Mathematics Interdisciplinary Research **7** (2022) 165 – 178

Original Scientific Paper

On Ordered Regular Semigroups with a Zero Element

Goncalo da Camara e Almeida Pinto ⋆

Abstract

In this paper, we study several conditions on ordered regular semigroups containing a zero element. In particular, we consider the natural and semigroup order and their connections to the properties of being principally ordered, Dubreil-Jacotin and BZS. We study also the set of biggest inverses in such a semigroup and we characterize subalgebras generated by two comparable idempotents.

Keywords: ordered semigroups, regular, naturally ordered, principally ordered, strong Dubreil-Jacotin, semilattice.

2020 Mathematics Subject Classification: 20M17, 06F05.

How to cite this article
G. A. Pinto, On ordered regular semigroups with a zero element, *Math. Interdisc. Res.* **7** (2022) 165 – 178.

1. Introduction

We are going to study several types of ordered regular semigroups, all of them with the common denominator of having a zero element, 0.

We recall (see, for example [1]) that the *natural order*, $\leq_n$ on the idempotents of a regular semigroup S, is defined by

$$e \leq_n f \iff e = ef = fe,$$

and that an ordered regular semigroup $(T, \leq)$ is said to be *naturally ordered* if the order extends the natural order, in the sense that if $e \leq_n f$ then $e \leq f$.

⋆Corresponding author (E-mail: gcapinto@gmail.com)
Academic Editor: Mohammad Shahryari
Received 17 August 2021, Accepted 23 April 2022
DOI:10.22052/mir.2022.243017.1301

A *strong Dubreil-Jacotin semigroup* (see, for example, [1]) is an ordered semigroup S for which there exists an ordered group G and an epimorphism $f : S \to G$ that is *residuated*, in the sense that the pre-image under f of every principal order ideal of G is a principal order ideal of S. In particular, the pre-image of the negative cone $N(G) = \{x \in G | x \leq 1\}$ is a principal order ideal $\xi^{\downarrow} = \{x \in S | x \leq \xi\}$ of S. To the top element ξ we call *bimaximum element.* This element is said to be *equiresidual* if, for every $x \in S$, the order ideals $\{y \in S | xy \leq \xi\}$ and $\{y \in S | yx \leq \xi\}$ coincide and have a greatest element denoted by $\xi : x$. When S is regular, the bimaximum element ξ is the biggest idempotent of S and if, e is an idempotent then $\xi : e = \xi$. The set of those elements $x \in S$, for which $x = x(\xi : x)x$ is called the set of *perfect* elements.

A semigroup S, with a *zero* element 0, is called *BZS semigroup* if for all $x \in S$ we have that $x^2 = x$ or $x^2 = 0$. These semigroups were first studied in ring theory and introduced to semigroup theory in [10].

Throughout this paper we consider S a regular semigroup, $E(S)$ the set of idempotents of S, and for every $x \in S$, we denote the set of inverses of x in S by $V(x)$.

2. Principally Ordered Naturally Ordered BZS Semigroups

We start considering an ordered regular semigroup S, with a zero element 0. To pursue our objectives we need to remind several concepts and previous results.

With this in mind, we recall that an ordered regular semigroup S, is said to be *principally ordered* if, for every $x \in S$ there exists $x^* = \max\{y \in S \mid xyx \leq x\}$. The basic properties of the operation $x \to x^*$ in such semigroups were established in [3] and [4] and are listed in [1, Theorem 13.26]. In particular, we recall for the readers convenience that, in such a semigroup, the following properties hold, and will be used throughout in what follows:

(P_1) $\big(\forall x \in S\big)$ $x = xx^*x$.

(P_2) every $x \in S$ has a biggest inverse, namely $x^0 = x^*xx^*$.

(P_3) $\big(\forall x \in S\big)$ $x^0 \leq x^*$.

(P_4) $\big(\forall x \in S\big)$ $xx^0 = xx^*$ and $x^0x = x^*x$.

(P_5) $\big(\forall e \in E(S)\big)$ $e \leq e^*$.

(P_6) $\big(\forall x \in S\big)$ $x \leq x^{**}$.

(P_7) $\big(\forall x \in S\big)$ $x^* = x^{***}$.

(P_8) $\big(\forall x \in S\big)$ $x^{0*} = x^{**} = x^{*0}$.

In any ordered regular semigroup S, in which every $x \in S$ has a biggest inverse x^0, it was proven in [11], and is stated in [1, Theorem 13.22], that

(P_9) $\big(\forall x \in S\big)$ $(xx^0)^0 = x^{00}x^0$ and $(x^0x)^0 = x^0x^{00}$.

(P_{10}) $\big(\forall x \in S\big)$ $x^0 = x^{000}$.

We recall that Green's relations $\mathcal{R}$ and $\mathcal{L}$ are said to be *regular* if for every $x, y \in S$, such that $x \leq y$, we have that $xx^0 \leq yy^0$ and $x^0x \leq y^0y$.

Similarly, Green's relations $\mathcal{R}$ and $\mathcal{L}$ are *weakly regular* if for every $e, f \in E(S)$, such that $e \leq f$, we have that $ee^0 \leq ff^0$ and $e^0e \leq f^0f$.

Clearly, if Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular, they are also weakly regular.

Properties (P_9) and (P_{10}) hold in a principally ordered regular semigroup since by (P_2), biggest inverses in such a semigroup exist.

In the sequel we shall refer to the application $x \to x^*$ as being *antitone*, if $x \leq y$ implies that $y^* \leq x^*$.

An important property that holds in a principally ordered regular semigroup S is, that the application $x \to x^*$ is antitone if and only if, S is naturally ordered.

Similarly, we say that the application $x \to x^*$ is *weakly isotone* if, for every $e, f \in E(S)$ such that $e \leq f$, we have that $e^* \leq f^*$.

It can be easily shown that in every principally ordered regular semigroup with a zero element 0, the element 0^* is the biggest element of the semigroup.

Also, in every naturally ordered principally ordered regular semigroup S, the zero element is its smallest element (in fact, an idempotent) and the following holds:

$$\big(\forall e \in E(S)\big)\, 0 \leq e \leq e^* \leq 0^*.$$

Lemma 2.1. *If S is a principally ordered naturally ordered regular BZS semigroup with a zero element 0, then*

(1) $0^* = 0^{**} \in E(S)$.

(2) $\big(\forall x \in S\big)\ x^{**} = x^* = 0^* \in E(S)$.

(3) $\big(\forall x \in S\big)\ x^* = x^*0^* = 0^*x^* = 0^*x^*0^*$.

(4) $\big(\forall x \in S\big)\ x^0 = x^{00} = 0^*x0^* \in E(S)$.

(5) *Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular.*

Proof. (1) Since S is a BZS semigroup, we have that $(0^*)^2 = 0^*$ or $(0^*)^2 = 0$. In the first case $0^* \in E(S)$ which means that $0^* \leq (0^*)^* \leq 0^*$. Thus, $0^* = 0^{**} \in E(S)$.

In the second case, $(0^*)^2 = 0$ implies that

$$0^* \cdot 0^* \cdot 0^* = 0 \cdot 0^* = 0 \leq 0^* \implies 0^* \leq 0^{**}.$$

Since 0^* is the biggest element of S, we can conclude that $0^* = 0^{**}$. Therefore, by property (P_1), $0^* = 0^*0^{**}0^* = 0^*0^*0^* = 00^* = 0 \in E(S)$.

(2) For any $x \in S$ we have, since S is a BZS semigroup, that $x^2 = x$ or $x^2 = 0$. If $x^2 = x$ then, $0 \leq x \leq x^* \leq 0^*$ and therefore, since the application $x \to x^*$

is antitone, we obtain $0^{**} \leq x^{**} \leq x^* \leq 0^*$, which means, by (1), that $x^{**} = x^* = 0^*$.

If $x^2 = 0$ then,

$$x \cdot xx^* \cdot x = 0 \leq x \implies xx^* \leq x^* \implies x = xx^*x \leq x^*x.$$

Similarly, $x \leq xx^*$ and using (P_8) and (P_6), we obtain

$$x \leq xx^* \leq x^*xx^* = x^0 \implies x^{0*} \leq x^* \implies x^{**} \leq x^* \implies x \leq x^*.$$

Thus, $0 \leq x \leq x^* \leq 0^*$ and the result follows like in the first case.

Finally by (1), we have that $x^*x^* = 0^*0^* = 0^* = x^*$, which means that x^* is an idempotent.

(3) For any $x \in S$, we have using successively (P_1), (2), [1 and Theorem 13.27((1) and (2))] and(P_7), that

$$\begin{aligned} x^* &= x^*x^{**}x^* \leq 0^*x^* \\ &= 0^*x^*x^{**}x^* \leq 0^*x^*0^* \\ &= ((x^*x^{**})^* \cdot x^*x^{**}) \cdot x^* \cdot (x^{**}x^* \cdot (x^{**}x^*)^*) \\ &\leq x^{***}x^{**} \cdot x^* \cdot x^{**}x^{***} \\ &= x^*x^{**} \cdot x^* \cdot x^{**}x^* = x^*. \end{aligned}$$

Therefore, $x^* = 0^*x^* = 0^*x^*0^*$ and similarly it is equal to x^*0^*.

(4) For $x \in S$ we have, by (P_2), (2), (P_4) and (P_9), that

$$\begin{aligned} x^0 = x^*xx^* &= 0^*x0^* \\ &= (xx^0)^*xx^0x(x^0x)^* \\ &= (xx^0)^0xx^0x(x^0x)^0 \\ &= x^{00}x^0xx^0x^{00} \\ &= x^{00}, \end{aligned}$$

By (2) we obtain that

$$x^0x^0 = x^*xx^*x^*xx^* = x^*xx^*xx^* = x^0,$$

and x^0 is an idempotent.

(5) Let $x, y \in S$ be such that $x \leq y$. Then, using (P_4) and (2), we have that

$$xx^0 = xx^* = x0^* \leq y0^* = yy^* = yy^0$$

and similarly, $x^0x \leq y^0y$. Therefore, $\mathcal{R}$ and $\mathcal{L}$ are regular.

□

It is an immediate consequence of Lemma 2.1(2) that in a naturally ordered principally ordered BZS regular semigroup S, the subset S^* is a singleton, where the unique element is an idempotent, and therefore it is the trivial group.

In the next two Theorems, we are going to consider the set of the biggest inverses S^0, of S which, by Lemma 2.1(4) is a subset of $E(S)$.

In order to reach our target, we need first to remind the definition of several types of bands.

A band B is said to be *left regular* (see for example [8]) if $xyx = xy$, for all $x, y \in B$. Similarly, B is a *right regular* band if $xyx = yx$, for all $x, y \in B$. A band B for which $xy = yx$, for all $x, y \in B$, is said to be a *semilattice.*

In [3, Theorem 4] Blyth and Pinto proved that if S is a naturally ordered principally ordered regular semigroup, then S^0 is a subsemigroup of S. With the additional hypothesis of having the BZS property, we can prove that S^0 is a semilattice.

Theorem 2.2. *If S is a naturally ordered principally ordered BZS regular semigroup, then the subsemigroup S^0 of S is a semillatice, for which*

$$\big(\forall x, y \in S\big)\, x^0y^0 = (x0^*y)^0 = y^0x^0.$$

Proof. Let x, y be any elements of S. We have, by (P_2) and Lemma 2.1(2), that

$$x^0y^0 = x^*xx^*y^*yy^* = 0^*x0^*0^*y0^* = 0^*x0^*y0^* = (x0^*y)^*x0^*y(x0^*y)^* = (x0^*y)^0,$$

which means that S^0 is a subsemigroup of S.

Also, in a naturally ordered regular semigroup with a biggest idempotent α, it is proved in [1, Theorem 13.16 and Theorem 13.17], that biggest inverses exist and that $y^0x^0 = (x\alpha y)^0$ for all elements x, y.

Since S is a naturally ordered principally ordered BZS regular semigroup we have, by Lemma 2.1(1), that S has a biggest idempotent 0^*. Therefore,

$$x^0y^0 = (x0^*y)^* = y^0x^0,$$

which means that S^0 is a semillatice. □

A natural question to raise is: when is S itself a semillatice?

Theorem 2.3. *Let S be a naturally ordered principally ordered BZS regular semigroup. The following statements are equivalent:*

(1) $S = S^0$.

(2) $\big(\forall x \in S\big)\ x = x^0$.

(3) *S is a left regular band and a right regular band.*

(4) *S is a semillatice.*

Proof. (1) $\implies$ (2) : For all $x \in S$, we have by (1), that there exist $y \in S$ such that $x = y^0$. Then, by (P_{10}) we have that $y^0 = y^{000}$, and therefore, using Lemma 2.1(4) we conclude that $x = y^0 = y^{000} = (y^0)^{00} = x^{00} = x^0$.
(2) $\implies$ (1) : From (2) we have that $S \subseteq S^0$. Since the reverse inclusion always hold, we can say that $S = S^0$.
(2) $\implies$ (3) : From Lemma 2.1(4), we can immediately say that S is a band. In [11] Saito proved (can be seen in [1, Theorem 13.25]) that in a naturally ordered regular semigroup S, with biggest inverses on which $\mathcal{R}$ and $\mathcal{L}$ are weakly regular, the equalities $(x^0xy)^0x^0 = (xy)^0 = y^0(xyy^0)^0$ holds for all $x, y \in S$.

By Lemma 2.1(5) we have, under our hypothesis, that $\mathcal{R}$ and $\mathcal{L}$ are weakly regular. Then, using (2) in the first equality we immediately obtain that $(xxy)x = xy$, Since S is a band, we can conclude that $xyx = xy$ that is, S is a left regular band. Using the second equality, interchanging x with y, we similarly obtain $xyx = yx$, meaning that S is a right regular band.
(3) $\implies$ (4) : If S is a left regular band and a right regular band, we have for any $x, y \in S$, that $xy = xyx = yx$, which means that S is a semilattice.
(4) $\implies$ (2) : For any $x \in S$, we have, by Lemma 2.1(4) that $x^0 \in E(S)$, and using the hypothesis that S is a semillatice, we have that

$$x = xx^0x = xx^0x^0x^0x = x^0xx^0xx^0 = x^0,$$

which proves (2). □

We now turn our attention to relations (in an ordered regular BZS semigroup) between the concepts of being principally ordered and naturally ordered. First of all, it is easy to see that, in every principally ordered regular BZS semigroup S, the following two conditions are equivalent:

Theorem 2.4. *If S is a principally ordered regular BZS semigroup, then the following statements are equivalent:*

(1) *S is naturally ordered;*

(2) $\big(\forall e, f \in E(S)\big)\ e^* = f^*$.

Proof. (1) $\implies$ (2): This follows from Lemma 2.1(2).
(2) $\implies$ (1): Let us consider $e, f \in E(S)$, such that $e \leq_n f$. Then, $e = ef = fe$ and we obtain by (2), that $e = ef = fef \leq fe^*f = ff^*f = f$ which means that S is naturally ordered. □

Theorem 2.5. *Let S be a naturally ordered regular BZS semigroup. The following statements are equivalent:*

(1) *S is principally ordered;*

(2) *S is a strong Dubreil-Jacotin semigroup.*

Proof. (1) $\Longrightarrow$ (2): We know that 0^* is the biggest element of S, but from Lemma 2.1(1), 0^* is an idempotent of S. Therefore, we can conclude that S has a biggest idempotent. The result follows by [1, Theorem 13.28].
(2) $\Longrightarrow$ (1): It follows directly from [1, Theorem 13.28]. □

An obvious conclusion of the previous results, can be summarise in the following:

Corollary 2.6. *If S is naturally ordered principally ordered regular BZS semigroup, then S is a strong Dubreil-Jacotin semigroup with a biggest and a smallest elements* (*in fact, idempotents*).

Let us now present some examples to illustrate this type of semigroups.

Example 2.7. In [2], Blyth and McFadden presented, using the following Hasse diagram (see Figure 1) and Cayley table:

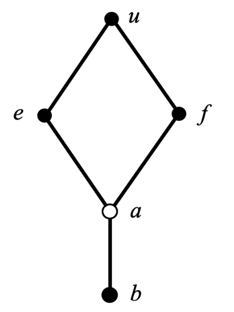


Figure 1:

	u	e	f	a	b
u	u	u	f	f	b
e	e	e	a	a	b
f	u	b	f	b	b
a	e	b	a	b	b
b	b	b	b	b	b

an ordered semigroup N_5, the smallest naturally ordered non-orthodox regular semigroup with a biggest idempotent.

From [3] and [10] we know that N_5 is a naturally ordered principally ordered regular BZS semigroup with a zero element b. In fact, b is the smallest element (idempotent) of N_5 and $u = b^*$ is the biggest element (idempotent) of N_5.

Example 2.8. Consider the rectangular band (see for example [9]) $S = \mathbb{Z} \times \mathbb{Z}$. Let us define the cartesian order in S, where we consider the usual order in the integers $\mathbb{Z}$. With this order we clearly obtain an ordered semigroup, in fact a band, and therefore a BZS semigroup. If $(i, \lambda), (j, \mu) \in S$ are such that $(i, \lambda) \leq_n (j, \mu)$, then

$$(i, \lambda)(j, \mu) = (i, \lambda) = (j, \mu)(i, \lambda),$$

from which we obtain $(i, \mu) = (i, \lambda) = (j, \lambda)$, that is $(i, \lambda) = (j, \mu)$ and the natural order coincides with the equality relation, therefore S is naturally ordered with the following Hasse diagram, see Figure 2.

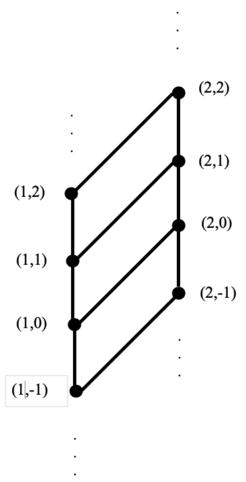


Figure 2:

If we add a bottom element 0, multiplying in the following way:

$$(i, \lambda) \cdot 0 = 0 = 0 \cdot (i, \lambda), \qquad 0 \cdot 0 = 0,$$

then the Hasse diagram of the new semigroup $S \cup \{0\}$ becomes, see Figure 3.

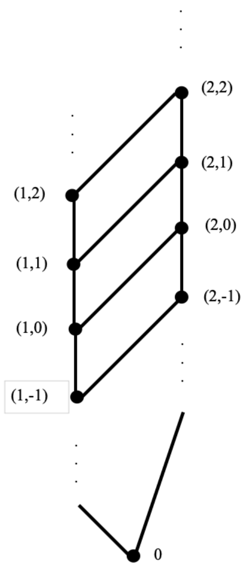
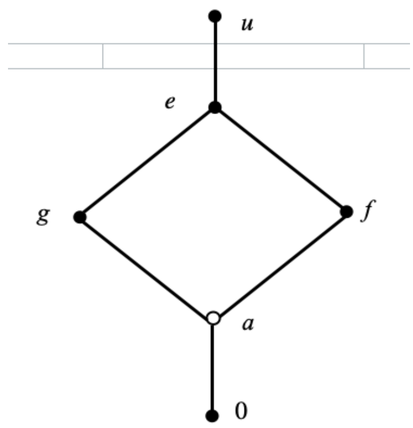


Figure 3:

Routine calculations allows us to state that $S \cup \{0\}$ is a naturally ordered regular BZS semigroup that is not principally ordered. In fact, $\{x \in S \mid 0 \cdot x \cdot 0 \leq 0\} = S \cup \{0\}$, which means that $\max\{x \in S \mid 0 \cdot x \cdot 0 \leq 0\}$ does not exist. Therefore by Theorem 2.5, $S \cup \{0\}$ is not a strong Dubreil-Jacotin semigroup.

Example 2.9. In [7], Blyth and Pinto presented a semigroup constructed using the isotone mappings from a three element chain into itself, preserving the bottom element. It is denoted by Res **3** and it can be defined by the following Hasse diagram (see Figure 4) and Cayley table:



	u	e	f	g	a	0
u	u	u	u	g	g	0
e	u	e	f	g	a	0
f	f	f	f	a	a	0
g	u	g	0	g	0	0
a	f	a	0	a	0	0
0	0	0	0	0	0	0

Figure 4:

It is a principally ordered regular BZS semigroup with zero element 0. We have that $u^* = a^* = f^* = g^* = 0^* = u$ and $e^* = e$. Note, that 0^* is neither the biggest element, nor the biggest idempotent of S. This is possible, since S is not naturally ordered. In fact, $u \leq_n e$, but $u \not\leq e$.

Example 2.10. The set of 2×2 real matrices $S = \{I, O, E_{11}, E_{12}, E_{21}, E_{22}\}$, where O is the zero matrix and

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with the usual product of matrices is a BZS inverse ordered semigroup with a partial order defined by the following Hasse diagram, see Figure 5.

We obtain a naturally ordered BZS inverse semigroup. It can be checked directly from the Hasse diagram that S has a biggest idempotent I, that is not a biggest element. Now, S is not principally ordered since for example, E_{12}^* does not exist. Therefore by Theorem 2.5, S is not a strong Dubreil-Jacotin semigroup.

Let us now, briefly illustrate what happens if the ordered regular semigroup S with a zero element 0, has an identity element 1, that is, if S is a *monoid*.

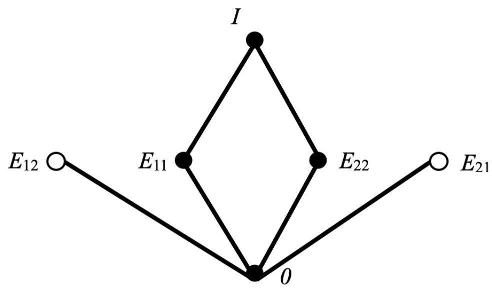


Figure 5:

Lemma 2.11. *Let S be an ordered regular monoid with a zero element* 0.

(1) *For all $e \in E(S)$, we have that $e \leq_n 1$.*

(2) *If S is naturally ordered, then $e \leq 1$ for all $e \in E(S)$.*

(3) *If S is naturally ordered and principally ordered, then for all $e \in E(S)$, $e \leq 1 \leq 0^*$.*

(4) *If S is naturally ordered principally ordered and BZS, then $1 = 0^*$.*

Proof. (1) For all $e \in E(S)$, we have since 1 is the identity element of S that $e \cdot 1 = 1 \cdot e = e$, and therefore $e \leq_n 1$.

(2) This follows immediately from (1).

(3) We have that, under these hypothesis, 0^* is the biggest element of S, and the result follows from (2).

(4) If S also has the BZS property, then by Lemma 2.1(1), 0^* is an idempotent and from (3), we have $0^* \leq 1 \leq 0^*$, and therefore $1 = 0^*$.

□

We remember that the semigroup in Example 2.9 denoted by Res **3**, is a principally ordered regular BZS semigroup with a zero element 0 and an identity element e, but it is not naturally ordered. We have that $0^* = u \neq e$, which means that the hypothesis that semigroup is naturally ordered in Lemma 2.11(4) is in fact, necessary to obtain that 0^* equals the identity element.

Also, in Example 2.9 we have a biggest idempotent (different from the identity element) u, that is the biggest element of the semigroup Res **3** while in Example 2.10, we have a biggest idempotent but not a biggest element. So, again natural order as well as principally ordered, play an important role.

In [1, Exercise 13.19] it is mentioned that in a strong Dubreil-Jacotin regular semigroup S, the set of perfect elements $P(S) = \{x \in S \mid x(\xi : x)x = x\}$, is a regular subsemigroup of S. With this in mind, we can prove the following result.

Theorem 2.12. *If S is a strong Dubreil-Jacotin regular BZS semigroup, then the set of perfect elements $P(S)$, is a strong Dubreil-Jacotin if and only if, the following statements hold:*

(1) *$P(S)$ is naturally ordered;*

(2) (2) *$P(S)$ is principally ordered.*

Proof. First, let us note that $0(\xi : 0)0 = 0$, and so $0 \in P(S)$. We know, by hypothesis that the BZS property holds in S, and therefore it also holds in $P(S)$. Let us assume that $P(S)$ is a strong Dubreil-Jacotin ordered regular semigroup. Take e, f idempotents in $P(S)$, such that $e \leq_n f$, that is $e = ef = fe$. Then,

$$e = fe = fef \leq f\xi f = f(\xi : f)f = f,$$

which means that $P(S)$ is naturally ordered. From Theorem 2.5, we can conclude that $P(S)$ is principally ordered.

Conversely, let us assume that $P(S)$ is a naturally ordered principally ordered BZS regular semigroup. Then, by Corollary 1, $P(S)$ is a strong Dubreil-Jacotin emigroup. □

If we look into the semigroup Res **3** in Example 2.10, it is a strong Dubreil-Jacotin regular BZS semigroup with bimaximum element $\xi = u$, that is not naturally ordered. Routine calculations allow us to conclude that $P(\text{Res } \mathbf{3}) = \{u, f, g, a, 0\}$ is a naturally ordered principally ordered regular BZS semigroup, which is in fact, an ordered isomorphic and semigroup isomorphic copy of N_5 in Example 2.7.

In the next two theorems we describe in a naturally ordered principally ordered regular BZS semigroup S, the structure generated by a pair of comparable idempotents of S. More specifically, if e, f are comparable idempotents of S, we present the Hasse diagrams of the subsemigroup generated by the set $\{e, f, e^0, f^0\}$, and of the subalgebra $(S, ^*)$ generated by $\{e, f\}$. To prove these statements, we shall use results of Blyth and Pinto, proved in [5] and [6].

Theorem 2.13. *Let S be a naturally ordered principally ordered regular BZS semigroup. If $e, f \in S$ are such that $e \leq f$ then the subsemigroup generated by the set $\{e, f, e^0, f^0\}$ has the Hasse diagram, see Figure* 6.

Proof. In a principally ordered regular semigroup every element x, has a biggest inverse x^0. By Lemma 2.1(5) Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular and therefore weakly regular. In [5, Theorem 8] the above Hasse diagram is obtained in a naturally ordered regular semigroup with biggest inverses on which $\mathcal{R}$ and $\mathcal{L}$ are weakly regular. Since, all the hypothesis of [5, Theorem 8] hold the result follows. □

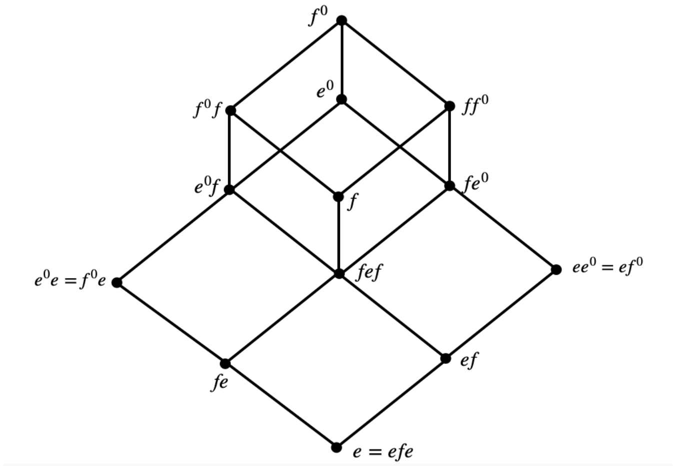


Figure 6:

Theorem 2.14. *Let S be a naturally ordered principally ordered regular BZS semigroup. If $e, f \in S$ are such that $e < f$ then the subalgebra T of $(S,^*)$ generated by $\{e, f\}$ is a band having at most 14 elements. In the case where T has precisely 14 elements it is represented by the Hasse diagram, see Figure 7.*

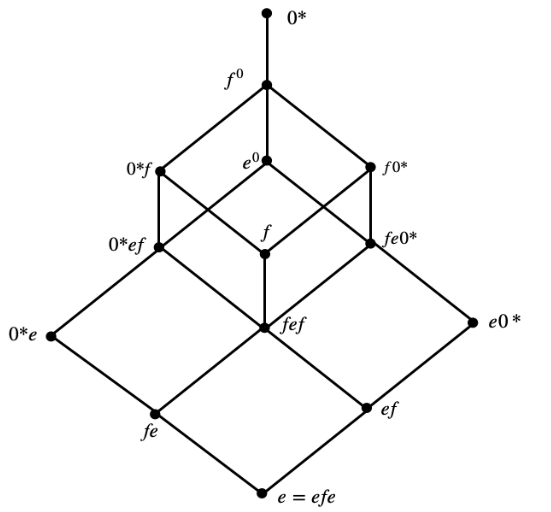


Figure 7:

where lines with positive slope are $\mathcal{R}$ related, lines with negative slopes are $\mathcal{L}$ related, *and vertical lines also denote the natural order.*

Proof. We have, by Theorem 2.4, that in a naturally ordered principally ordered regular BZS semigroup S, $e^* = 0^* = f^*$ for all idempotent $e, f \in S$.

Also, for any $e, f \in E(S)$ such that $e < f$ we have that $e^* = f^*$ and, in particular, the application $x \to x^*$ satisfies, for any idempotents e, f in S, that

$$e \leq f \quad \Longrightarrow \quad e^* \leq f^*,$$

which means that $x \to x^*$ is weakly isotone.

In [6, Theorem 1] the above Hasse diagram is obtained in a principally ordered regular semigroup on which for any $e, f \in E(S)$ such that $e < f$ we have that $e^* = f^*$. Since, all the hypothesis of [6, Theorem 1] hold the result follows. □

If $e = 0$ in Theorem 2.13, the Hasse diagram collapses to at most 5 elements that are ordered in the following Hasse diagram, see Figure 8.

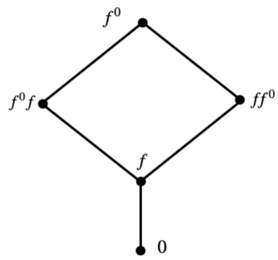


Figure 8:

Also, if $e = 0$ in Theorem 2.14, the Hasse diagram collapses to at most 6 elements that are ordered in the following Hasse diagram, see Figure 9.

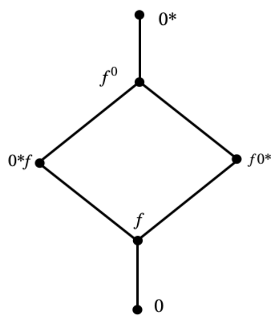


Figure 9:

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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Goncalo da Camara e Almeida Pinto
Department of Mathematics,
College of Science,
Sultan Qaboos University,
Muscat, Oman
e-mail: gcapinto@gmail.com