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# Commutativity Degree of Certain Finite AC-Groups

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**Abstract**

For a finite group $G$, the probability of two elements of $G$ that commute is the commutativity degree of $G$ denoted by $P(G)$. As a matter of fact, if $\mathcal{C} = \{(a, b) \in G\times G \mid ab = ba\}$, then $P(G) = \frac{|\mathcal{C}|}{|G|^2}$. In this paper, we are going to find few formulas for $P(G)$ independent of $|\mathcal{C}|$; for some AC-groups, and also in some special cases of finite minimal non-abelian groups. Moreover, the study will present implications for certain qualified finite groups.

Keywords: AC-group, commutativity degree, minimal non-abelian group.

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## 1. Introduction

For a finite group $G$, the statistical results go back to 1965, with a number of papers [6, 7, 8, 9, 10, 11, 12]. In [9], it has been proven, for a finite group $G$,

$$|\mathcal{C}| = |\{(a, b) \in G \times G \mid ab = ba\}| = |G|k(G),$$

where $k(G)$ is the number of conjugacy classes of $G$. Moreover, let $C_G(a)$ be the centralizer of $a \in G$, then clearly $(a, b) \in \mathcal{C}$ if and only if $b \in C_G(a)$. Therefore, we also have $|\mathcal{C}| = \sum_{a\in G} |C_G(a)|$. In 1973, Gustafson, in [17], re-proved the results in

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[9]. He also, in that paper, considered the probability of two group elements that commuted, for the first time, and proved some valuable results. Since then, the investigations as to this probability have been interesting topics to research, and up to now, many results have been published, see [4, 5, 13, 16, 18], for instance.

In a finite group $G$, the probability of two elements of $G$ which commute, is also known as the commutativity degree of $G$, and denoted by $P(G)$. This probability equals to $\frac{|\mathcal{C}|}{|G|^2}$. Therefore, by the above mentioned discussion, we also have,

$$P(G) = \frac{\sum_{a \in G} |C_G(a)|}{|G|^2} = \frac{k(G)}{|G|}.$$

In this paper, we consider the commutativity degree of those types of AC-groups that every centralizer of their non-central elements has a certain cardinality, and minimal non-abelian groups, so we present the definition of these types of groups, at the following.

**Definition 1.1.** If every non-central element of a group has an abelian centralizer, then the group is an AC-group.

**Definition 1.2.** Let $G$ be an AC-group. If for every non-central element $x$, we have $|C_G(x)| = n_i$, for the natural numbers $n_i$, $1 \leq i \leq k$, then $G$ is called $(n_1, n_2, \ldots, n_k)$-regular. In particular, if $n_i = k$, for $1 \leq i \leq k$, we simply call it $k$-regular AC-group.

A non-abelian group $H$ is a minimal non-abelian group, if all its non-trivial subgroups are abelian. Clearly, every minimal non-abelian group is an AC-group.

Moreover, our results would be related to the cardinality of a specific subset of a finite group $H$, which is called clique number of $H$. The clique number of $H$ is actually based on the definition of an important algebraic graph which called as non-commuting graph. The non-commuting graph associated with a finite group $H$ is a simple graph as follows: The vertices of the graph are all elements of $H$, and, an edge unites two elements of $H$ that do not commute, hence, in non-commuting graph, a clique defines as the pairwise non-commuting subset of $H$(for more information see [3], for example). The number of vertices in a maximum clique or, as we prefer in this paper, the pairwise non-commuting subset of $H$ with the maximal cardinality announced as clique number and labeled by $\omega(H)$.

Throughout this paper, for a group $H$, we use the notations $Z(H)$ and $N_H(\langle a \rangle)$ for the center of $H$, and the normalizer of $a \in H$, respectively. Moreover, for two elements $a, b \in H$, the commutator of $a$ and $b$ is denoted by $[a, b]$. For all further unexplained notations, one can see [15], for example.

Finally, in this paper, in Section 2, some technical lemmas and theorems are provided which are used throughout the paper. We present in Section 3, our main results. First, the results related to the minimal non-abelian groups will be presented, and in Subsection 3.1, the results as to AC-groups will be presented.

## 2. Auxiliary Results

In this section, we collect some lemmas and theorems which are needed throughout this paper. We begin with a very important lemma related to AC-groups from [19], that is applied in several results.

**Theorem 2.1.** [19] *A group $H$ is an* AC*-group if and only if $C_H(h) = C_H(k)$, for all elements $h, k \in H \setminus Z(H)$ with the condition $[h, k] = 1$.*

**Lemma 2.2.** [2] *For an* AC*-group $H$,*

(i) *If $x, y \in H \setminus Z(H)$ with distinct centralizers, then $C_H(x) \cap C_H(y) = Z(H)$.*

(ii) *If $H = \cup_{i=1}^{k} C_H(h_i)$, where $C_H(h_i)$ and $C_H(h_j)$ are distinct, for $1 \leq i < j \leq k$, then the set of $\{h_1, h_2, \ldots, h_k\}$ is a pairwise non-commuting elements with the maximal cardinality.*

In what follows, we present some lemmas and theorems as to the minimal non-abelian groups.

**Lemma 2.3.** [14] *Let $H$ be a finite minimal non-abelian group, and $p \neq q$ be two prime divisors of $|H|$. If $H = \mathcal{P}\mathcal{Q}$, for a cyclic Sylow $p$-subgroup $\mathcal{P}$, and an elementary abelian minimal normal Sylow $q$-subgroup $\mathcal{Q}$, then $C_H(b) = Z(H) \times \mathcal{Q}$, for any $1 \neq b \in \mathcal{Q}$.*

**Theorem 2.4.** [14] *Let $G$ be a finite minimal non-abelian group, and $G = \mathcal{P}\mathcal{Q}$, for a cyclic Sylow $p$-subgroup $\mathcal{P}$ and an elementary abelian minimal normal Sylow $q$-subgroup $\mathcal{Q}$, for two distinct prime divisors $p$ and $q$ of $|G|$. Then $\omega(G) = |\mathcal{Q}| + 1$.*

**Remark 1**. If $b \in \mathcal{Q}$, by Lemma 2.3, then clearly $\mathcal{Q} \subseteq C_G(b)$. Therefore every set of pairwise non-commuting elements of $G$ with the maximal cardinality consists of only one element of $\mathcal{Q}$.

In Subsection 3.1, we illustrate the results for the group $PSL(2, q)$, where $q$ is a power of a prime $p$. Therefore, at the following, the structure of these kinds of finite groups would be needed.

**Proposition 2.5.** [1, 20] *Let $q$ be a power of a prime $p$, $G = \mathrm{PSL}(2, q)$, and $k = gcd(q-1, 2)$. Therefore*

(i) *$G$ has an elementary abelian Sylow $p$-subgroup $\mathcal{P}$, with order $q$, and its Sylow $p$-subgroups number is $q + 1$.*

(ii) *$G$ has a cyclic subgroup $\mathcal{A}$, with order $t = \frac{q-1}{k}$. In addition, for every non-trivial element $a \in \mathcal{A}$, $N_G(\langle a \rangle)$ is a dihedral group of order $2t$.*

(iii) *$G$ has a cyclic subgroup $\mathcal{B}$, with order $s = \frac{q+1}{k}$. In addition, for every non-trivial element $b \in \mathcal{B}$. $N_G(\langle b \rangle)$ is a dihedral group of order $2s$.*

(iv) *a partition for* $G$ *is the set* $\{\mathcal{P}^g, \mathcal{A}^g, \mathcal{B}^g \mid g \in G\}$.

*Moreover, assume that* $\alpha \in G \setminus Z(G)$, *therefore*

(v) *if* $q \equiv 0 \pmod 4$, *then*

$$C_G(\alpha) = \begin{cases} \mathcal{A}^g & \alpha \in \mathcal{A}^g, \ g \in G, \\ \mathcal{B}^g & \alpha \in \mathcal{B}^g, \ g \in G, \\ \mathcal{P}^g & \alpha \in \mathcal{P}^g, \ g \in G. \end{cases}$$

# 3. Main Results

In this section, we consider our main results. Our first focus is on the proof of the commutativity degree of minimal non-abelian groups, and then we investigate the commutativity degree of AC-groups.

**Theorem 3.1.** *Suppose that* $H$ *be a finite minimal non-abelian group, and* $H = \mathcal{P}\mathcal{Q}$, *for a cyclic Sylow* $p$*-subgroup* $\mathcal{P}$, *and an elementary abelian minimal normal Sylow* $q$*-subgroup* $\mathcal{Q}$, *where* $p$ *and* $q$ *are prime divisors of* $|H|$. *Then*

$$P(H) = \frac{|\mathcal{P}|^2 + |Z(H)|^2(|\mathcal{Q}| - 1)}{|\mathcal{P}|^2|\mathcal{Q}|}.$$

*Proof.* Assume that $|\mathcal{Q}| = q^m$, for a positive integer $m$. Then by Theorem 2.4 and Remark 1, there is a maximal subset of elements $\{a_1, \ldots, a_{q^m}, b\}$ that every two disjoint elements do not commute, with $b \in \mathcal{Q}$ and $a_i \notin \mathcal{Q}$, for $1 \leq i \leq q^m$, in which $H = \bigcup_{i=1}^{q^m} C_H(a_i) \cup C_H(b)$. If for some elements, say $x \in \mathcal{Q}$, we have $x \in C_H(a_i)$, for some $1 \leq i \leq q^m$, then by Theorem 2.1, $C_H(a_i) = C_H(b)$, which is a contradiction, hence $C_H(a_i) \subseteq \mathcal{P}$. Now, by Lemma 2.3, we have $|C_H(b)| = |\mathcal{Q}||Z(H)|$ and $|C_H(a_i)| = |\mathcal{P}|$, for $1 \leq i \leq q^m$. So for the calculating of $P(H) = \frac{\sum_{h\in H}|C_H(h)|}{|H|^2}$, we have

$$\sum_{h\in H} |C_H(h)| = \sum_{h\in Z(H)} |C_H(h)| + \sum_{h\notin Z(H)} |C_H(h)|,$$

and so

$$\begin{aligned}\sum_{h\notin Z(H)} |C_H(h)| &= \sum_{h\in C_H(a_1)\setminus Z(H)} |C_H(h)| + \cdots + \sum_{h\in C_H(a_{q^m})\setminus Z(H)} |C_H(h)| \\ &+ \sum_{h\in C_H(b)\setminus Z(H)} |C_H(h)| \\ &= |C_H(a_1) \setminus Z(H)||\mathcal{P}| + \cdots + |C_H(a_{q^m}) \setminus Z(H)||\mathcal{P}| \\ &+ |C_H(b) \setminus Z(H)||\mathcal{Q}||Z(H)| \\ &= |\mathcal{Q}|(|\mathcal{P}| - |Z(H)||\mathcal{P}| + (|\mathcal{Q}||Z(H)| - |Z(H)|)|\mathcal{Q}||Z(H)| \\ &= |\mathcal{Q}||\mathcal{P}|^2 - |\mathcal{Q}||\mathcal{P}||Z(H)| + |\mathcal{Q}|^2|Z(H)|^2 - |Z(H)|^2|\mathcal{Q}|.\end{aligned}$$

Therefore, an easy calculation shows that

$$P(H) = \frac{|\mathcal{P}|^2 + |Z(H)|^2(|\mathcal{Q}| - 1)}{|\mathcal{P}|^2|\mathcal{Q}|},$$

as required. □

## 3.1 On the Commutativity Degree of AC-Group

Assume that $G$ be a finite $k$-regular AC-group. In this subsection, the results would be on the investigation of the commutativity degree of $G$.

**Lemma 3.2.** *If $\{a_1, a_2, \ldots, a_{\omega(G)}\}$ be the maximal subset of a group $G$, such that $[a_i, a_j] \neq 1$, for every $1 \leq i \neq j \leq \omega(G)$, then $G = \bigcup_{i=1}^{\omega(G)} C_G(a_i)$. Note that by eliminating every $C_G(a_i)$, for $1 \leq i \leq \omega(G)$, the equality does not hold.*

*Proof.* It is clear that $\bigcup_{i=1}^{\omega(G)} C_G(a_i) \subseteq G$. Now, assume that $g \in G \setminus \bigcup_{i=1}^{\omega(G)} C_G(a_i)$, so, $ga_i \neq a_i g$, for $1 \leq i \leq \omega(G)$. Therefore, $\{a_1, a_2, \ldots, a_{\omega(G)}, g\}$ is a subset that every its two elements do not commute, which is a contradiction. □

**Lemma 3.3.** *Let $K$ be a $k$-regular AC-group. Therefore*

$$\omega(K) = \frac{|K| - |Z(K)|}{k - |Z(K)|}.$$

*Proof.* Suppose that $\{a_1, a_2, \ldots, a_{\omega(K)}\}$ be the maximal subset of a group $K$, with the condition $[a_i, a_j] \neq 1$, for every $1 \leq i \neq j \leq \omega(K)$. By Lemma 3.2, $K = \bigcup_{i=1}^{\omega(K)} C_K(a_i)$. On the other hand, by Lemma 2.2,

$$|K \setminus Z(K)| = \sum_{i=1}^{\omega(K)} |C_K(a_i) \setminus Z(K)|,$$

which concludes that,

$$|K| = |Z(K)| + \omega(K)(k - |Z(K)|) = |Z(K)| + \omega(K)(k - |Z(K)|),$$

therefore $\omega(K) = \frac{|K|-|Z(K)|}{k-|Z(K)|}$, as required. □

Now, we are ready to consider the commutativity degree of AC-groups. Our first focus is on the commutativity degree of $k$-regular AC-groups, and then, as an illustration to the results, we consider the commutativity degree of the projective special linear group $\mathrm{PSL}(2, q)$, where $q$ is a power of a prime $p$.

**Theorem 3.4.** *For a finite AC-group $H$, if $H$ is a $k$-regular group, then*

$$P(H) = \frac{|H||Z(H)| + k|H \setminus Z(H)|}{|H|^2}.$$

*Proof.* Since $P(H) = \frac{\sum_{h\in H}|C_H(h)|}{|H|^2}$, by Lemma 2.2, we have

$$\begin{aligned} P(H) &= \frac{\sum_{h\in Z(H)}|C_H(h)| + \sum_{h\notin Z(H)}|C_H(h)|}{|H|^2} \\ &= \frac{|H||Z(H)| + k|H \setminus Z(H)|}{|H|^2}, \end{aligned}$$

as desired. □

Here, an illustrative and expressive example is given as an application of Lemma 3.3 and Theorem 3.4. Let $H$ be a finite $p$-group, and $|H| = p^n$, where $p$ is a prime number and $n$ is a positive integer. Moreover, assume that $\frac{|H|}{|Z(H)|} = p^2$. Obviously, for every element of $H$ which is not central, say $x$, we have $Z(H) \subsetneq C_H(x) \subsetneq H$, and $C_H(x) = \langle Z(H), x\rangle$, which concludes that $C_H(x)$ is abelian, so $H$ is an AC-group. Moreover, clearly $H$ is $p^{n-1}$-regular. For these types of groups, in [2], there was a complex process to find the $\omega(H) = p+1$, while by using Lemma 3.3, and a simple calculation, it is easy to see that

$$\omega(H) = \frac{|H| - |Z(H)|}{k - |Z(H)|} = \frac{p^n - p^{n-2}}{p^{n-1} - p^{n-2}} = p+1.$$

In addition, Theorem 3.4 deduces that

$$P(H) = \frac{(p^n \cdot p^{n-2}) + (p^n - p^{n-2})p^{n-1}}{p^{2n}} = \frac{p^2+p-1}{p^3}.$$

Theorem 3.4 could be extended to the $(m, n)$-regular AC-group, for natural numbers $m$ and $n$.

**Corollary 3.5.** *For every $(m, n)$-regular* AC*-group, say $\mathcal{K}$, the commutativity degree of $\mathcal{K}$ is:*

$$P(\mathcal{K}) = \frac{|\mathcal{K}||Z(\mathcal{K})| + lm(m - |Z(\mathcal{K})|) + kn(n - |Z(\mathcal{K})|)}{|\mathcal{K}|^2},$$

*where $l$ and $k$ are the number of centralizers of $\mathcal{K}$ with the orders of $m$ and $n$, respectively.*

*Proof.* Suppose that $X$ and $Y$ are the set of all non-central elements of $\mathcal{K}$ with the centralizers of orders $m$ and $n$, respectively. Clearly, $|X| = l(m - |Z(\mathcal{K})|)$ and $|Y| = k(n - |Z(\mathcal{K})|)$. By the similar process of Theorem 3.4, and by the following fact:

$$\sum_{x\in\mathcal{K}} |C_{\mathcal{K}}(x)| = \sum_{x\in Z(\mathcal{K})} |C_{\mathcal{K}}(x)| + \sum_{x\in X} |C_{\mathcal{K}}(x)| + \sum_{x\in Y} |C_{\mathcal{K}}(x)|,$$

the conclusion would be straightforward. □

Moreover, according to the above discussion, we can logically find a formula for all the cases in general.

**Theorem 3.6.** *Let $\mathcal{H}$ be a $(m_1, m_2, \ldots, m_k)$-regular* AC*-group, where $m_i$ and $k$ are positive integers, for $1 \leq i \leq k$. Therefore*

$$P(\mathcal{H}) = \frac{|\mathcal{H}||Z(\mathcal{H})|+\sum_{i=1}^{k} n_i m_i (m_i - |Z(\mathcal{H})|)}{|\mathcal{H}|^2},$$

*where $n_i$ is the number of centralizer of $\mathcal{H}$ with the order of $m_i$, for $1 \leq i \leq k$.*

*Proof.* Utilizing the same way of the proof of Corollary 3.5, the proof would be straightforward. □

In the following, we focus on the commutativity degree of some projective special linear groups.

**Theorem 3.7.** *The commutativity degree of the projective special linear group* $\mathrm{PSL}(2, q)$*, where $q$ is a power of a prime $p$, and $q \equiv 0 \pmod 4$, is* $\frac{q^2+q-1}{q(q-1)^2(q+1)}$.

*Proof.* By Proposition 2.5, obviously, $\mathrm{PSL}(2, q)$ is an AC-group. On the other hand, Proposition 2.5 forces that every $C_G(a)$, for $a \in G \setminus Z(G)$, is actually equal to the conjugation of some subgroups $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{P}$, which are introduced in Proposition 2.5. Therefor, by the orbit-stabilizer theorem, the number of $C_G(a)$, for $a \in G$, would be $[G : N_G(\langle a \rangle)]$. Hence if we deduce that $n_{\mathcal{A}}$ and $n_{\mathcal{B}}$ be the number of centralizers of $G$ with the orders of $|\mathcal{A}|$ and $|\mathcal{B}|$, respectively, then Proposition 2.5 and the fact that $|\mathrm{PSL}(2, q)| = q(q^2 - 1)$ assure us

$$n_{\mathcal{A}} = \frac{|\mathrm{PSL}(2,q)|}{|N_G(A)|} = \frac{q(q^2-1)}{2(q-1)} = \frac{q(q+1)}{2},$$

and similarly, $n_{\mathcal{B}} = \frac{q(q-1)}{2}$. The above discussion and Theorem 3.6 give us a straightforward calculation to find the commutativity degree of $\mathrm{PSL}(2, q)$, where $q$ is a power of a prime $p$, and $q \equiv 0 \pmod 4$, in fact, we have

$$P(G) = \frac{|\mathrm{PSL}(2,q)||Z(G)| + n_{\mathcal{A}}|\mathcal{A}|(|\mathcal{A}| - |Z(G)|) + n_{\mathcal{B}}|\mathcal{B}|(|\mathcal{B}| - |Z(G)| + n_{\mathcal{P}}|\mathcal{P}|(|\mathcal{P}| - |Z(G)|)}{|\mathrm{PSL}(2,q)|^2},$$

where $n_p$ is the number of Sylow $p$-subgroups of $\mathrm{PSL}(2, q)$, which is $q + 1$, by Proposition 2.5(i). By a simple calculation, we have:

$$P(G) = \frac{q(q^2-1)+\frac{q(q+1)}{2}(q-1)(q-2)+\frac{q(q+1)}{2}(q+1)q+q(q+1)(q-1)}{q^2(q^2-1)^2} = \frac{q^2+q-1}{q(q-1)^2(q+1)},$$

as required. □

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