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Continuity and Differentiability of Solutions with Respect to Initial Conditions and Peano Theorem for Uncertain Differential Equations

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Abstract

In this paper we study the dependence of solutions of uncertain initial value problems (UIVP) on the initial values. Introducing a contraction mapping and using Banach Fixed Point Theorem (BFPT), the existence and uniqueness (EaU) of solutions of the UIVP will be proven. We show that under appropriate assumptions, the solutions of UIVP are continues and differentiable with respect to initial conditions (ICs). The paper will be ended by proving a theorem about the existence of solutions of an autonomous UIVP under weaker conditions. This theorem is a generalization of Peano Theorem to UDEs.

Keywords: continuity, differentiability, uncertain differential equations, initial conditions, Peano theorem, fixed Point.

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1. Introduction

To study the behavior of uncertain phenomena, Liu has founded the uncertainty theory which is a branch of mathematics based on normality, monotonicity, self-

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duality, countable subadditivity, and product measure axioms ([1]). Uncertain differential equation (UDE) has been introduced by Liu ([1]) as a differential equation involving uncertain process. The theory of UDEs is developed by Yao Key ([2]). A solution of an UDE, which models a physical or natural phenomenon, is used to predict the behavior of the system with uncertain term.

Under Lipschitz continuous condition and linear growth condition, an EaU theorem of solutions of an UDE is proved by Chen and Liu ([3]). EaU of solutions of multi-dimensional UDE is proved by Ji and Zhou in [4]. Another EaU theorem on UDEs with local Lipschitz condition (LC) is proved by Gao in [5]. EaU theorem of a system of uncertain linear equations is also proved by the authors in [6]. Recently, the Liouville formula for the uncertain homogeneous linear system and explicit solutions of the system has been proven by the authors in [7].

In this paper, using BFPT, the EaU of solutions of an UDE will be proven. UDEs with an IC that have a unique solution, this unique solution is constantly changing relative to the ICs. This means that a minor mistake in the IC measurement only results in a minor mistake in the solutions. Continuity and differentiability of the solutions of an UIVP with respect to the ICs will also be considered in this work. At the end, an existence theorem for an autonomous UIVP will be presented without LC. This theorem is known as Peano Theorem in ordinary differential equations. To see the definitions of uncertain measure, uncertainty space, uncertain process, sample path, Liu process and Liu integral refer to [1, 8] and [9].

Definition 1.1. [8] Suppose f and g are two functions and C_t is a Liu process. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{1}$$

is called an UDE. The uncertain process X_t that satisfies (1) identically in t is also called a solution.

Equation (1) with the IC $X_{t_0} = X_0$ is called an UIVP. That is:

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \quad X_{t_0} = X_0.$$

The EaU of the solutions of the UIVP is proved in [3].

Remark 1. UDE (1) is autonomous if f and g be independent of t.

Remark 2. [8] The UIVP is equivalent to the uncertain integral equation

$$X_t = X_0 + \int_{t_0}^{t} f(s, X_s)\mathrm{d}s + \int_{t_0}^{t} g(s, X_s)\mathrm{d}C_s.$$

2. Continuity

In the classical theory of ordinary differential equations, it is well known that under suitable conditions, solutions of an initial value problem are continuous with respect to the ICs. This type of theorem concerning continuity of solutions with respect to ICs can be found in the literature. Similar theorems are also available for other types of equations such as dynamic equations on time scales or differential equations with impulses. The aim of this section is to prove a different version of the EaU theorem for the UIVP by using a contraction mapping. Moreover, the continuity of the solutions of the UIVP with respect to the ICs will be proven. Our result unifies and extends existing theorems for other types of equations. First, consider BFPT.

Theorem 2.1. [10] *Every contraction mapping on a complete metric space has a unique fixed point.*

Let $t \in \mathbb{R}$, $X_t \in \Gamma^n$ and D be an open set in (t, X_t)-space. Suppose also that $(t_0, X_0) \in D$ and f and g are continuous on D and satisfy local LC

$$|f(t, X_t) - f(t, Y_t)| + |g(t, X_t) - g(t, Y_t)| \leqslant L|X_t - Y_t|, \quad \forall X_t, Y_t \in \Gamma^n,$$

for some constant L. Assume that A and B are positive such that

$$D^{'} = \{(t, X_t) \in \mathbb{R} \times \Gamma^n \colon |t - t_0| \leqslant A, |X_t - X_0| \leqslant B\} \subset D.$$

Let

$$M = \max_{(t,X_t)\in D'} |f(t, X_t) + K_\gamma g(t, X_t)|,$$

and a, b be sufficiently small positive numbers such that $Ma \leqslant B - b$ and if $|\bar{X} - X_0| \leqslant b$, then

$$K_{\bar{X}} = \{(t, X_t) \colon |t - t_0| \leqslant a, |X_t - \bar{X}| \leqslant M|t - t_0|\} \subset D^{'}.$$

Let Y shows the space of continuous mappings $m : G \to \mathbb{R}^n$ where

$$G = \{(t, X_t) \colon |t - t_0| \leqslant a, |X_t - X_0| \leqslant b\},$$

and consider E as follows:

$$E = \{m \in Y \colon |m(t, X_t)| \leqslant M|t - t_0|\}.$$

One can easily show that d, defined by

$$d(m_1, m_2) = \max_{(t,X_t)\in G} |(m_1(t, X_t) - m_2(t, X_t)|, \quad m_1, m_2 \in E,$$

is a metric on E and (E, d) is a complete metric space.

Define mapping $S : E \to Y$ as,

$$Sm(t, X_t) = \int_{t_0}^{t} f(s, X_s + m(s, X_s))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s + m(s, X_s))\mathrm{d}C_s.$$

Now, we state the following lemma.

Lemma 2.2. *If a is sufficiently small, then $S : E \to E$ is a contraction mapping.*

Proof. Let $(s, X_s) \in G$. Then,

$$|X_s - X_0| \leqslant b, \quad |s - t_0| \leqslant a.$$

From

$$\begin{aligned} |X_s + m(s, X_s) - X_{t_0}| &\leqslant |X_s - X_{t_0}| + |m(s, X_s)| \\ &\leqslant b + Ma \leqslant B, \end{aligned}$$

it can be concluded that $(s, X_s + m(s, X_s))$ belongs to the domain of f; and therefore, S is well-defined. From the continuity of f, g and m, it is easy to see that, Sm is continuous and

$$|Sm(t, X_t)| \leqslant M|t - t_0|.$$

Hence, S maps E to E.

On the other hand, since f and g satisfy LC with Lipschitz constant L, then

$$\begin{aligned} |Sm_1(t, X_t) - Sm_2(t, X_t)| = &\left| \int_{t_0}^{t} f(s, X_s + m_1(s + X_s))\mathrm{d}s \right. \\ &+ \int_{t_0}^{t} g(s, X_s + m_1(s + X_s))\mathrm{d}C_s \\ &- \int_{t_0}^{t} f(s, X_s + m_2(s + X_s))\mathrm{d}s \\ &\left. - \int_{t_0}^{t} g(s, X_s + m_2(s + X_s))\mathrm{d}C_s \right| \\ &\leqslant L \int_{t_0}^{t} |m_1(s, X_s) - m_2(s, X_s)\mathrm{d}s| \\ &+ K_\gamma L \int_{t_0}^{t} |m_1(s, X_s) - m_2(s, X_s)\mathrm{d}s| \\ &\leqslant L(1 + K_\gamma) d(m_1, m_2)|t - t_0| \\ &\leqslant L(1 + K_\gamma) a d(m_1, m_2). \end{aligned}$$

If $L(1+K_\gamma)a < 1$, then S is a contraction mapping on E, where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. □

In the following, we prove a theorem that includes the EaU of the solutions of the UIVP and the continuity of the solutions with respect to the IC.

Theorem 2.3. *Let D be an open set in (t, X_t)-space, $(t_0, X_0) \in D$, f and g be continuous functions on D and satisfy LC with respect to X_t on D. There exist a, b such that if $|\bar{X} - X_0| \leqslant b$, then the initial value problem*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \quad X_{t_0} = \bar{X}, \tag{2}$$

has a unique solution $X_t(t_0,\bar{X})$ *on* $[t_0-a,t_0+a]$. *Moreover, this solution is a continuous function of* $(t,\bar{X})$ *on* $G^{'}=\{(t,\bar{X})\colon |t-t_0|<a, |\bar{X}-X_0|<b\}$.

Proof. Let a, b, G, E and S be as before. According to Lemma 2.2, $S\colon E\to E$ is a contraction mapping and E is complete metric space. Therefore, according to Theorem 2.1, there exists a unique fixed point of S in E called $m_0(t,X_t)$. Thus,

$$m_0(t,X_t)=\int_{t_0}^{t} f(s,X_s+m_0(s,X_s))\mathrm{d}s+g(s,X_s+m_0(s,X_s))\mathrm{d}C_s. \tag{3}$$

Now, replacing X_t with $\bar{X}$ in (3), for $|\bar{X}-X_0|\leqslant b$ and $|t-t_0|\leqslant a$, it can be concluded that

$$\bar{X}+m_0(t,\bar{X})=\bar{X}+\int_{t_0}^{t} f(s,\bar{X}+m_0(s,\bar{X}))\mathrm{d}s+\int_{t_0}^{t} g(s,\bar{X}+m_0(s,\bar{X}))\mathrm{d}C_s.$$

Now let $X_t=\bar{X}+m_0(t,\bar{X})$, then X_t is a solution of the equation

$$X_t=\bar{X}+\int_{t_0}^{t} f(s,X_s)\mathrm{d}s+\int_{t_0}^{t} g(s,X_s)\mathrm{d}C_s.$$

Therefore, according to Remark 1.2, X_t is a solution of (2). Since $m_0(t,\bar{X})$ is continuous with respect to $(t,\bar{X})$ and it is a unique fixed point of S in E, X_t is also continuous with respect to $(t,\bar{X})$ on $G^{'}$.

If Y_t be another solution of (2), then $m(t,\bar{X})=Y_t-\bar{X}$ is another fixed point of S which satisfies condition $|Y_t-\bar{X}|\leqslant M|t-t_0|$. Therefore, $Y_t-\bar{X}$ is a fixed point of S in E. Since S has a unique fixed point on E, we have $Y_t=X_t$. □

3. Differentiability

In this section another property of the solutions will be studied. We consider basic characteristic differentiability of the solutions with respect to ICs in the UDE. It is important sometimes to use the fact that if f has continuous second derivatives, then the solution is a differentiable function of X_{t_0}.

Theorem 3.1. *Let* D, a, b, f, g, (t_0,X_0), $\bar{X}$, G', $X_t(t_0,\bar{X})$ *be those which are defined in Theorem* 2.2. *Moreover, suppose that* f *and* g *have continuous second partial derivatives with respect to all variables on* D. *Then,* $X_t(t_0,\bar{X})$ *has continuous first partial derivatives with respect to each component of* $\bar{X}$ *and has continuous third partial derivative with respect to* t.

Proof. Since $X_t(t_0,\bar{X})$ is a solution of (2), it is continuous on $G^{'}$ and

$$\frac{\partial}{\partial t}X_t(t_0,\bar{X})=f(t,\bar{X})+g(t,\bar{X})\frac{\partial C_t}{\partial t}.$$

Also, f and g are continuous functions of $(t, \bar{X})$, thus $\frac{\partial X_t}{\partial t}$ is a continuous function of $(t, \bar{X})$ and

$$\frac{\partial^2 x_i}{\partial t^2} = \frac{\partial f_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\frac{\partial x_j}{\partial t} + \frac{\partial g_i}{\partial t}\frac{\partial C_t}{\partial t} + \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{\partial C_t}{\partial t} + g(t, \bar{X})\frac{\partial^2 C_t}{\partial t^2}, \quad 1 \leqslant i \leqslant n, \tag{4}$$

where x_i is the i^{th} component of X_t. Therefore, $\frac{\partial^2 X_t}{\partial t^2}$ exists and is a continuous function of $(t, \bar{X})$. Differentiating from both sides of (4) again, it can be concluded that $\frac{\partial^3 X_t}{\partial t^3}$ also exists and it is continuous.

By Remark 2, if $X_t(t_0, \bar{X})$ is a solution of (2) such that $X_0 = \bar{X}$, then

$$X_t(t_0, \bar{X}) = \bar{X} + \int_{t_0}^{t} f(s, X_s(t_0, \bar{X}))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s(t_0, \bar{X}))\mathrm{d}C_s. \tag{5}$$

Now, if such a solution $X_t(t_0, \bar{X})$ exists and is differentiable with respect to the components $\bar{x}_1$, $\bar{x}_2$,..., $\bar{x}_n$ of $\bar{X}$, then differentiating both sides of (5) with respect to $\bar{x}_i$, we obtain

$$\begin{aligned}\frac{\partial X_t}{\partial \bar{x}_i} = I_i &+ \int_{t_0}^{t} \left[\frac{\partial f}{\partial X_s}(s, X_s(t_0, \bar{X}))\right]\left[\frac{\partial X_s}{\partial \bar{x}_i}\right]\mathrm{d}s \\ &+ \int_{t_0}^{t} \left[\frac{\partial g}{\partial X_s}(s, X_s(t_0, \bar{X}))\right]\left[\frac{\partial X_s}{\partial \bar{x}_i}\right]\mathrm{d}C_s.\end{aligned} \tag{6}$$

Hear $I_i = (0, 0, \ldots, 1, 0, \ldots, 0)$, I_i has a 1 in the i^{th} position and 0 elsewhere, $\left[\frac{\partial X_t}{\partial \bar{x}_i}\right] = \left(\frac{\partial x_{1t}}{\partial \bar{x}_i}, \ldots, \frac{\partial x_{nt}}{\partial \bar{x}_i}\right)^T$ and $\left[\frac{\partial f}{\partial X_s}\left(s, X_s(t_0, \bar{X})\right)\right]$ and $\left[\frac{\partial g}{\partial X_s}\left(s, X_s(t_0, \bar{X})\right)\right]$ are $n \times n$ matrices whose element in the (i, j) position is $\frac{\partial f_i}{\partial x_j}\left(s, X_s(t_0, \bar{X})\right)$ and $\frac{\partial g_i}{\partial x_j}\left(s, X_s(t_0, \bar{X})\right)$ respectively; where x_j is the j^{th} component of X_t and f_i and g_i are the i^{th} components of f and g respectively. This suggests that in order to show the differentiability of $X_t(t_0, \bar{X})$ with respect to $\bar{x}_i$ and the continuity of that, it is sufficient to show that the system

$$\begin{aligned} X_t &= \bar{X} + \int_{t_0}^{t} f(s, X_s)\mathrm{d}s + \int_{t_0}^{t} g(s, X_s)\mathrm{d}C_s, \\ Y_t^i &= I_i + \int_{t_0}^{t} \frac{\partial f}{\partial X_s}(s, X_s)Y_s^i\mathrm{d}s + \int_{t_0}^{t} \left[\frac{\partial g}{\partial X_s}(s, X_s)Y_s^i\right]\mathrm{d}C_s, \quad (i = 1, \ldots, n). \end{aligned} \tag{7}$$

Since f and g have continuous second partial derivatives with respect to X_t, it follows that $\frac{\partial f_i}{\partial x_j}$ and $\frac{\partial g_i}{\partial x_j}$ satisfy LC with respect to X_t and the result (X_t, Y_t) on G'. Solving (7) for X_t and $Y_t^1, \ldots, Y_t^n$ is equivalent to solving (5) and (6) for $X_t(t_0, \bar{X})$ and $\frac{\partial X_t}{\partial \bar{x}_i}$, $i = 1, \ldots, n$. Therefore, by the EaU theorem ([3]), system

(7) has a unique solution. Let $\big(X_t(t_0,\bar{X}),Y_t(t_0,\bar{X})\big)$ denotes the solution. Then, $\big(X_t(t_0,\bar{X}),Y_t^1(t_0,\bar{X}),\ldots,Y_t^n(t_0,\bar{X})\big)$ is the limit of

$$\left\{\big(X_t^m(t_0,\bar{X}),(Y_t^1)^m(t_0,\bar{X}),\ldots,(Y_t^n)^m(t_0,\bar{X})\big)\right\}_{m=1}^{\infty},$$

on $G=\{(t,X_t)\colon |t-t_0|\leqslant a, |X_t-\bar{X}|\leqslant b\}$ where

$$\big(X_t(t_0,\bar{X}),Y_t^1(t_0,\bar{X}),\ldots,Y_t^n(t_0,\bar{X})\big)=(\bar{X},1,\ldots,1),$$

and

$$X_t^{m+1}(t_0,\bar{X})=\bar{X}+\int_{t_0}^{t} f\left(s,X_s^m(t_0,\bar{X})\right)\mathrm{d}s+\int_{t_0}^{t} g\left(s,X_s^m(t_0,\bar{X})\right)\mathrm{d}C_s, \qquad (8)$$

and

$$\begin{aligned}(Y_t^i)^{m+1}(t_0,\bar{X})=I_i&+\int_{t_0}^{t}\frac{\partial f}{\partial X_s}(s,X_s^m(t_0,\bar{X}))(Y_s^i)^m(t_0,\bar{X})\mathrm{d}s\\&+\int_{t_0}^{t}\frac{\partial g}{\partial X_s}(s,X_s^m(t_0,\bar{X}))(Y_s^i)^m(t_0,\bar{X})\mathrm{d}C_s.\end{aligned}$$

Now, it is sufficient to prove $(Y_t^i)^m=\frac{\partial X_t^m}{\partial\bar{x}_i}$. Let for some fixed m and for all $(t,X_t)\in IntG$, it is true that $\frac{\partial X_t^m}{\partial\bar{x}_i}=(Y_t^i)^m$, for $i=1,\ldots,n$. Now differentiating both sides of (8) with respect to $\bar{x}_i$, we get

$$\begin{aligned}\frac{\partial X_t^{m+1}}{\partial\bar{x}_i}&=I_i+\int_{t_0}^{t}\frac{\partial f}{\partial X_s}\left(s,{X_s}^m(t_0,\bar{X})\right)\left(\frac{\partial {X_s}^m}{\partial\bar{x}_i}(t_0,\bar{X})\right)\mathrm{d}s\\&+\int_{t_0}^{t}\frac{\partial g}{\partial X_s}\left(s,{X_s}^m(t_0,\bar{X})\right)\left(\frac{\partial {X_s}^m}{\partial\bar{x}_i}(t_0,\bar{X})\right)\mathrm{d}C_s\\&=I_i+\int_{t_0}^{t}\frac{\partial f}{\partial X_s}\left(s,{X_s}^m(t_0,\bar{X})\right)(Y_s^i)^m(t_0,\bar{X})\mathrm{d}s\\&+\int_{t_0}^{t}\frac{\partial g}{\partial X_s}\left(s,{X_s}^m(t_0,\bar{X})\right)(Y_t^i)^m(t_0,\bar{X})\mathrm{d}C_s\\&=(Y_t^i)^{m+1}(t_0,\bar{X}).\end{aligned}$$

Thus, for all m and for all $(t,X_t)\in IntG$ it is true that $(Y_t^i)^m=\frac{\partial X_t^m}{\partial\bar{x}_i}$. But the sequence $\left\{(Y_t^i)^m(t_0,\bar{X})\right\}_{m=1}^{\infty}$ converges uniformly on G to $(Y_t^i)(t_0,\bar{X})$. Thus, $\frac{\partial X_t(t_0,\bar{X})}{\partial\bar{x}_i}=(Y_t^i)(t_0,\bar{X})$. Also, since $(Y_t^i)(t_0,\bar{X})$ is continuous with respect to $(t,t_0,\bar{X})$, then $\frac{\partial X_t}{\partial\bar{x}_i}$ is also continuous. $\square$

4. Existence without Uniqueness

In this section, we prove the existence of solutions for an uncertain autonomous differential equation if f and g be continuous, and do not satisfy LC. Proving such an existence result helps us to find as clear a picture as possible of what conditions are needed to insure existence. Also, the used technique in the proof is the basis for numerical methods.

Theorem 4.1. (*Existence without uniqueness*) *Let $\bar{G}$ be the closure of a bounded open set G in $\mathbb{R}^n$ and $f = (f_1, f_2, \dots, f_n)^T : \bar{G} \longrightarrow \mathbb{R}^n$ and $g = (g_1, g_2, \dots, g_n)^T : \bar{G} \longrightarrow \mathbb{R}^n$ be continuous on $\bar{G}$. Let t_0 be fixed and $X_{t_0} = (X_{1t_0}, \dots, X_{nt_0})^T$ be a fixed point in G. If $M = \sup\{|f_i(X_t)| : X_t \in \bar{G}, 1 \leqslant i \leqslant n\}$, $N = \sup\{|g_i(X_t)| : X_t \in \bar{G}, 1 \leqslant i \leqslant n\}$, and $d = \inf \|X_0 - Q\|$ for $Q \in \bar{G} \setminus G$, then autonomous UIVP*

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}C_t, \quad X_{t_0} = X_0, \tag{9}$$

has a solution that its domain includes the interval $[t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}, t_0 + \frac{d}{\sqrt{n}(M+K_\gamma N)}]$, where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$.

Proof. Let $\epsilon > 0$. Since f and g are continuous on $\bar{G}$ and $\bar{G}$ is compact, there exists $\delta > 0$ such that for $X_t, Y_t \in \bar{G}$, if $|X_t - Y_t| = \sum_{i=1}^{n} |x_{it} - y_{it}| < \delta$, then

$$|f(X_t) - f(Y_t)| < \frac{\epsilon}{2}, \quad |g(X_t) - g(Y_t)| < \frac{\epsilon}{2}.$$

Now, we defined a sequence $X_{1t}, X_{2t}, \dots$ of uncertain process such that attains the solution of (9). If $f(X_0) = g(X_0) = 0$, then $X_t = X_0$ with domain $\mathbb{R}$ is a solution of (9). If $f(X_0) \neq 0$ or $g(X_0) \neq 0$, then let $t_1 = t_0 + \frac{\delta}{2n(M+K_\gamma N+1)}$ and

$$X_{1t} = X_0 + f(X_0)(t - t_0) + g(X_0)(C_t - C_{t_0}), \quad t_0 \leqslant t \leqslant t_1,$$

and $X_1 = X_{1t_1}$. If $X_1 \in G$, $f(X_1) = g(X_1) = 0$, then let $t_1 \leqslant t < \infty$ and $X_{2t} = X_1$, and if $X_1 \in G$ and $f(X_1) \neq 0$ or $g(X_1) \neq 0$, then let $t_2 = t_1 + \frac{\delta}{2(M+K_\gamma N+1)n}$ and

$$X_{2t} = X_1 + f(X_1)(t - t_1) + g(X_1)(C_t - C_{t_1}), \quad t_1 \leqslant t \leqslant t_2,$$

and $X_{2t_2} = X_2$. Suppose that by induction, $t_1, t_2, \dots, t_k$ and $X_{1t}, \dots X_{kt}$ and also $X_1, \dots, X_k$ are defined. If $X_k \in G$ and $f(X_k) = g(X_k) = 0$, then let $t_k \leqslant t < \infty$ and $X_{(k+1)t} = X_k$. If $X_k \in G$ and $f(X_k) \neq 0$ or $g(X_k) \neq 0$, then let $t_{k+1} = t_k + \frac{\delta}{2(M+K_\gamma N+1)n}$ and

$$X_{(k+1)t} = X_k + f(X_k)(t - t_k) + g(X_k)(C_t - C_{t_k}), \quad t_k \leqslant t \leqslant t_{k+1}. \tag{10}$$

Now, let $J = \bigcup_{k \geqslant 0} [t_k, t_{k+1}]$ and define $X_{\epsilon t}$ as follows

$$X_{\epsilon t} = X_{kt}, \quad t \in [t_{k-1}, t_k], \quad k \geqslant 1.$$

First, we show that the domain of $X_{\epsilon t}$ contains the interval $[t_0, t_0 + \frac{d}{(M+K_\gamma N)\sqrt{n}}]$. If $X_{\epsilon t} = X_{kt}$ for $k = 1$ to m, then the domain of $X_{\epsilon t}$ is $[t_0, t_0 + \frac{m\delta}{2(M+K_\gamma N+1)n}]$. Now, let $X_m \notin G$, then we must have

$$\begin{aligned}
d &\leqslant \| X_m - X_0 \| \leqslant \sum_{k=1}^{m} \| X_k - X_{k-1} \| \\
&= \sum_{k=1}^{m} \| f(X_{k-1})(t_k - t_{k-1}) + g(X_{k-1})(C_{t_k} - C_{t_{k-1}}) \| \\
&\leqslant \sum_{k=1}^{m} \| f(X_{k-1})(t_k - t_{k-1}) \| + \sum_{k=1}^{m} \| g(X_{(k-1)})(C_{t_k} - C_{t_{k-1})} \| \\
&\leqslant \sum_{k=1}^{m} \sqrt{n} M(t_k - t_{k-1}) + \sum_{k=1}^{m} \sqrt{n} N |C_{t_k} - C_{t_{k-1}}|.
\end{aligned}$$

Therefore, a necessary condition for $X_m \notin G$ is that m be large enough so that

$$\begin{aligned}
d &\leqslant \sqrt{n} M \frac{m\delta}{2(M + K_\gamma N + 1)n} + \sqrt{n} N \frac{K_\gamma m\delta}{2(M + K_\gamma N + 1)n} \\
&= \frac{m\delta}{2(M + K_\gamma N + 1)n} (\sqrt{n} M + K_\gamma \sqrt{n} N),
\end{aligned}$$

or

$$\frac{d}{\sqrt{n}(M + K_\gamma N)} \leqslant \frac{m\delta}{2(M + K_\gamma N + 1)n}.$$

Thus, $[t_0, t_0 + \frac{d}{(M+K_\gamma N)\sqrt{n}}] \subseteq [t_0, t_0 + \frac{m\delta}{2(M+K_\gamma N+1)n}]$. Similarly, we can define the uncertain variable $X_{\epsilon t}$ on the interval $[t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}]$. That way, we will obtain continuous uncertain variable $X_{\epsilon t}$ with domain $[t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}, t_0 + \frac{d}{\sqrt{n}(M+K_\gamma N)}]$ which is continuous and piecewise smooth. Now, we want to extract a sequence of $X_{\epsilon t}$ such that converges to the solution of (9).

Claim: For $t \in I = [t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}, t_0 + \frac{d}{\sqrt{n}(M+K_\gamma N)}]$, we have

$$X_{\epsilon t} = X_0 + \int_{t_0}^{t} f(X_{\epsilon s}) \mathrm{d}s + \int_{t_0}^{t} g(X_{\epsilon s}) \mathrm{d}C_s + \int_{t_0}^{t} h_\epsilon(s) \mathrm{d}s + \int_{t_0}^{t} p_\epsilon(s) \mathrm{d}C_s,$$

where $h_\epsilon(s) = f(X_j) - f(X_{(j+1)s})$ and $p_\epsilon(s) = g(X_j) - g(X_{(j+1)s})$ for $s \in [t_j, t_{j+1}]$. Also, for $s \in I$ we have $|q_\epsilon(s)| = |h_\epsilon(s) + p_\epsilon(s)| < \epsilon$.

Proof of the claim: Let $t \in [t_0, t_0 + \frac{d}{\sqrt{n}(M+K_\gamma N)}]$. From (10) it can be concluded that

$$X_{j+1} - X_j = \int_{t_j}^{t_{j+1}} f(X_j) \mathrm{d}s + \int_{t_j}^{t_{j+1}} g(X_j) \mathrm{d}C_s, \quad j = 0, 1, \ldots, k - 1.$$

Similarly, for $t_k \leqslant t \leqslant t_{k+1}$,

$$X_{\epsilon t} - X_k = \int_{t_k}^{t} f(X_k)\mathrm{d}s + \int_{t_k}^{t} g(X_k)\mathrm{d}C_s.$$

Thus,

$$\begin{aligned}
X_{\epsilon t} - X_0 &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(X_j)\mathrm{d}s + \int_{t_k}^{t} f(X_k)\mathrm{d}s \\
&+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} g(X_j)\mathrm{d}C_s + \int_{t_k}^{t} g(X_k)\mathrm{d}C_s \\
&= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(X_{js})\mathrm{d}s + \int_{t_k}^{t} f(X_{ks})\mathrm{d}s \\
&+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} g(X_{js})\mathrm{d}C_s + \int_{t_k}^{t} g(X_{ks})\mathrm{d}C_s \\
&+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} [-f(X_{js}) + f(X_j)]\mathrm{d}s + \int_{t_k}^{t} [-f(X_{ks}) + f(X_k)]\mathrm{d}s \\
&+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} [-g(X_{js}) + g(X_j)]\mathrm{d}C_s + \int_{t_k}^{t} [-g(X_{ks}) + g(X_k)]\mathrm{d}C_s \\
&= \int_{t_0}^{t} f(X_{\epsilon s})\mathrm{d}s + \int_{t_0}^{t} g(X_{\epsilon s})\mathrm{d}C_s + \int_{t_0}^{t} h_\epsilon(s)\mathrm{d}s + \int_{t_0}^{t} p_\epsilon(s)\mathrm{d}c_s,
\end{aligned}$$

where for $s \in [t_j, t_{j+1}]$ and $j = 0, 1, \ldots, k$, we have

$$\begin{aligned}
|X_{js} - X_j| &= |f(X_j)(t - t_j) + g(X_j)(C_t - C_{t_j})| \\
&\leqslant |f(X_j)|(t_{j+1} - t_j) + |g(X_j)|(C_{t_{j+1}} - C_{t_j}) \\
&\leqslant \frac{\sqrt{n}M\delta}{2(M + K_\gamma N + 1)n} + \frac{\sqrt{n}K_\gamma N\delta}{2(M + K_\gamma N + 1)n} \\
&= \frac{\delta}{2}\left(\frac{\sqrt{n}M}{(M + K_\gamma N + 1)n} + \frac{\sqrt{n}K_\gamma N}{(M + K_\gamma N + 1)n}\right) \leqslant \delta.
\end{aligned}$$

Therefore,

$$|q_\epsilon(s)| = |f(X_j) - f(X_{js}) + g(X_j) - g(X_{js})| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Proof for $t \in [t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}, t_0]$ can be done in the same manner. If $t \in I$, then for $k \geqslant 1$ we have $t_0 \leqslant t \leqslant t_k$ or there exists a k such that $t \geqslant t_k$. In the latter case for $s > t_k$, we have $q_\epsilon(s) = 0$. Thus, in every case, the above clime is true.

Now, let $m = 1, 2, \dots$ and $\epsilon = \frac{1}{m}$. We display $X_{\epsilon t}$ match it with X_t^m. That way, we will get sequence $(X_t^m)_{m=1}^{\infty}$ of uncertain variables with domain $[t_0 - \frac{d}{\sqrt{n}(M+K_\gamma N)}, t_0 + \frac{d}{\sqrt{n}(M+K_\gamma N)}]$ that is uniformly bounded and equicontinuous. We have from (9),

$$|X_t^m| \leqslant |X_0| + M|t - t_0| + N|C_t - C_{t_0}| + \frac{1}{m}|t - t_0| + \frac{1}{m}|C_t - C_{t_0}|,$$

$$|X_{t'}^m - X_{t''}^m| \leqslant M|t' - t''| + N|C_{t'} - C_{t''}| + \frac{1}{m}|t' - t''| + \frac{1}{m}|C_{t'} - C_{t''}|,$$

Thus, according to Arzela-Ascoli Theorem ([11]), there exists a subsequence $(X_t^{m_k})_{k=1}^{\infty}$ of $(X_t^m)_{m=1}^{\infty}$ which is uniformly convergence on I. Let $X_t = \lim\limits_{m_k \to \infty} X_t^{m_k}$ for $t \in I$ Thus,

$$X_t^{m_k} = X_0 + \int_{t_0}^{t} f(X_s^{m_k})\mathrm{d}s + \int_{t_0}^{t} g(X_s^{m_k})\mathrm{d}C_s + \int_{t_0}^{t} h_{m_k}(s)\mathrm{d}s + \int_{t_0}^{t} p_{m_k}(s)\mathrm{d}C_s,$$

and

$$\lim_{m_k \to \infty} \int_{t_0}^{t} h_{m_k}(s)\mathrm{d}s = \lim_{m_k \to \infty} \int_{t_0}^{t} p_{m_k}(s)\mathrm{d}C_s = 0.$$

Functions f and g are uniformly continuous on $\bar{G}$ and the sequence $(X_t^{m_k})_{m=1}^{\infty}$ is uniformly convergence on I. Thus, $\{f(X_s^{m_k})\}_{k=1}^{\infty}$ and $\{g(X_s^{m_k})\}_{k=1}^{\infty}$ are uniformly convergence. Therefore, if $m_k \to \infty$, then

$$X_t = X_0 + \int_{t_0}^{t} f(X_s)\mathrm{d}s + \int_{t_0}^{t} g(X_s)\mathrm{d}C_s, \quad t \in I.$$

Or in other word

$$\frac{\partial X_t}{\partial t} = f(X_t) + g(X_t)\frac{\partial C_t}{\partial t}, \quad X_{t_0} = X_0, \quad t \in I.$$

□

Remark 3. Theorem 4.2 is known as Peano Theorem in ordinary differential equations.

5. Conclusion

In this work the continuity and differentiability of solutions on ICs of the UIVP have been considered. Also, the EaU of solutions of an UIVP have been proved under the weaker conditions. Moreover, a generalization of Peano Theorem to autonomous UIVP has been presented.

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