Mathematics Interdisciplinary Research **6** (2021) 171 − 183

Original Scientific Paper

Inverse Nodal Problem for Polynomial Pencil of a Sturm-Liouville Operator from Nodal Parameters

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Abstract

A Sturm-Liouville problem with n-potential functions in the second order differential equation and which contains spectral parameter depending on linearly in one boundary condition is considered. The asymptotic formulas for the eigenvalues, nodal parameters (nodal points and nodal lengths) of this problem are calculated by the Prüfer's substitutions. Also, using these asymptotic formulas, an explicit formula for the potential functions are given. Finally, a numerical example is given.

Keywords: eigenvalues, eigenfunctions, Prüfer's substitutions, Sturm-Liouville problem, inverse nodal problem.

2010 Mathematics Subject Classification: Primary 34A55, Secondary 34L05, 34L20.

How to cite this article
S. Goktas and E. Biten, Inverse nodal problem for polynomial pencil of a Sturm-Liouville operator from nodal parameters, *Math. Interdisc. Res.* **6** (2021) 171 − 183.

1. Introduction

Consider the following Sturm-Liouville problem

$$-u'' + \big(q_0(x) + \lambda q_1(x) + \cdots + \lambda^{n-1} q_{n-1}(x)\big)\, u = \lambda^{2n} u, \qquad x \in [0, \pi] \qquad (1)$$

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Academic Editor: Seyfollah Mosazadeh
Received 8 May 2021, Accepted 28 September 2021
DOI: 10.22052/mir.2021.242239.1286

$$u(0) = 0, \tag{2}$$

$$(\alpha_1\lambda + \beta_1) u'(\pi) + (\alpha_2\lambda + \beta_2) u(\pi) = 0, \tag{3}$$

where $n \in \mathbb{N}$, λ is a spectral parameter and $q_m(x) \in C^1[0,\pi], \left(m = \overline{0, n-1}\right)$. Here, for $j = 1, 2, 3, \ldots, k-1$, $k \geq 2$, $\left\{x_j^k\right\}$ are nodal points (zeros of the eigenfunction) and $l_j^k = x_{j+1}^k - x_j^k$ are nodal lengths of (1)-(3).

In mathematical physics problems, partial differential equations are encountered. In order to make a mathematical description of a physical process, some conditions that uniquely determine this process are also needed. These conditions are called initial or boundary conditions. Such problems are transformed into ordinary differential equations containing a parameter by some methods. One of these equations is the Sturm-Liouville equation. In the present study, the Sturm-Liouville equation is different from the classical Sturm-Liouvile equation in that it contains more than one function. In addition, one of the boundary conditions of the problem has an linearly eigenparameter.

The most important contribution to the development of inverse problems for the Sturm-Liouville operator was the work formulated and examined by Ambartsumyan [2] also these type problems have many applications in physical problems such as vibration of a string, quantum mechanics and geophysics. However, unlike this study, it turned out that a set of eigenvalues is not sufficient to determine the form of the operator. For example; In addition to the condition Ambarsumyan's "knowing a unique set of eigenvalues", Borg [4] stated that knowing a second set of eigenvalues or fulfilling the $q(x) = q(1-x)$ condition is sufficient to determine the form of the operator; Gel'fand and Levitan [7] stated that the potential function can be determined individually by using a set of norming constants; Hoschtadt [12] showed that the potential function can be obtained as an absolute sum over the index set of eigenfunctions.

Problems with the eigenvalue parameter linearly contained in the boundary conditions have been studied extensively because of the important application in probability theory, physics and so on. Physical applications of the eigenparameter dependent Sturm-Liouville problems are considered by numerous authors for its wide applications in mechanics, mathematical physics, and engineering [1, 13, 14, 15, 19, 21, 25].

In recent years, the theory of inverse nodal problems, a new class of inverse problems, has attracted the attention of authors. The inverse nodal problem was first addressed by McLaughlin [22]. It has been proved that only the knowledge of nodal points is uniquely sufficient to determine the potential function in the Sturm-Liouville problem [11, 22]. Subsequently, some remarkable results have been obtained by some authors. For example, some authors restructured the potential function and its derivatives with the help of nodal points in their studies [6, 16, 17, 20, 28, 32]. Additionally, some authors discussed the inverse nodal problem when the boundary conditions depend on eigenparameter [5, 27, 29, 33]. In addition to these studies, the inverse nodal problem has been studied for the discontinuous Sturm-Liouville operator with boundary conditions depending on the eigenparameter [18, 23, 24, 30, 31].

Some authors have studied cases of the Sturm-Liouville operator includes more than one potential function differently from the classical Sturm-Liouville operator [8, 9, 10, 26].

2. Main Results

2.1 Asymptotics of Eigenvalues and Nodal Parameters

The aim of this subsection is to give asymptotic formulas for eigenvalues and nodal parameters of Prob.(1)-(3). As methodology, we use the following modified Prüfer's substitution for solution of the inverse nodal problem:

$$\begin{aligned} u\left(x\right) &= r\left(x\right)\sin\left(\lambda^{n}\theta\left(x\right)\right), \\ u'\left(x\right) &= \lambda^{n} r\left(x\right)\cos\left(\lambda^{n}\theta\left(x\right)\right), \end{aligned} \tag{4}$$

or

$$\frac{u'(x)}{u(x)} = \lambda^{n}\cot\left(\lambda^{n}\theta(x)\right), \tag{5}$$

where $r(x)$ is amplitude and $\theta(x,\lambda)$ is Prüfer's variable [3, 8].

If the following equalities

$$\left(\frac{u'}{u}\right)' = -\frac{\lambda^{2n}\theta'\left(x\right)}{\sin\left(\lambda^{n}\theta\left(x\right)\right)}, \quad \left(\frac{u'}{u}\right)^{2} = \lambda^{2n}.\cot^{2}\left(\lambda^{n}\theta\left(x\right)\right),$$

obtained from (5) are taken into account in the equality

$$\frac{u''}{u} = \left(\frac{u'}{u}\right)' + \left(\frac{u'}{u}\right)^{2},$$

we have

$$\frac{u''}{u} = \frac{\lambda^{2n}\left(-\theta'\left(x\right) + \cot^{2}\left(\lambda^{n}\theta\left(x\right)\right)\right)}{\sin^{2}\left(\lambda^{n}\theta\left(x\right)\right)}.$$

From the last equality and Equation (1),

$$\theta'\left(x\right) = 1 - \frac{1}{\lambda^{2n}}\left(\sum_{m=0}^{n-1}\lambda^{m}q_{m}\left(x\right)\right)\sin^{2}\left(\lambda^{n}\theta\left(x\right)\right), \tag{6}$$

is obtained.

Theorem 2.1. The asymptotic formulas of eigenvalues for Prob.(1)-(3) have the representation

$$\begin{aligned}\lambda_k^n &= \left(k-\frac{1}{2}\right)+\frac{\alpha_2}{\alpha_1\pi\left(k-\frac{1}{2}\right)}\\&+\frac{1}{2\pi\left(k-\frac{1}{2}\right)}\int_0^\pi q_0\left(x\right)dx+\frac{1}{2\pi\left(k-\frac{1}{2}\right)^{1-\frac{1}{n}}}\int_0^\pi q_1\left(x\right)dx\\&+\frac{1}{2\pi\left(k-\frac{1}{2}\right)^{1-\frac{2}{n}}}\int_0^\pi q_2\left(x\right)dx+\cdots+\frac{1}{2\pi\left(k-\frac{1}{2}\right)^{1-\frac{n-1}{n}}}\int_0^\pi q_{n-1}\left(x\right)dx\\&+O\left(\frac{1}{\left(k-\frac{1}{2}\right)^{1+\frac{1}{n}}}\right),\end{aligned}\tag{7}$$

as $k\to\infty$.

Proof. Let us consider $\lambda=\lambda_k$. $\theta(0)=0$ by (2). On the other hand, $\frac{u'(\pi)}{u(\pi)}=\frac{-(\alpha_2\lambda_k+\beta_2)}{\alpha_1\lambda_k+\beta_1}$ by (3). Considering Prüfer's substitution (4) in the last equality , we have

$$\lambda_k^n\theta\left(\pi\right)=arccot\left(-\frac{\alpha_2}{\alpha_1\lambda_k^n}+O\left(\frac{1}{\lambda_k^{n+1}}\right)\right).$$

In the right side of the last equality, by using Taylor expansion of arccot function, we have seen that

$$\theta\left(\pi\right)=\frac{\left(k-\frac{1}{2}\right)\pi}{\lambda_k^n}-\frac{\alpha_2}{\alpha_1\lambda_k^{2n}}+O\left(\frac{1}{\lambda_k^{2n+1}}\right).\tag{8}$$

Now, integrating both sides of (6) with respect to x on $[0,\pi]$, we get

$$\begin{aligned}\frac{\left(k-\frac{1}{2}\right)\pi}{\lambda_k^n}+O\left(\frac{1}{\lambda_k^{2n+1}}\right)&=\pi+\frac{\alpha_2}{\alpha_1\lambda_k^{2n}}-\frac{1}{2\lambda_k^{2n}}\sum_{m=0}^{n-1}\int_0^\pi q_m\left(x\right)dx\\&+\sum_{m=0}^{n-1}\frac{1}{2\lambda_k^{2n-m}}\int_0^\pi q_m\left(x\right)\left[1-2\sin^2\left(\lambda_k^n\theta\left(x\right)\right)\right]dx.\end{aligned}\tag{9}$$

Using the identity

$$1-2\sin^2\left(\lambda_k^n\theta(x)\right)=\cos\left(2\lambda_k^n\theta(x)\right)=\frac{1}{2\lambda_k^n\theta'(x)}\frac{d}{dx}\left[\sin(2\lambda_k^n\theta(x)\right],$$

we get

$$\frac{\left(k-\frac{1}{2}\right)\pi}{\lambda_k^n}+O\left(\frac{1}{\lambda_k^{2n+1}}\right)=\pi+\frac{\alpha_2}{\alpha_1\lambda_k^{2n}}-\frac{1}{2\lambda_k^{2n-m}}\sum_{m=0}^{n-1}\int_0^\pi q_m(x)dx$$
$$+\sum_{m=0}^{n-1}\frac{1}{2\lambda_k^{2n-m}}\int_0^\pi\frac{q_m(x)}{2\lambda_k^n\theta'(x)}\frac{d}{dx}\left[\sin\left(\lambda_k^n\theta(x)\right)\right].$$

For the last n integral in the last equality, we should apply integration by parts

$$\int_0^\pi\frac{q_m(x)}{2\lambda_k^n\theta'}\frac{d}{dx}\left[\sin(2\lambda_k^n\theta(x))\right]dx=\frac{q_m(x)}{2\lambda_k^n\theta'}\sin\left(2\lambda_k^n\theta(x)\right)\Big|_0^\pi$$
$$-\frac{1}{2\lambda_k^n}\int_0^\pi\sin(2\lambda_k^n\theta(x))\frac{d}{dx}\left[\frac{q_m(x)}{\theta'}\right]dx=O\left(\frac{1}{\lambda_k^n}\right),$$

for $m=\overline{0,n-1}$ and after collecting all terms, we have

$$\lambda_k^n=\left(k-\frac{1}{2}\right)\left[1+\frac{\alpha_2}{\alpha_1\pi\lambda_k^{2n}}+\sum_{m=0}^{n-1}\frac{1}{2\lambda_k^{2n-m}\pi}\int_0^\pi q_m(x)dx+O\left(\frac{1}{\lambda_k^{2n+1}}\right)\right].$$

For $k\to\infty$, $\lambda_k\approx\left(k-\frac{1}{2}\right)^{1/n}$, then

$$\lambda_k^n=\left(k-\frac{1}{2}\right)+\frac{\alpha_2}{\alpha_1\pi\left(k-\frac{1}{2}\right)}+\sum_{m=0}^{n-1}\frac{1}{2\pi\left(k-\frac{1}{2}\right)^{1-\frac{m}{n}}}\int_0^\pi q_m(x)dx+O\left(\frac{1}{\left(k-\frac{1}{2}\right)^{1+\frac{1}{n}}}\right).$$

This completes the proof. □

Theorem 2.2. The asymptotic formulas of nodal points for Prob.(1)-(3) have the representation

$$x_j^k=\frac{j\pi}{k-\frac{1}{2}}+\frac{j\alpha_2}{\alpha_1\left(k-\frac{1}{2}\right)^3}+\sum_{m=0}^{n-1}\frac{1}{2\left(k-\frac{1}{2}\right)^{2-\frac{m}{n}}}\int_0^{x_j^k}q_m(x)dx+O\left(\frac{1}{k^{2+\frac{1}{n}}}\right),\quad(10)$$

as $k\to\infty$.

Proof. For $x=x_j^k$ on Prüfer's substitution (4),

$$u\left(x_j^k\right)=r\left(x_j^k\right)\sin\left(\lambda_k^n\theta\left(x_j^k\right)\right),$$

while $r(x_j^k) \neq 0$. Since x_j^k are nodal points so $u(x_j^k) = 0$, therefore

$$\sin\left(\lambda_k^n \theta\left(x_j^k\right)\right) = 0,$$

so $\theta\left(x_j^k\right) = \frac{j\pi}{\lambda_k^n}$.

Now, integrating (6) with respect to x on $\left[0, x_j^k\right]$ and letting $\theta(0) = 0$ and $\theta\left(x_j^k\right) = \frac{j\pi}{\lambda_k^n}$,

$$\int_0^{x_j^k} \theta'(x)dx = \int_0^{x_j^k} dx - \sum_{m=0}^{n-1} \frac{1}{\lambda_k^{2n-m}} \int_0^{x_j^k} q_m(x)\sin^2\left(\lambda_k^n \theta(x)\right) dx,$$

or

$$\begin{aligned} x_j^k = \frac{j\pi}{\lambda_k^n} &+ \sum_{m=0}^{n-1} \frac{1}{2\lambda_k^{2n-m}} \int_0^{x_j^k} q_m(x)dx \\ &- \sum_{m=0}^{n-1} \frac{1}{\lambda_k^{2n-m}} \int_0^{x_j^k} \frac{q_m(x)}{2\lambda_k^n \theta'(x)} \frac{d}{dx}\left[\sin\left(2\lambda_k^n \theta(x)\right)\right]. \end{aligned} \tag{11}$$

Applying integration by parts for the last n integral in the equality (11),

$$\int_0^{x_j^k} \frac{q_m(x)}{2\lambda_k^n \theta'(x)} \frac{d}{dx}\left[\sin\left(2\lambda_k^n \theta(x)\right)\right] = O\left(\frac{1}{\lambda_k^n}\right), \tag{12}$$

for $m = \overline{0, n-1}$. By inserting (12) into (11), we get

$$x_j^k = \frac{j\pi}{\lambda_k^n} + \sum_{m=0}^{n-1} \frac{1}{\lambda_k^{2n-m}} \int_0^{x_j^k} q_m(x)dx + O\left(\frac{1}{\lambda_k^{2n+1}}\right),$$

and considering (7), we have

$$\begin{aligned} x_j^k = &\frac{j\pi}{\left(k-\frac{1}{2}\right)\left(1 + \frac{\alpha_2}{\alpha_1 \pi \left(k-\frac{1}{2}\right)^2 + O\left(\frac{1}{k^{1+\frac{1}{n}}}\right)}\right)} \\ &+ \sum_{m=0}^{n-1} \frac{1}{2\left(k-\frac{1}{2}\right)^{2-\frac{m}{n}}} \int_0^{x_j^k} q_m(x)dx + O\left(\frac{1}{k^{2+\frac{1}{n}}}\right). \end{aligned}$$

By using the properties of Big- O and for $k \to \infty$, this completes the proof. $\square$

Theorem 2.3. The asymptotic formulas of nodal lengths for Prob.(1)-(3) have the representation

$$l_j^k = \frac{\pi}{\left(k-\frac{1}{2}\right)} + \frac{\alpha_2}{\alpha_1\left(k-\frac{1}{2}\right)^3} + \sum_{m=0}^{n-1} \frac{1}{2\left(k-\frac{1}{2}\right)^{2-\frac{m}{n}}} \int\limits_{x_j^k}^{x_{j+1}^k} q_m(x)dx + O\left(\frac{1}{k^{3+\frac{1}{n}}}\right), \tag{13}$$

as $k \to \infty$.

Proof. For large $n \in \mathbb{N}$, integrating (6) on $[x_j^k, x_{j+1}^k]$, we have

$$l_j^k = \frac{\pi}{\lambda_k^n} + \sum_{m=0}^{n-1} \frac{1}{\lambda_k^{2n-m}} \int\limits_{x_j^k}^{x_{j+1}^k} q_m(x)\sin^2\left(\lambda_k^n \theta(x)\right) dx,$$

or

$$\begin{aligned} l_j^k = {} & \frac{\pi}{\lambda_k^n} + \sum_{m=0}^{n-1} \frac{1}{2\lambda_k^{2n-m}} \int\limits_{x_j^k}^{x_{j+1}^k} q_m(x)dx \\ & - \sum_{m=0}^{n-1} \frac{1}{\lambda_k^{2n-m}} \int\limits_{x_j^k}^{x_{j+1}^k} \frac{q_m(x)}{2\lambda_k^n \theta'(x)} \frac{d}{dx}\left[\sin\left(2\lambda_k^n \theta(x)\right)\right]. \end{aligned} \tag{14}$$

By a similar process used in Theorem 2.1 and using trigonometric identity we obtain that

$$\int\limits_{x_j^k}^{x_{j+1}^k} \frac{q_m(x)}{2\lambda_k^n \theta'(x)} \frac{d}{dx}\left[\sin\left(2\lambda_k^n \theta(x)\right)\right] = O\left(\frac{1}{\lambda_k^{2n}}\right),$$

for $m = \overline{0, n-1}$. If these values are written in the last equality, the following equality is obtained:

$$l_j^k = \frac{\pi}{\lambda_k^n} + \sum_{m=0}^{n-1} \frac{1}{2\lambda_k^{2n-m}} \int\limits_{x_j^k}^{x_{j+1}^k} q_m(x)dx + O\left(\frac{1}{\lambda_k^{3n+1}}\right), \tag{15}$$

and considering (7), we have

$$l_j^k = \frac{\pi}{\left(k-\frac{1}{2}\right)\left(1+\frac{\alpha_2}{\alpha_1\pi\left(k-\frac{1}{2}\right)^2+O\left(\frac{1}{k^{1+\frac{1}{n}}}\right)}\right)}$$
$$+\sum_{m=0}^{n-1}\frac{1}{2\left(k-\frac{1}{2}\right)^{2-\frac{m}{n}}}\int_{x_j^k}^{x_{j+1}^k} q_m(x)dx + O\left(\frac{1}{k^{3+\frac{1}{n}}}\right).$$

By using the properties of Big- O and for $k\to\infty$, this completes the proof. □

2.2 Reconstruction of Potential Functions

This section is devoted to give an explicit formula for the potential functions of Equation (1) by using nodal lengths.

Theorem 2.4. Let $q_m(x)$ $(m=\overline{0,n-1})$ are real-valued functions defined on $[0,\pi]$. Then

$$q_0(t) = 2\lim_{k\to\infty}\left\{\lambda_k^2 - \frac{\pi}{l_j^k}\lambda_k\right\},$$
$$q_m(t) = 2\lim_{k\to\infty}\left\{\lambda_k^{m+2} - \lambda_k^m + \frac{\pi}{l_j^k}(1-\lambda_k)\right\},$$

for $j=j_k(x)=\max\left\{j : x_j^k < x\right\}$.

Proof. Applying the mean value theorem for integrals in the asymptotic formula (15) calculated for nodal lengths, with fixed n, there exists $t\in\left(x_j^k, x_{j+1}^k\right)$, we can obtain

$$l_j^k = \frac{\pi}{\lambda_k^n} + \sum_{m=0}^{n-1}\frac{1}{2\lambda_k^{2n-m}} q_m(t)\, l_j^k + O\left(\frac{1}{\lambda_k^{3n+1}}\right),$$

or

$$\sum_{m=0}^{n-1}\lambda_k^m q_m(t) = 2\lambda_k^{2n}\left\{1-\frac{\pi}{\lambda_k^n l_j^k}\right\} + O\left(\frac{1}{\lambda_k^{n+1}}\right), \tag{16}$$

where $\int_{x_j^k}^{x_{j+1}^k} q_m(x)dx = q_m(t)\, l_j^k$.

By (16):
For $n=1$

$$q_0(t) = 2\lambda_k^2\left\{1-\frac{\pi}{\lambda_k l_j^k}\right\} + O\left(\frac{1}{\lambda_k^2}\right), \tag{17}$$

and $k \to \infty$

$$q_0(t) = \lim_{k\to\infty} 2\lambda_k^2 \left\{ 1 - \frac{\pi}{\lambda_k l_j^k} \right\}.$$

For $n = 2$,

$$q_0(t) + \lambda_k q_1(t) = 2\lambda_k^4 \left\{ 1 - \frac{\pi}{\lambda_k^2 l_j^k} \right\} + O\left(\frac{1}{\lambda_k^3}\right),$$

from the formula (17)

$$q_1(t) = 2\left\{ \lambda_k^3 - \lambda_k + \frac{\pi}{l_j^k}(1-\lambda_k) \right\} + O\left(\frac{1}{\lambda_k^4}\right), \tag{18}$$

and $k \to \infty$,

$$q_1(t) = 2\lim_{k\to\infty} \left\{ \lambda_k^3 - \lambda_k + \frac{\pi}{l_j^k}(1-\lambda_k) \right\}.$$

For $n = 3$,

$$q_0(t) + \lambda_k q_1(t) + \lambda_k^2 q_2(t) = 2\lambda_k^6 \left\{ 1 - \frac{\pi}{\lambda_k^3 l_j^k} \right\} + O\left(\frac{1}{\lambda_k^4}\right),$$

from the formula (17) and (18)

$$q_2(t) = 2\left\{ \lambda_k^4 - \lambda_k^2 + \frac{\pi}{l_j^k}(1-\lambda_k) \right\} + O\left(\frac{1}{\lambda_k^6}\right),$$

and $k \to \infty$,

$$q_2(t) = 2\lim_{k\to\infty} \left\{ \lambda_k^4 - \lambda_k^2 + \frac{\pi}{l_j^k}(1-\lambda_k) \right\}.$$

By processing in this way we obtain that

$$q_m(t) = 2\lim_{k\to\infty} \left\{ \lambda_k^{m+2} - \lambda_k^m + \frac{\pi}{l_j^k}(1-\lambda_k) \right\},$$

for $(m = \overline{4, n-1})$. Here, the asymptotic formulas λ_k^m and l_j^k are defined by the formulas (7) and (13), respectively. Proof is completed. □

2.3 A Numerical Example

This section aims to give a numerical conclusion to embody the eigenvalues and nodal parameters for Prob.(1)-(3).

Let $\alpha_1 = \alpha_2 = 1$.

Let us consider (1)-(3) for $n = 3$. In Table 1, Table 2 and Table 3, the behavior of eigenvalues, nodal points and nodal lengths were examined when $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$, $q_m(x) = 0$, $m \geq 3$, respectively.

Table 1. The eigenvalues of Prob.(1)-(3) for $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
1.76044	1.59816	1.64676	1.72779	1.81372	1.89757	1.97748	2.05311	2.12464	2.19239

In Table 1, It is seen that as k grows for $k > 1$, the sequence of eigenvalues increases. This is in accordance with the general theory.

Table 2. The nodal points of Prob.(1)-(3) for $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$.

x_j^k	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=1$	**1.63411**	**1.00928**	**0.744862**	**0.594912**	**0.496931**	**0.427386**	**0.375261**	**0.334648**
$j=2$	4.13642	**2.2181**	**1.55822**	**1.21935**	**1.00862**	**0.862952**	**0.755418**	**0.672404**
$j=3$	8.18577	3.77343	**2.48805**	**1.89313**	**1.54458**	**1.31179**	**1.14342**	**1.01509**
$j=4$	14.461	5.82224	3.58237	**2.63605**	**2.11434**	**1.77899**	**1.54221**	**1.36451**
$j=5$	23.6409	8.5115	4.88916	3.46793	**2.72742**	**2.26964**	**1.95475**	**1.72249**
$j=6$	36.4044	11.9882	6.45641	4.40857	3.39332	**2.78884**	**2.38398**	**2.09085**
$j=7$	53.4302	16.3993	8.33212	5.47779	4.12158	3.34168	**2.83285**	**2.47139**
$j=8$	75.3972	21.8918	10.5643	6.69538	4.92171	3.93325	3.30431	**2.86595**
$j=9$	102.984	28.6126	13.2009	8.08116	5.80324	4.56864	3.80132	3.27633
$j=10$	136.87	36.7088	16.29	9.65493	6.77567	5.25295	4.32681	3.70435

In Table 2, it is seen that as the value of k increases, the nodal points oscillate on $[0, \pi]$. Morover, the k-th eigenfunction has exactly $|k-2|$ zeroes on $[0, \pi]$. So, the problem is stable and results are accurate and explicit.

Table 3. The nodal length of Prob.(1)-(3) for $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$.

l_j^k	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=1$	**1.40117**	**0.842973**	**0.631566**	**0.513938**	**0.43642**	**0.380528**	**0.337935**
$j=2$	2.61601	**1.0694**	**0.704293**	**0.544949**	**0.452062**	**0.389336**	**0.343297**
$j=3$	6.11863	1.55736	**0.836556**	**0.595963**	**0.476256**	**0.402429**	**0.351055**
$j=4$	15.3745	2.55746	1.06657	**0.675886**	**0.511706**	**0.420791**	**0.361616**
$j=5$	37.9682	4.52613	1.45578	0.797935	**0.56223**	**0.445763**	**0.375523**
$j=6$	89.2697	8.25984	2.09987	0.981142	0.633039	**0.479115**	**0.393477**
$j=7$	198.392	15.0968	3.14441	1.25238	0.731114	0.523132	**0.416357**
$j=8$	417.338	27.2088	4.80608	1.64899	0.865673	0.58072	0.445254
$j=9$	834.402	48.0097	7.40124	2.22223	1.04876	0.655541	0.481503
$j=10$	1593.05	82.7119	11.3833	3.04153	1.29598	0.75217	0.526732

The numerical results obtained in this special case under consideration show that the formulas found are suitable.

3. Conclusion

In this study, the Sturm-Liouville equation is different from the classical Sturm-Liouvile equation since it has more than one potential function. Moreover, there is a linearly eigenparameter on one boundary condition. Using the Prüfer's substitutions, asymptotic formulas of eigenvalues and nodal parameters were obtained. With the knowledge of these asymptotic formulas, the form of the linear operator dealt with in the study has been found. It has been shown with a numerical output that this substitutionsn is a very effective method.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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