lhd G$ such that x is not in N_x and G/N_x is n-nilpotent. Therefore for an integer $c > 0$, $\gamma_c^n(G/N_x) = 1$ and hence $(\gamma_c^n(G)N_x)/N_x = 1$ which means $x \in N_x$. This is a contradiction and the assertion is shown. Conversely if $\bigcap_{t=1}^{\infty} \gamma_t^n(G) = 1$, then for any $x \in G$ there exists a positive integer i such that $x \notin \gamma_i^n(G)$ and $G/\gamma_i^n(G)$ is n-nilpotent. Thus G is residually n-nilpotent. □

As an immediate corollary we obtain

Corollary 2.9. *Let G be a group. Then G/G_n is residually n-nilpotent if and only if G is residually n-nilpotent.*

Theorem 2.10. *A principal factor of a locally n-nilpotent group G is n-central.*

Proof. Let N be a minimal normal subgroup of G. We argue that N is a subgroup of $Z^n(G)$. If $N \not\leq Z^n(G)$, there exist $a \in N$ and $g \in G$ such that $b = [a, g^n] \neq 1$. Since $b \in N$, we have $N = b^{G^n}$ by minimality of N. Thus $a \in \langle b^{g_1^n}, \ldots, b^{g_t^n} \rangle$ for certain $g_i^n \in G^n$. Let $H = \langle a, g^n, g_1^n, \ldots, g_t^n \rangle$ which is an n-nilpotent subgroup of G and set $A = a^{H^n}$. It follows therefore that $b \in [A, H^n]$, hence $b^{g_i^n} \in [A, H^n]$ for any $1 \leq i \leq t$ and consequently $a \in [A, H^n]$. We could derive $A = [A, H^n]$ and $A = [A,_r H^n] = 1$ for all r. Indeed, by n-nilpotency of H, we have $A = 1$ and $a = 1$. But this means that $[a, g^n] = 1$. □

3. n-Nilpotency of n-Abelian Groups

In this section we derive some additional properties of the n-nilpotent groups which are related to the n-abelian groups. Our first theorem deals with the case of torsion-free groups. As we know the properties of the center of a nilpotent group are often reflected in the entire group. Such result for n-nilpotent groups is the following:

Theorem 3.1. *Let an n-abelian group G be n-nilpotent. Then G is torsion-free if and only if $Z^n(G)$ is torsion-free.*

Proof. The direct side of the proof is clear. For the other side, let $Z^n(G)$ be torsion-free. Since $Z(G^n) \leq Z^n(G)$, then $Z(G^n)$ is torsion-free. Moreover G is n-nilpotent, thus G^n is nilpotent and so G^n is torsion-free. Now proof has been completed if we show that for any $x \in G$ and $x^n = 1$, then $x = 1$. Let $x \in G$ and $x^n = 1$. Then $[x^n, g^n] = 1$ for any $g \in G$. As G is n-abelian we have $[x, g^n] = 1$ therefore $x \in Z^n(G)$ and the proposition follows at once. □

Remark 1. Of course, if G is a n-nilpotent group of class c and $\exp(Z^n(G)) = e$, then $\exp(G)|ne^c$ (see Proposition 2.4). Now let G be an n-abelian n-nilpotent group of class at most c and let $\exp(Z^n(G)) = e$. Then by induction on i, we can see $\exp(Z_i^n(G))|e^i$. Hence it follows that $\exp(G)|e^c$.

The following proposition shows the relationship between n-nilpotency of two normal n-nilpotent subgroups and their products.

Proposition 3.2. *Let N and M be two normal n-nilpotent subgroups of an n-abelian group G. If c and d are the n-nilpotent classes of N and M respectively, then $L = MN$ is n-nilpotent of class at most $c + d$.*

Proof. By induction on i we can show that $\gamma_i^n(L)$ equals the product of $[X_1, X_2^n, \ldots, X_i^n]$ such that for any $1 \leq j \leq i$, X_j's are M or N. For $i = 1$ this is clear. Now for $i + 1$ we have

$$\gamma_{i+1}^n(L) = [\gamma_i^n(L), L^n] = [\gamma_i^n(L), M^n][\gamma_i^n(L), N^n].$$

So by induction hypothesis $\gamma_i^n(L)$ is equal to the product of all $[X_1, X_2^n, \ldots, X_i^n]$. Thus we have the same for $\gamma_{i+1}^n(L)$. To complete the proof set $i = c + d + 1$. Then in $[X_1, X_2^n, \ldots, X_i^n]$ either M occurs at least $d+1$ times or N occurs at least $c+1$ which yields $[X_1, X_2^n, \ldots, X_i^n]$ is contained in either $\gamma_{d+1}^n(M)$ or $\gamma_{c+1}^n(N)$. Since both of which equal 1, $\gamma_i^n(L) = 1$ and L is n-nilpotent with class at most $c+d$. $\square$

An extensive class of generalized n-nilpotent groups is the class of locally n-nilpotent groups. We assert an interesting result for this class as follows:

Theorem 3.3. *Let n-abelian group G, be locally n-nilpotent. Then the elements of finite order in G form a fully-invariant subgroup T (the torsion-subgroup of G) such that G/T is torsion-free.*

Proof. Since G is locally n-nilpotent therefore G^n is locally nilpotent so T_{G^n} that containing all elements of finite order in G^n is a fully invariant subgroup of G^n. Obviously if $g \in T$, then $g^n \in T_{G^n}$. Consequently, $g^{-1} \in T$. All it remains is to show that for $g_1, g_2 \in T$, $g_1g_2 \in T$. First $g_1^n, g_2^n \in T_{G^n}$ and therefore $g_1^n g_2^n \in T_{G^n}$. Directly from the definition of n-abelian groups,$(g_1g_2)^n \in T_{G^n}$. So that $g_1g_2 \in T$ and T is a fully-invariant subgroup of G, as asserted. $\square$

Remark 2. Let G be an n-abelian group and let $F_i = \gamma_i^n(G)/\gamma_{i+1}^n(G)$ for a positive integer i. Define two maps $f : G_{ab} \times F_i \longrightarrow G_{ab}$ by the rule $f(yG', x\gamma_{i+1}^n) = y^{x^n}G'$ and $g : F_i \times G_{ab} \longrightarrow F_{i+1}$ by the rule $g(yG', x\gamma_{i+1}^n) = x^{y^n}\gamma_{i+1}^n$ where $y^{x^n} = x^{-n}yx^n$ and $x^{y^n} = y^{-n}xy^n$. So we can easily see that F_i and G_{ab} act on each other trivially for all $i \geq 0$. Hence

$$F_i \otimes G_{ab} \cong (F_i)_{ab} \otimes_Z G_{ab}.$$

Indeed we have faced to nonabelian tensor product with trivial actions which is isomorphic to their abelian tensor product.

Proposition 3.4. *Let G be an n-abelian group and let $F_i = \gamma_i^n(G)/\gamma_{i+1}^n(G)$. Then the mapping $a(\gamma_{i+1}^n(G)) \otimes gG' \longmapsto [a, g^n]\gamma_{i+2}^n(G)$ is a well-defined epimorphism from $F_i \otimes G_{ab}$ to F_{i+1}.*

Proof. As G is n-abelian, the map $f : F_i \times G_{ab} \longrightarrow F_{i+1}$ by the rule $f(a(\gamma_{i+1}^n(G)), gG') = [a, g^n]\gamma_{i+2}^n(G)$ is well-defined. Now let $a_1, a_2 \in \gamma_i^n(G)$ and $g \in G$. Then we have

$$\begin{aligned}
[a_1a_2, g^n] &= [(a_1a_2)^n, g] = [a_1^n a_2^n, g] \\
&= [a_1^n, g][a_1^n, g, a_2^n][a_2^n, g] \\
&= [a_1^n, g][[a_1^n, g], a_2^n][a_2^n, g] \\
&= [a_1^n, g][[a_1, g^n], a_2^n][a_2^n, g] \\
&\equiv [a_1, g^n][a_2, g^n] \quad mod \quad \gamma_{i+2}^n(G).
\end{aligned}$$

Similarly for $g_1, g_2 \in G$ and $a \in \gamma_i^n(G)$ we obtain

$$[a, (g_1g_2)^n] = [a, g_1^n g_2^n] \equiv [a, g_1^n][a, g_2^n] \quad mod \quad \gamma_{i+2}^n(G).$$

This shows that f is a bilinear map. Therefore it induces an epimorphism $\omega_i : F_i \otimes G_{ab} \longrightarrow F_{i+1}$ by the rule $\omega_i(a(\gamma_{i+1}^n(G)) \otimes gG') = [a, g^n]\gamma_{i+2}^n(G)$. $\square$

In view of the previous proposition, we have the following important result:

Theorem 3.5. *Let $\wp$ be a group-theoretical property which is inherited by images of tensor products and extensions. If G is an n-abelian n-nilpotent group such that G/G_n has $\wp$, then G has $\wp$.*

Proof. Since there is natural epimorphism from G/G_n into G/G', therefore G/G' has $\wp$. Thus by Proposition 3.4, $\gamma_i^n(G)/\gamma_{i+1}^n(G)$ has $\wp$ for any $i \geq 0$. By n-nilpotency of G, $\gamma_{c+1}^n(G) = 1$ for some c and $\gamma_c^n(G)/\gamma_{c+1}^n(G) = \gamma_c^n(G)$ has $\wp$. Furthermore, $\wp$ is inherited by extension hence G has $\wp$. □

It is known that an extension of a nilpotent (1-nilpotent) group by another nilpotent group may not be nilpotent in general. Hall [4] obtained a criterion under which such an extension can be nilpotent. Thus, the following question arises:

- Under what circumstances, an extension of a n-nilpotent group by another is n-nilpotent?

Our next result gives simple, but important answer to this question.

Theorem 3.6. *Let G be an n-abelian group and N a normal subgroup of G such that N and G/N_n are n-nilpotent. Then G is n-nilpotent.*

Proof. Since N and G/N_n are n-nilpotent, so N^n and $(G/N_n)^n \cong G^n/(G^n \cap N_n)$ are nilpotent. Furthermore, G is n-abelian thus $G^n \cap N_n = (N^n)'$. Hence G^n is nilpotent and it follows evidently that G is n-nilpotent. □

We now focus our attention on locally n-nilpotent groups that is a generalization of n-nilpotent groups. Recall that the product of two normal n-nilpotent subgroups of an n-abelian group is n-nilpotent. In the following we show that the corresponding statement holds for locally n-nilpotent groups. We begin with the following lemma:

Lemma 3.7. *Let G be an n-abelian group. Then G is a locally n-nilpotent group if and only if G^n is a locally nilpotent group.*

Proof. Let the n-abelian group G be locally n-nilpotent. It is easy to show that G^n is locally nilpotent. On the other hand, for the locally nilpotent group G^n consider a subgroup H generated by some elements $g_1, g_2, \ldots, g_t$. Since G is n-abelian, this forces H^n to be a finitely generated subgroup of G^n and therefore H^n is nilpotent. Thus H is n-nilpotent as we wanted to show. □

Lemma 3.8. *Let H and K be two normal locally n-nilpotent subgroups of an n-abelian group G. Then $J = HK$ is a locally n-nilpotent group.*

Proof. By assumption H^n and K^n are normal locally nilpotent subgroups of G^n, this reduces to $J^n = H^nK^n$ is locally nilpotent. We could derive from Lemma 3.7 that $J = HK$ is a locally n-nilpotent group. □

Theorem 3.9. *In any n-abelian group G, there is a unique maximal locally n-nilpotent subgroup containing all normal locally n-nilpotent subgroups of G.*

Proof. Since the union of a chain of locally n-nilpotent subgroups is locally n-nilpotent, each normal locally n-nilpotent subgroup is contained in a maximal normal locally n-nilpotent subgroup, by Zorn's Lemma. We could derive uniqueness of this subgroup from Lemma 3.8. □

Lemma 3.10. *Let G be an n-abelian group, $H \leq G$, $N \trianglelefteq G$ and $G = HN_n$. Then $G = H\gamma_i^n(N)$ for any $i \leq 2$.*

Proof. Of course if $G = H\gamma_i^n(N)$, then $N = (H \cap N)\gamma_i^n(N)$. We prove our theorem by induction on i. For $i = 2$ the statement is true. Assume that $G = H\gamma_i^n(N)$. We shall prove that $G = H\gamma_{i+1}^n(N)$.

$$\begin{aligned} G = HN_n = H[N, N^n] &= H[(H \cap N)\gamma_i^n(N), N^n] \\ &= H[N \cap H, N^n][\gamma_i^n(N), N^n] \\ &= H[N \cap H, N^n]\gamma_i^n(N) \\ &= H[N \cap H, (H \cap N)\gamma_i^n(N)]\gamma_{i+1}^n(N) \\ &= H[N \cap H, (N \cap H)^n][N \cap H, (\gamma_i^n(N))^n]\gamma_{i+1}^n(N) \\ &= H\gamma_{i+1}^n(N). \end{aligned}$$

□

The lemma has the following useful consequence:

Proposition 3.11. *Let an n-abelian group G be locally n-nilpotent and let G/G_n be finitely generated. Then $\gamma_c^n(G) = \gamma_{c+1}^n(G) = \cdots$ for some c and $G/\gamma_c^n(G)$ is n-nilpotent.*

Proof. Let X be a finite set such that $G/G_n = \langle X \rangle = H$. Then $G = HG_n$ and H is an n-nilpotent group of class $c - 1$, say. Furthermore by using the previous lemma $G = H\gamma_i^n(G)$ for all positive integers i. Let "bars" denote the quotient groups modulo $\gamma_{c+1}^n(G)$. Then since $G = H\gamma_{c+1}^n(G)$, we have $\bar{G} = \bar{H}$ which means that $\bar{G}$ has n-nilpotent class at most $c - 1$ and so $\gamma_c^n(G) = \gamma_{c+1}^n(G) = \cdots$. Of course $G/\gamma_c^n(G)$ is n-nilpotent. □

The Frattini subgroup $\Phi(G)$ of a group G is defined as the intersection of all maximal proper subgroups of the group G, if G has any maximal subgroups; otherwise $\Phi(G) = G$. Our next results concern the Frattini subgroup, n-nilpotent and locally n-nilpotent groups are worth noting.

Lemma 3.12. *Let G be a group and H be a subgroup of G such that $\Phi(G) \leq H \triangleleft G$ and H^n be finite. If $H/\Phi(G)$ is n-nilpotent, then H is n-nilpotent.*

Proof. Since $H/\Phi(G)$ is n-nilpotent, so $(H/\Phi(G))^n = (H^n\Phi(G))/\Phi(G)$ is nilpotent. Furthermore $H^n\Phi(G)$ is nilpotent. Obviously H^n is nilpotent, thus H is n-nilpotent. □

Now we easily attain a necessary condition for n-nilpotency of a group G such that G^n is finite.

Theorem 3.13. *Let G be a group such that G^n is finite. If $G_n \leq \Phi(G)$, then G is n-nilpotent.*

Proof. Certainly G/G_n is n-nilpotent and so there exists an epimorphism from G/G_n onto $G/\Phi(G)$ which obligated $G/\Phi(G)$ to be n-nilpotent and by the previous lemma, G is n-nilpotent. □

Theorem 3.14. *Let G be an n-abelian group.*
(i) *If G is locally n-nilpotent, then $G_n \leq \Phi(G)$.*
(ii) *If G is finitely generated n-nilpotent, then $G_n \leq \Phi(G)$.*

Proof. (i) Clearly G^n is locally nilpotent and therefore $(G^n)' \leq \Phi(G^n) \leq \Phi(G)$. Hence $G_n \leq \Phi(G)$.
(ii) Let $cl_n(G) = c$. We use induction on c to achieve our aim. If $c = 1$, then $G_n = 1 \leq \Phi(G)$, as asserted. Putting

$$\gamma_c^n(G) = \langle [u, v^n] | u \in \gamma_{c-1}^n(G), v \in G\rangle,$$

we have obviously $G/\gamma_c^n(G)$ is n-nilpotent of class at most $c-1$. Now if $G = \langle X, G_n\rangle$, then

$$\frac{G}{\gamma_c^n(G)} = \langle x\gamma_c^n(G), y\gamma_c^n(G) | x \in G, y \in G_n\rangle.$$

So by the induction hypothesis $G/\gamma_c^n(G) = \langle x\gamma_c^n(G) | x \in X\rangle$. Hence $G = \langle X\rangle$. Thus (ii) also holds. □

In the following we determine the relationship between some subgroups of an n-abelian group G.

Theorem 3.15. *If G is an n-abelian group, then $G_n \cap Z^n(G) \leq \Phi(G)$.*

Proof. Set $S = G_n \cap Z^n(G)$. Suppose that there exist a maximal subgroup M of G such that S is not in M. Hence $G = SM$ by maximality of M. So for any $x, y \in G$, there is some elements $s, s' \in S$ and $m, m' \in M$ such that $x = sm$ and $y = s'm'$. Then we have

$$\begin{aligned}
[x, y^n] =& [x, (s'm')^n] \\
=& [x, (s')^n(m')^n] \\
=& [x, (s')^n]^{(m')^n}[x, (m')^n] \\
=& [x, (m')^n] \\
=& [sm, (m')^n] \\
=& [m, (m')^n].
\end{aligned}$$

Since $[m, (m')^n] \in M$ therefore $G_n \leq M$ and this is contradiction. $\square$

Let X be a nonempty set, F a free group on X and $\mathcal{V}$ a variety with set of laws $S = \{[x_1, x_2^n, \ldots, x_{c+1}^n]\}$. We denote this variety by $\mathfrak{N}_{\mathfrak{c}}^{\mathbf{n}}$. To close this section we obtain the structures of verbal and marginal subgroups of n-abelian groups, for the variety $\mathfrak{N}_{\mathfrak{c}}^{\mathbf{n}}$.

Proposition 3.16. *By the above assumption $V(G) = \gamma_{c+1}^n(G)$ and $V^*(G) = Z_c^n(G)$, for any n-abelian group G.*

Proof. This is easy to show that $V(G) = \gamma_{c+1}^n(G)$. We show that $V^*(G) = Z_c^n(G)$. let $x \in V^*(G)$, thus $[x1, g_1^n, \ldots, g_c^n] = [1, g_1^n, \ldots, g_c^n] = 1$, for any $g_1, \ldots, g_c \in G$. Therefore $x \in Z_c^n(G)$ and so $V^*(G) \leq Z_c^n(G)$. We use induction on c to show that $Z_c^n(G) \leq V^*(G)$. For $c = 1$ if $x \in Z^n(G)$, then for any $y, g \in G$, we have $[xy, g^n] = [x, g^n]^y[y, g^n] = [y, g^n]$, and

$$[y, (xg)^n] = [y^n, xg] = [y^n, g][y^n, x]^g = [y, g^n].$$

Therefore $x \in V^*(G)$.
Now suppose that, for all $i < c$, $Z_i^n(G) \leq V^*(G)$. We prove that $Z_c^n(G) \leq V^*(G)$. Let $x \in Z_c^n(G)$. Then $xZ^n(G) \in Z_{c-1}^n(G/Z^n(G))$ and so by induction hypothesis, $xZ^n(G) \in V^*(G/Z^n(G))$. It follows that, for all $g_j \in G$, $1 \leq j \leq c$ and all $1 \leq i < c$, we have

$$[g_1, g_2^n, \ldots, (g_ix)^n, \ldots, g_c^n] \equiv [g_1, g_2^n, \ldots, g_i^n, \ldots, g_c^n] \pmod{Z^n(G)}.$$

Now let $X = [g_1, g_2^n, \ldots, (g_ix)^n, \ldots, g_c^n]$ and $Y = [g_1, g_2^n, \ldots, g_i^n, \ldots, g_c^n]$. Then $X = Yz$, for an element $z \in Z^n(G)$. Thus

$$[X, g_{c+1}^n] = [Yz, g_{c+1}^n] = [Y, g_{c+1}^n]^z[z, g_{c+1}^n] = ([Y, g_{c+1}]^n)^z = [Y, g_{c+1}^n].$$

So

$$[g_1, g_2^n, \ldots, (g_ix)^n, \ldots, g_c^n, g_{c+1}^n] = [g_1, g_2^n, \ldots, g_i^n, \ldots, g_c^n, g_{c+1}^n] \tag{1}$$

The missing case is $i = c + 1$. Put $y = g_c^{-1}$ and $z = [g_1, g_2^n, \ldots, g_{c-1}^n]$ then

$$[g_1, g_2^n, \ldots, g_c^n, x^n] = [z, y^{-n}, x^n],$$

and therefore by equality $[z, y^{-1}, x]^y[y, x^{-1}, z]^x[x, z^{-1}, y]^z = 1$ we have,

$$[z, y^{-n}, x^n] = (([x^n, z^{-1}, y^n]^z)^{-1}([y^n, x^{-n}, z]^{x^n})^{-1})^{y^{-n}}.$$

The equality $Z_c^n(G)/Z_{c-1}^n(G) = Z^n(G/Z_{c-1}^n(G))$ implies that

$$[Z_c^n(G), G^n] \leq Z_{c-1}^n(G).$$

It follows that

$$[g_1, g_2^n, \ldots, g_c^n, x^n] = [z, y^{-n}, x^n] = 1. \tag{2}$$

Now by (2) we are in a position to show that

$$[g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n x^n] = [g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n].$$

Of course

$$[g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n x^n] = [g_1, g_2^n, \ldots, g_c^n, x^n][g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n]^{x^n}.$$

So

$$([g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n x^n])^{x^{-n}} = [g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n]$$

and

$$([g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n x^n])^{x^{-n}} = [g_1^{x^{-n}}, (g_2^n)^{x^{-n}}, \ldots, (g_c^n)^{x^{-n}}, (g_{c+1}^n)^{x^{-n}}(x^n)^{x^{-n}}].$$

Therefore by (1), $[g_1, g_2^n, \ldots, g_c^n, x^n g_{c+1}^n] = [g_1, g_2^n, \ldots, g_c^n, g_{c+1}^n]$. This complete the proof. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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