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On a Maximal Subgroup $2^6 : (3^{\cdot}S_6)$ of M_{24}

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Abstract

The Mathieu group M_{24} has a maximal subgroup of the form $\bar{G} = N{:}G$, where $N = 2^6$ and $G = 3^{\cdot}S_6 \cong 3.PGL_2(9)$. Using Atlas, we can see that M_{24} has only one maximal subgroup of type $2^6{:}(3^{\cdot}S_6)$. The group is a split extension of an elementary abelian group, $N = 2^6$ by a non-split extension group, $G = 3^{\cdot}S_6$. The Fischer matrices for each class representative of G are computed which together with character tables of inertia factor groups of G lead to the full character table of $\bar{G}$. The complete fusion of $\bar{G}$ into the parent group M_{24} has been determined using the technique of set intersections of characters.

Keywords: Mathieu group, conjugacy classes, irreducible characters, Fischer matrices, fusions.

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1. Introduction

The Mathieu group M_{24} is a sporadic simple group of order 244823040 and its character table is given in Atlas [8, p. 56]. M_{24} is the largest of the five Mathieu groups and has nine conjugacy classes of maximal subgroups. In this paper, we will determine the conjugacy classes, inertia factor groups, calculate the Fischer matrices and hence the ordinary character table of $2^6{:}(3^{\cdot}S_6)$ using *coset analysis*

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technique and the *theory of Fischer matrices*. These methods have been used extensively in literature (see for instance: [2, 3, 4, 6, 7, 15, 16]).

The fusion of $\bar{G}$ into M_{24} has been completely determined using the technique of set intersections. This technique has also been used by Ali [1], Chileshe [7], Moori [13] and Mpono [14]. Using the two 11×11 matrices over $\mathbb{F}_2$ that generate M_{24}, supplied by [18], we were able to construct $\bar{G}$ inside GAP [10]. The two $11-$dimensional matrices, m_1 and m_2 over $\mathbb{F}_2$ that generate $\bar{G}$ are given in Table 1 with $o(m_1) = o(m_2) = 4$ and $o(m_1m_2) = 6$.

Since $\bar{G}$ can be constructed in GAP, it is easy to obtain all its normal subgroups. The group $2^6{:}(3^{\cdot}S_6)$ has 3 non-trivial normal subgroups but only one of these groups is of order 64 and is an elementary abelian 2-group. This group is the $N = 2^6$ generated by six 11×11 matrices γ_1, γ_2, γ_3, γ_4, γ_5 and γ_6 each of order 2. Up to isomorphism, $2^6{:}(3^{\cdot}S_6)$ has only one subgroup of order 2160 which is a non-split extension of the form $G = 3^{\cdot}S_6$ complement to $N = 2^6$. We note that $G = 3^{\cdot}S_6 \cong 3.PGL_2(9)$ and hence the two groups have the same character table. The character table of $3.PGL_2(9)$ has been determined by Whitley [17]. The group G is generated by three matrices a, b and c given in Table 2 with $o(a) = 2 = o(ab)$, $o(b) = 6 = o(ac)$, $o(c) = 3$, $o(bc) = 5$ and $o(abc) = 4$.

2. Conjugacy Classes of $\bar{G} = 2^6{:}(3^{\cdot}S_6)$

In this section, we calculate the conjugacy classes of $\bar{G}$ using the coset analysis technique (for more details, see [1, 7, 13, 14, 16, 17]). Corresponding to 16 conjugacy classes of $3^{\cdot}S_6$, we get 33 conjugacy classes of $\bar{G}$. We also use Programmes **A** and **B** for the f's and power maps of elements of $\bar{G}$ (see Chileshe [7]). We list the conjugacy classes of $\bar{G}$ in Table 3, where in this table:

- k is the number of fixed points of g on N which is equal to the number of orbits under the action of N on Ng. The action of N on the identity coset N produces 64 orbits and therefore, $k = 64$.
- f_j is the number of orbits that come together under the action of $C_G(g)$ on the orbits, $Q_1, Q_2, \ldots, Q_k$. In this case, the action of $C_G(1A) = G$ on $Q_1, Q_2, \ldots, Q_k$ produces 3 orbits of lengths 1, 18 and 45. Thus $f_1 = 1$, $f_2 = 18$ and $f_3 = 45$.
- the weights, (w_j's) for the row orthogonalities of the Fischer matrices are computed using the formula, $w_j = \dfrac{f_j \times |N|}{k}$.

3. Fischer Matrices and Character Table of $\bar{G}$

The theory behind Fischer matrices has been explained in several research papers. If readers require a review on the method of Fischer matrices, they are encouraged to consult the following papers [1, 9, 14, 17].

For any group extension $\bar{G} = N.G$, the Fischer matrix on a class representative g of G is denoted by$M(g)$. Further more, $a_j^{(i,y)}$ denote the (i, j)th entry of the Fischer matrix where y ranges over class representatives of the inertia factor groups say H that fuse to g in G. Next, we state an important theorem that explains the computation of the Fischer matrix $M(1_G)$.

Theorem 3.1. [14] *Let $\bar{G} = N.G$. Then the Fischer matrix $M(1_G)$ is the matrix with rows equal to orbit sums of the action of $\bar{G}$ on the irreducible characters of N with duplicate columns discarded. Further more, for this matrix, we have that $a_j^{(i,1_G)} = [G{:}H_i]$ satisfies the row orthogonality relation*

$$\sum_{j=1}^{t} a_j^{(i,1_G)} a_j^{(i',1_G)} |C_{\bar{G}}(x_j)|^{-1} = \delta_{ii'} |C_{H_i}(1_G)|^{-1} = \delta_{ii'} |H_i|^{-1},$$

where the H_i's are the inertia factor groups of G.

Results of Theorem 3.1 led to the construction of **Programme C** (given in Section **5**) to compute the Fischer matrix on the identity class of G. Note that this programme can work for both split and non-split extensions since the orbit sums from which the Fischer matrix is formed, are as a result of the action of the whole $\bar{G}$ on the irreducible characters of N.

We now discuss how to obtain Fischer matrices on non-identity classes of G. First we state and prove the following Corollary.

Corollary 3.2. *Let $\bar{G}$ be a group extension of N by G such that N is an elementary abelian p-group and $\bar{g} \in \bar{G}$. Then the map $\vartheta_{\bar{g}} : N \longrightarrow N$ given by $\vartheta_{\bar{g}}(n) = n\bar{g}n^{-1}\bar{g}^{-1}$ is an endomorphism of N.*

Proof. We have to show that $\vartheta_{\bar{g}}$ is a well defined homomorphism. For $n_1, n_2 \in N$, there is a $\bar{g} \in \bar{G}$ such that

$$n_1 = n_2 \Rightarrow n_1\bar{g}n_1^{-1}\bar{g}^{-1} = n_2\bar{g}n_2^{-1}\bar{g}^{-1} \Rightarrow \vartheta_{\bar{g}}(n_1) = \vartheta_{\bar{g}}(n_2),$$

and thus, $\vartheta_{\bar{g}}$ is well defined. Furthermore,

$$\begin{aligned}
\vartheta_{\bar{g}}(n_1n_2) &= (n_1n_2)\bar{g}(n_1n_2)^{-1}\bar{g}^{-1} \\
&= n_1n_2\bar{g}n_1^{-1}n_2^{-1}\bar{g}^{-1} \\
&= n_1n_2\bar{g}n_1^{-1}\bar{g}^{-1}\bar{g}n_2^{-1}\bar{g}^{-1} \\
&= n_1(\bar{g}n_1^{-1}\bar{g}^{-1})n_2\bar{g}n_2^{-1}\bar{g}^{-1} \\
&= (n_1\bar{g}n_1^{-1}\bar{g}^{-1})(n_2\bar{g}n_2^{-1}\bar{g}^{-1}) \\
&= \vartheta_{\bar{g}}(n_1)\vartheta_{\bar{g}}(n_2).
\end{aligned}$$

Therefore, $\vartheta_{\bar{g}}$ is a group homomorphism and the proof follows. □

Remark 1. *In view of a vector space N which is invariant under the action of $\bar{G}$, it is shown in [12] that $Im(\vartheta_{\bar{g}})$ and $ker(\vartheta_{\bar{g}})$ are $C_{\bar{g}}$-submodules of N. Recall that $C_{\bar{g}} = N.C_G(g)$. The factor group $N/Im(\vartheta_{\bar{g}})$ is prominent in the determination of the entries of Fischer matrices. Denote $M = Im(\vartheta_{\bar{g}})$. From coset analysis, we know that N acts on $N\bar{g}$ where $\bar{g}$ is a lifting of $g \in G$ under the homomorphism $\lambda : \bar{G} \longrightarrow G$ such that $N \subset Ker(\lambda)$. Suppose the action of N on $N\bar{g}$ results into the set of orbits Ω_i, then the action of M on $N\bar{g}$ by left multiplication results into the same set of orbits Ω_i. In this manner, the action of $C_{\bar{g}}$ on Ω_i is the same as the action of $C_{\bar{g}}$ on the module N/M. Therefore, the elements of N/M can be identified with the orbits of M on $N\bar{g}$.*

For a split extension of N by G, it is shown (see [12]) that the Fischer matrix at a non-identity element of G is the matrix with rows that are orbit sums of C_g acting on the rows of all the irreducible characters of N/M with duplicate columns discarded.

Corollary 3.3. *Let N be an elementary abelian subgroup of $\bar{G}$ such that $\bar{G} = N{:}G$, where G is a complement of N in $\bar{G}$. Let $M = \{ngn^{-1}g^{-1} \mid n \in N\}$. Then $[N{:}M] = k$, where k is the number of elements in a fixed subspace of N by a class representative $g \in G$.*

Proof. From coset analysis, we know that when N acts on Ng, we obtain k orbits denoted by $Q_1,\ Q_2, \ldots, Q_k$. Suppose $B_1,\ B_2, \ldots, B_k$ are orbits for the action of M on Ng by left multiplication. These orbits are the same as those obtained when N acted on Ng. The orbits can be identified with the elements of N/M. Thus it follows that $|N/M| = [N{:}M] = k$. □

If N is an elementary abelian $p-$group such that $|N| = p^n$, then $k = p^q$ for $0 \leq q \leq n$ and k in coset analysis is the number of elements of N fixed by a class representative g of G. Let $Q_1,\ Q_2, \ldots, Q_k$ be the orbits obtained from the action of N on Ng and that the action of C_g on the listed orbits results into $f_i = 1$ for all $i \in \{1,\ 2, \ldots, k\}$. Then we have that the action of C_g on the irreducible characters of N/M is trivial. In this case, the Fischer matrix $M(g)$ coincides with the character table of N/M, a group of order k. Thus for $\bar{G} = N{:}G$, the set stabilizer $C_g = N{:}C_G(g)$ and the action of C_g on the character table of N/M is simply carried out as that of $C_G(g)$.

Further more, the discussion of Remark 1 led to the construction of **programme D** given in Section **5**. This programme determines the Fischer matrices on non-identity classes of G and is applied on a group extension $\bar{G} = N.G$ such that N is an elementary abelian $p-$group.

We now calculate the Fischer matrices $M(g)$ of $\bar{G} = 2^6{:}(3^{\cdot}S_6)$ where g is a representative of a conjugacy class of $G = 3^{\cdot}S_6$. The size of the Fischer matrix $M(g)$ is the number of conjugacy classes of $\bar{G}$ that correspond to $[g]_{3^{\cdot}S_6}$ and thus range from 1 to 4. We also recall that the fusion of the conjugacy classes of the inertia factor groups into $3^{\cdot}S_6$ is critical for the row labels of the Fischer matrices. Since $3^{\cdot}S_6$ has 3 orbits from its action on the conjugacy classes, we have that it has 3 orbits from the action on $\mathrm{Irr}(2^6)$ but the lengths may be different. Thus, to act $3^{\cdot}S_6$ on $\mathrm{Irr}(2^6)$, we need to write the 11×11 matrix generators of G in terms of 6×6 matrices. To do this, we use **Programme A** in [5, p.105]. Since G is generated by a, b and c then the action of a on the 6 generators of N is the matrix a_1 described below

$$a_1 : \gamma_1 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5, \quad \gamma_2 \longrightarrow \gamma_4\gamma_6,$$

$$\gamma_3 \longrightarrow \gamma_2\gamma_4\gamma_5, \quad \gamma_4 \longrightarrow \gamma_1\gamma_3\gamma_4,$$

$$\gamma_5 \longrightarrow \gamma_1\gamma_6, \quad \gamma_6 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4.$$

Similarly, for the action of b we have b_1 described by

$$b_1 : \gamma_1 \longrightarrow \gamma_3\gamma_4\gamma_5\gamma_6, \quad \gamma_2 \longrightarrow \gamma_2\gamma_5,$$

$$\gamma_5 \longrightarrow \gamma_1\gamma_2\gamma_5, \quad \gamma_6 \longrightarrow \gamma_3\gamma_6,$$

$$\gamma_3 \longrightarrow \gamma_6, \quad \gamma_4 \longrightarrow \gamma_3\gamma_5.$$

Also, for the action of c we obtain c_1 described by

$$c_1 : \gamma_5 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4\gamma_6, \quad \gamma_6 \longrightarrow \gamma_1\gamma_2\gamma_5\gamma_6,$$

$$\gamma_3 \longrightarrow \gamma_1\gamma_2\gamma_4\gamma_5\gamma_6, \quad \gamma_4 \longrightarrow \gamma_4\gamma_5\gamma_6,$$

$$\gamma_1 \longrightarrow \gamma_2\gamma_3\gamma_4, \quad \gamma_2 \longrightarrow \gamma_4\gamma_5.$$

Hence $G = \langle a_1, b_1, c_1 \rangle$, where a_1, b_1 and c_1 are the 6×6 matrices given in Table 4 with $o(a_1) = 2 = o(a_1b_1)$, $o(b_1) = 6 = o(a_1c_1)$, $o(c_1) = 3$, $o(b_1c_1) = 5$ and $o(a_1b_1c_1) = 4$.

Therefore, when $3^{\cdot}S_6$ acts on $\mathrm{Irr}(2^6)$ we have that $64 = 1 + p + q$, since the identity character is fixed. Using GAP [10], we obtain $p = 18$ and $q = 45$. Thus the action of $3^{\cdot}S_6$ on $\mathrm{Irr}(2^6)$ produces 3 orbits of lengths 1, 18 and 45 with corresponding point stabilizers $3^{\cdot}S_6$, S_5 and $2 \times S_4$. These are the inertia factor groups.

The fusions of the inertia factor groups $H_2 = S_5$ and $H_3 = 2 \times S_4$ are given in Tables 5 and 6 respectively. Also, the character tables of H_1, H_2 and H_3 computed in GAP [10] are given in Tables 7, 9 and 8 respectively.

It is important to note that Programmes C and D only produce candidates for Fischer matrices. However, the following information supplied in [14] together with the partial character tables of the inertia factor groups, help to correctly label the candidates for Fischer matrices:

- the centralizer orders of the class representatives of $\bar{G}$ as computed from coset analysis corresponding to each class representative g of G,
- if χ is a character of any group H and $h \in H$, then we have $|\chi(h)| \leq \chi(1_H)$, where 1_H is the identity element of H,

- if χ is a character of any group H and h is a $p-$singular element of H, where p is a prime, then we have $\chi(h) \equiv \chi(h^p) \mod p$,

- for any irreducible character χ of a group H and for $h_i \in C_i$ then $d_i = \dfrac{b_i\chi(h_i)}{\chi(1_H)}$ is an algebraic integer, where C_i is the ith conjugacy class of H and $b_i = |C_i| = [H : C_H(h_i)]$. Clearly if $d_i \in \mathbb{Q}$, then $d_i \in \mathbb{Z}$,

- let H be a group such that $\bar{G} \leq H$. We use the fusion of conjugacy classes of $\bar{G}$ into H and restrictions of characters to $\bar{G}$ to correctly label the columns of the Fischer matrices.

Furthermore, for the Fischer matrix on the identity class, the rows could be well labeled by associating to each row the stabilizer of its orbit representative (since each row is an orbit sum, with appropriate duplicates discarded). These stabilizers are subgroups of the group on top in the group extension and therefore are the inertia factor groups. All the Fischer matrices of $\bar{G}$ are listed in Table 10.

Remark 2. *By using the centralizer orders on class representatives of $\bar{G}$ and the power maps of the elements of $\bar{G}$ the columns of the Fischer matrices are identified correctly. For instance, for $M(1A)$ we have that columns* 1, 2 *and* 3 *correspond to* $1A$, $2A_1$ *and* $2A_2$ *respectively. For $M(2A)$, columns* 1, 2, 3 *and* 4 *correspond to* $2A_3$, $2A_4$, $4A_1$ *and* $4A_2$ *respectively. For $M(2B)$, columns* 1, 2 *and* 3 *correspond to* $2A_6$, $4A_5$ *and* $4A_6$ *respectively. For $M(2C)$, columns* 1, 2 *and* 3 *correspond to* $2A_5$, $4A_3$ *and* $4A_4$ *respectively. For $M(4A)$, columns* 1, 2 *and* 3 *correspond to* $4A_7$, $4A_8$ *and* $8A_1$ *respectively. And for the rest of the Fischers matrices, identification of their columns reveals that they coincide with the classes of $\bar{G}$ as obtained from coset analysis.*

We used the Fischer matrices and the character tables of the inertia factor groups H_1, H_2 and H_3 together with their fusions into $3{}^{\cdot}S_6$ given in Tables 5 and 6 to obtain the character table of $\bar{G} = 2^6{:}(3{}^{\cdot}S_6)$. The character table of $\bar{G} = 2^6{:}(3{}^{\cdot}S_6)$ is partitioned into 3 blocks BL_1, BL_2 and BL_3 corresponding to the inertia factor groups H_1, H_2 and H_3 respectively. Clearly,

$$BL_1 = \{\chi_i : 1 \leq i \leq 16\},$$
$$BL_2 = \{\chi_i : 17 \leq i \leq 23\},$$
$$BL_3 = \{\chi_i : 24 \leq i \leq 33\},$$

where $\mathrm{Irr}(\bar{G}) = \bigcup_{i=1}^{3} BL_i$. For instance, on the classes $1A$, $2A_1$ and $2A_2$ of $\bar{G}$ in the first, second and third block we have:

$$\begin{bmatrix}1\\1\\5\\5\\5\\5\\6\\6\\9\\9\\10\\10\\12\\16\\18\\30\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1&1&1\\1&1&1\\5&5&5\\5&5&5\\5&5&5\\5&5&5\\6&6&6\\6&6&6\\9&9&9\\9&9&9\\10&10&10\\10&10&10\\12&12&12\\16&16&16\\18&18&18\\30&30&30\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\4\\4\\5\\5\\6\end{bmatrix}\begin{bmatrix}18 & -6 & 2\end{bmatrix}=\begin{bmatrix}18&-6&2\\18&-6&2\\72&-24&8\\72&-24&8\\90&-30&10\\90&-30&10\\108&-36&12\end{bmatrix} \text{ and}$$

$$\begin{bmatrix}1\\1\\1\\1\\2\\2\\3\\3\\3\\3\end{bmatrix}\begin{bmatrix}45 & 5 & -3\end{bmatrix}=\begin{bmatrix}45&5&-3\\45&5&-3\\45&5&-3\\45&5&-3\\90&10&-6\\90&10&-6\\135&15&-9\\135&15&-9\\135&15&-9\\135&15&-9\end{bmatrix} \text{ respectively.}$$

This way, we are able to fill all the portions of the character table of $\bar{G}$ given as Table 11.

4. Fusion of $\bar{G} = 2^6 : (3^{\cdot}S_6)$ into $M24$

In this section, we determine the fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24}. The permutation character for the action of M_{24} on the cosets of $2^6{:}(3^{\cdot}S_6)$ in M_{24} is given by

$$\chi(M_{24}|\bar{G}) = \chi_1 + \chi_7 + \chi_9 + \chi_{14},$$

where $\chi_i \in \mathrm{Irr}(M_{24})$ (refer to Atlas [8, p.56] or Table $A.2$ in [5, p.111] for character values of M_{24}). We use the permutation character values on the classes of M_{24} and the known power maps for M_{24} and $\bar{G}$ to determine the partial fusions of the conjugacy classes of $\bar{G}$ into M_{24}. To get a complete fusion of $\bar{G}$ into M_{24}, we restrict some irreducible characters of M_{24} to $\bar{G}$. In order to achieve this, we employ the technique of set intersections as follows.

If ζ is a character of $3^{\cdot}S_6$ afforded by the regular representation of $3^{\cdot}S_6$, then

$$\zeta = \sum_{i=1}^{16} d_i\phi_i, \text{ where } \phi_i \in \mathrm{Irr}(3^{\cdot}S_6) \text{ and } d_i \text{ is the degree of } \phi_i.$$

Thus, ζ can be taken as a character of $2^6{:}(3^{\cdot}S_6)$ containing 2^6 in its kernel such that

$$\zeta(g) = \begin{cases} |3^{\cdot}S_6|, & \text{if } g \in 2^6, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for ψ a character of $2^6{:}(3^{\cdot}S_6)$, ρ_1 an identity character of 2^6 and ψ_{2^6} a restriction of ψ to 2^6, we obtain the following:

$$\begin{aligned}
\langle \zeta, \psi \rangle_{2^6:(3^{\cdot}S_6)} &= \frac{1}{|2^6{:}(3^{\cdot}S_6)|}(\zeta(1A)\psi(1A) + 18\zeta(2A_1)\psi(2A_1) + 45\zeta(2A_2)\psi(2A_2)) \\
&= \frac{1}{|2^6{:}(3^{\cdot}S_6)|}(|3^{\cdot}S_6|\psi(1A) + 18|3^{\cdot}S_6|\psi(2A_1) + 45|3^{\cdot}S_6|\psi(2A_2)) \\
&= \frac{|3^{\cdot}S_6|}{|2^6{:}(3^{\cdot}S_6)|}(\psi(1A) + 18\psi(2A_1) + 45\psi(2A_2)) \\
&= \frac{1}{64}(\psi(1A)\rho_1(1A) + 18\psi(2A_1)\rho_1(2A_1) + 45\psi(2A_2)\rho_1(2A_2)) \\
&= \langle \psi_{2^6}, \rho_1 \rangle.
\end{aligned}$$

Similarly, we obtain $\psi_{2^6} = \delta_1\vartheta_1 + \delta_2\vartheta_2 + \delta_3\vartheta_3$, where for $i \in \{1, 2, 3, 4\}$, the δ_i's are non-negative integers and ϑ_i's are the sums of the irreducible characters of 2^6 which are in one orbit under the action of $3^{\cdot}S_6$ on $\mathrm{Irr}(2^6)$. Let $\rho_j \in \mathrm{Irr}(2^6)$, for $j \in \{1, 2, \ldots, 64\}$. Since the groups 2^6 and $3^{\cdot}S_6$ were constructed inside $2^6{:}(3^{\cdot}S_6)$ and the orbits for the action of $3^{\cdot}S_6$ on $\mathrm{Irr}(2^5)$ are of lengths $1, 18$ and 45, we have the following:

$$\vartheta_1 = \sum_{i=1}^{1} \rho_j = \rho_1, \ \deg(\vartheta_1) = 1,$$

$$\vartheta_2 = \sum_{i=2}^{19} \rho_j, \ \deg(\vartheta_2) = 18 \ \text{ and}$$

$$\vartheta_3 = \sum_{i=20}^{64} \rho_j, \ \deg(\vartheta_3) = 45.$$

Therefore, $\psi_{2^6} = \delta_1\rho_1 + \delta_2 \sum_{i=2}^{19} \rho_j + \delta_3 \sum_{i=20}^{64} \rho_j$ and so $\langle \psi_{2^6}, \psi_{2^6} \rangle = {\delta_1}^2 + 18{\delta_2}^2 + 45{\delta_3}^2$. Clearly,

$$\delta_1 = \langle \psi_{2^6}, \rho_1 \rangle = \langle \zeta, \psi \rangle_{2^6:(3^{\cdot}S_6)}.$$

In a similar way, we obtain

$$\langle \psi_{2^6}, \psi_{2^6} \rangle = \frac{1}{64}(\psi(1A)\psi(1A) + 18\psi(2A_1)\psi(2A_1) + 45\psi(2A_2)\psi(2A_2)).$$

We now apply the above results to the restrictions $\psi_1 = \chi_3$ and $\psi_2 = \chi_2$, where χ_2 and χ_3 are irreducible characters of M_{24} of degrees 23 and 45 respectively (see the character table of M_{24} in Atlas [8, p.56] or Table A.2 in [5, p.111]). From the partial fusions of $2^6{:}(3^{\cdot}S_6)$ into M_{24} established already, we note that $1A \mapsto 1A$, $2A_1 \mapsto 2B$ and $2A_2 \mapsto 2A$. Thus, for ψ_1, we obtain

$$\begin{aligned}
\delta_1 = \langle \zeta, \psi_1 \rangle_{2^6:(3^{\cdot}S_6)} &= \frac{1}{64}(\psi_1(1A) + 18\psi_1(2A_1) + 45\psi_1(2A_2)) \\
&= \frac{1}{64}(\chi_3(1A) + 18\chi_3(2B) + 45\chi_3(2A)) \\
&= \frac{1}{64}(45 + 18(5) + 45(-3)) = 0.
\end{aligned}$$

Since $\deg(\psi_1) = 45$, it follows that $\delta_1 + 18\delta_2 + 45\delta_3 = 45$. This way, $\delta_1 = 0$ implies that $\delta_2 = 0$ and $\delta_3 = 1$. Therefore, $(\psi_1)_{2^6:(3^{\cdot}S_6)}$ becomes an irreducible character of degree 45 when restricted to $2^6{:}(3^{\cdot}S_6)$. Thus, based on the partial fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24}, which has already been computed, we obtain

$$(\psi_1)_{2^6:(3^{\cdot}S_6)} = \chi_{25}.$$

Next, we restrict $\psi_2 = \chi_2 \in \mathrm{Irr}(M_{24})$ to an irreducible character of $\bar{G} = 2^6{:}(3^{\cdot}S_6)$ as follows:

$$\begin{aligned}
\delta_1 = \langle \zeta, \psi_2 \rangle_{2^6:(3^{\cdot}S_6)} &= \frac{1}{64}(\psi_2(1A) + 18\psi_2(2A_1) + 45\psi_2(2A_2)) \\
&= \frac{1}{64}(\chi_2(1A) + 18\chi_2(2B) + 45\chi_2(2A)) \\
&= \frac{1}{64}(23 + 18(-1) + 45(7)) = 5.
\end{aligned}$$

Since $\deg(\psi_2) = 23$, we have $\delta_1 + 18\delta_2 + 45\delta_3 = 23$. Thus, $\delta_1 = 5$ implies that $\delta_2 = 1$ and $\delta_3 = 0$. This way, $(\psi_2)_{2^6:(3^{\cdot}S_6)}$ becomes the sum of two irreducible characters of degrees 5 and 18 in $2^6{:}(3^{\cdot}S_6)$. Therefore, based on the partial fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24} which has already been computed, we obtain

$$(\psi_2)_{2^6:(3^{\cdot}S_6)} = \chi_3 + \chi_{17}.$$

For the values of χ_3 and χ_{17} on the classes of $2^6{:}(3^{\cdot}S_6)$, see Table 11.

Using the partial fusion which has already been determined, the values of ψ_1 and ψ_2 on the classes of M_{24} and the values of $(\psi_1)_{2^6:(3^{\cdot}S_6)} = \chi_{25}$ and $(\psi_2)_{2^6:(3^{\cdot}S_6)} = \chi_3 + \chi_{17}$ on the classes of $2^6{:}(3^{\cdot}S_6)$, we are able to complete the fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24}. For instance, consider the classes $2A_6$, $4A_{10}$, $12A_1$ and $15A_2$ of $2^6{:}(3^{\cdot}S_6)$. The following holds:

$$\chi_{25}(2A_6) = \psi_1(2A) = -3,$$

$$\chi_{25}(4A_{10}) = \psi_1(4B) = 1.$$

Similarly,

$$\chi_3(12A_1) + \chi_{17}(12A_1) = \psi_2(12B) = -1,$$

$$\chi_3(15A_2) + \chi_{17}(15A_2) = \psi_2(15B) = 0.$$

This way, for the classes $2A_6$, $4A_{10}$, $12A_1$ and $15A_2$ of $2^6{:}(3^{\cdot}S_6)$, we obtain the following fusions into M_{24}.

$$2A_6 \mapsto 2A,\ 4A_{10} \mapsto 4B,\ 12A_1 \mapsto 12B \text{ and } 15A_2 \mapsto 15B.$$

The complete fusion of all the classes of $2^6{:}(3^{\cdot}S_6)$ into the parent group M_{24} is shown in Tables 12 and 13.

5. GAP Programmes and Tables

In this section, we present GAP programmes and Tables cited in the main work.

Programme C

```
I:=Irr(N);; o:=Orbits(G,I);; FM:=[];; for i in [1..Size(o)]
do Append(FM,[AsList(Sum(o[i]))]);od;
M1:=TransposedMat(FM);; M2:=AsDuplicateFreeList(M1);;
FM:=TransposedMat(M2);; Display(FM);
this is the Fischer Matrix on the identity
element of G
```

Programme D

```
C:=List(ConjugacyClasses(G),Representative);; M:=[];;
g:=C[i];; for n in N do
Add(M, n*g*Inverse(n)*Inverse(g));; od;
M:=AsGroup(M);; cent:=Centralizer(G, g);;
O:=RightCosets(N,M);; D:=O;; B:=[];;
for j in [1..Size(N)] do
B[j]:=[];;od; j:=1; while D <> [] do
x:=Representative(D[1]);for i in [1..Size(O)] do
y:=x^cent; if Intersection(y, O[i]) <> [] then
```

```
Add(B[j],O[i]); fi;od; D:=Difference(D,B[j]);
j:=j+1; od; i:=1; while B[i] <> [] do
Print(Size(B[i]));Print(" - " );i:=i+1; od;
I:=Irr(N);; IM:=[];; for i in [1..Size(I)] do
if IsSubgroup(Kernel(I[i]), M) then Add(IM,I[i]);
fi; od; oo:=Orbits(cent,IM);; FM:=[];;
for i in [1..Size(oo)] do
Append(FM,[AsList(Sum(oo[i]))]);od; M1:=TransposedMat(FM);;
M2:=AsDuplicateFreeList(M1);;
FM:=TransposedMat(M2);; Display(FM);
```

Table 1: Generators of $\bar{G} = 2^6{:}(3^{\cdot}S_6)$.

$m_1 = \begin{pmatrix} 1&1&0&1&0&1&1&1&1&1&1 \\ 1&0&1&1&0&1&1&1&1&1&0 \\ 1&1&1&1&1&1&0&1&0&0&1 \\ 0&0&1&0&0&0&1&1&1&1&0 \\ 1&1&1&0&1&0&1&0&1&1&1 \\ 1&1&1&0&0&1&0&0&1&0&1 \\ 1&0&0&1&0&1&0&0&1&0&1 \\ 0&0&1&1&0&1&0&0&0&1&0 \\ 0&0&1&1&0&1&1&0&0&0&0 \\ 1&0&0&1&0&1&0&1&0&0&1 \\ 0&1&0&0&1&0&0&1&1&1&1 \end{pmatrix}$
$m_2 = \begin{pmatrix} 1&0&1&0&0&0&0&0&1&1&1 \\ 1&1&0&0&1&1&1&1&1&0&0 \\ 1&1&1&0&1&0&0&1&1&1&1 \\ 1&1&0&1&1&0&1&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0&0&1 \\ 0&1&0&0&1&0&0&1&0&1&1 \\ 1&1&1&0&0&0&1&1&0&0&1 \\ 0&0&1&1&0&1&0&0&1&1&0 \\ 0&1&0&0&1&1&1&0&0&1&0 \\ 1&1&1&1&0&1&1&1&0&1&1 \\ 0&0&0&1&1&1&0&1&1&0&1 \end{pmatrix}$

Table 2: Matrix Generators of $G = 3^{\cdot}S_6$.

$$a = \begin{pmatrix} 0&0&0&0&0&0&0&0&1&0&1\\ 1&1&1&1&0&1&0&0&1&0&0\\ 1&1&0&1&1&1&1&0&1&1&1\\ 0&1&1&1&1&0&1&0&1&0&0\\ 0&0&0&1&1&1&0&0&1&1&0\\ 0&1&0&0&1&1&1&0&0&1&1\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&1&0&1&1&0&1&1\\ 0&1&0&1&1&1&1&0&0&0&1\\ 0&1&1&1&1&1&1&0&0&0&0\\ 1&1&0&1&1&1&1&0&0&0&1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0&1&0&0&0&0&1&1&1&1&1\\ 0&1&1&1&1&0&0&0&0&0&1\\ 0&0&0&1&1&0&0&1&0&0&1\\ 1&0&0&1&1&1&1&0&0&0&0\\ 0&1&0&1&1&0&0&1&0&0&0\\ 1&1&1&0&0&1&1&1&0&0&0\\ 0&0&1&0&1&0&0&1&0&1&1\\ 0&1&0&0&1&0&1&1&0&1&1\\ 0&0&0&1&0&0&1&1&1&1&0\\ 1&0&1&1&0&0&0&1&1&1&1\\ 1&0&1&1&1&0&0&0&1&0&1 \end{pmatrix}$$

$$c = \begin{pmatrix} 0&1&1&1&1&1&0&0&0&1&0\\ 1&1&0&1&0&1&1&1&0&0&1\\ 0&1&1&0&0&0&1&0&0&1&0\\ 1&1&0&1&0&1&1&1&1&0&1\\ 1&1&0&1&0&1&1&1&1&1&1\\ 1&0&1&0&1&0&0&0&0&1&0\\ 1&0&0&0&0&0&1&1&1&1&0\\ 0&0&1&1&0&1&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1&0&0\\ 1&1&1&1&0&0&0&0&0&1&1\\ 1&1&1&1&0&1&1&0&1&1&1 \end{pmatrix}$$

Table 3: Conjugacy Classes of $\bar{G} = 2^6{:}(3^{\cdot}S_6)$.

$[g]_G$	k	f_j	w_j	$[g]_{\bar{G}}$	$\lvert C_{\bar{G}}(g)\rvert$	2	3	5	Size of $[g]_{\bar{G}}$
$1A$	64	$f_1 = 1$	$w_1 = 1$	$1A$	138240	1A			1
		$f_2 = 18$	$w_2 = 18$	$2A_1$	7680	1A			18
		$f_3 = 45$	$w_3 = 45$	$2A_2$	3072	1A			45
$2A$	16	$f_1 = 1$	$w_4 = 4$	$2A_3$	768	$1A$			180
		$f_2 = 3$	$w_5 = 12$	$2A_4$	256	$1A$			540
		$f_3 = 6$	$w_6 = 24$	$4A_1$	128	$2A_2$			1080
		$f_4 = 6$	$w_7 = 24$	$4A_2$	128	$2A_2$			1080
$2C$	8	$f_1 = 1$	$w_8 = 8$	$2A_5$	384	$1A$			360
		$f_2 = 3$	$w_9 = 24$	$4A_3$	128	$2A_2$			1080
		$f_3 = 4$	$w_{10} = 32$	$4A_4$	96	$2A_1$			1440
$2B$	8	$f_1 = 1$	$w_{11} = 8$	$2A_6$	384	$1A$			360
		$f_2 = 6$	$w_{12} = 48$	$4A_5$	384	$2A_2$			360
		$f_3 = 6$	$w_{13} = 48$	$4A_6$	64	$2A_2$			2160
$3A$	1	$f_1 = 1$	$w_{14} = 64$	$3A_1$	1080		$1A$		128
$3B$	4	$f_1 = 1$	$w_{15} = 16$	$3A_2$	72		$1A$		1920
		$f_2 = 3$	$w_{16} = 48$	$6A_1$	24	$3A_2$	$2A_1$		5760
$3C$	4	$f_1 = 1$	$w_{17} = 16$	$3A_3$	72		$1A$		1920
		$f_2 = 3$	$w_{18} = 48$	$6A_2$	24	$3A_3$	$2A_2$		5760
$4A$	4	$f_1 = 1$	$w_{19} = 16$	$4A_7$	32	$2A_4$			4320
		$f_2 = 1$	$w_{20} = 16$	$4A_8$	32	$2A_3$			4320
		$f_3 = 2$	$w_{21} = 32$	$8A_1$	16	$4A_1$			8640
$4B$	4	$f_1 = 1$	$w_{22} = 16$	$4A_9$	96	$2A_3$			1440
		$f_2 = 3$	$w_{23} = 48$	$4A_{10}$	32	$2A_4$			4320
$5A$	4	$f_1 = 1$	$w_{24} = 16$	$5A_1$	60			$1A$	2304
		$f_2 = 3$	$w_{25} = 48$	$10A_1$	20	$5A_1$		$2A_1$	6912
$6A$	1	$f_1 = 1$	$w_{26} = 64$	$6A_3$	24	$3A_1$	$2A_3$		5760
$6B$	2	$f_1 = 1$	$w_{27} = 32$	$6A_4$	12	$3A_2$	$2A_5$		11520
		$f_2 = 1$	$w_{28} = 32$	$12A_1$	12	$6A_1$	$4A_4$		11520
$6C$	2	$f_1 = 1$	$w_{29} = 32$	$6A_5$	12	$3A_3$	$2A_6$		11520
		$f_2 = 1$	$w_{30} = 32$	$12A_2$	12	$6A_2$	$4A_5$		11520
$12A$	1	$f_1 = 1$	$w_{31} = 64$	$12A_3$	12	$6A_3$	$4A_9$		11520
$15A$	1	$f_1 = 1$	$w_{32} = 64$	$15A_1$	15		$5A_1$	$3A_1$	9216
$15B$	1	$f_1 = 1$	$w_{33} = 64$	$15A_2$	15		$5A_1$	$3A_1$	9216

Table 4: 6×6 Matrix Generators of $G = 3^{\cdot}S_6$.

Generators of G	*Generators of G*
$a_1 = \begin{bmatrix} 1&1&1&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&1&1&0 \\ 1&0&1&1&0&0 \\ 1&0&0&0&0&1 \\ 1&1&1&1&0&0 \end{bmatrix}$	$b_1 = \begin{bmatrix} 0&0&1&1&1&1 \\ 0&1&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&1&0 \\ 1&1&0&0&1&0 \\ 0&0&1&0&0&1 \end{bmatrix}$
$c_1 = \begin{bmatrix} 0&1&1&1&0&0 \\ 0&0&0&1&1&0 \\ 1&1&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&1&1&1&0&1 \\ 1&1&0&0&1&1 \end{bmatrix}$	

Table 5: Fusion of the classes from H_2 to $G = 3^{\cdot}S_6$.

$[h]_{H_2}$	$\mapsto$	$[g]_G$	$[h]_{H_2}$	$\mapsto$	$[g]_G$	$[h]_{H_2}$	$\mapsto$	$[g]_G$
$1a$		$1A$	$3a$		$3B$	$5a$		$5A$
$2a$		$2A$	$4a$		$4A$	$6a$		$6B$
$2b$		$2B$						

Table 6: Fusion of the Classes from H_3 to $G = 3^{\cdot}S_6$.

$[h]_{H_3}$	$\mapsto$	$[g]_G$	$[h]_{H_3}$	$\mapsto$	$[g]_G$	$[h]_{H_3}$	$\mapsto$	$[g]_G$
$1a$		$1A$	$2d$		$2B$	$4a$		$4B$
$2a$		$2A$	$2e$		$2C$	$4b$		$4A$
$2b$		$2A$	$3a$		$3C$	$6a$		$6C$
$2c$		$2C$						

Table 7: Character Table of $G = H_1 = 3^{\cdot}S_6$.

$[g]_G$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$4A$	$4B$	$5A$	$6A$	$6B$	$6C$	$12A$	$15A$	$15B$
$C_G(g)$	2160	48	48	48	1080	18	18	8	24	15	24	6	6	12	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	1	1	1	−1	1	1	1	−1	−1	1	1	1
χ_3	5	1	3	−1	5	2	−1	1	−1	0	1	0	−1	−1	0	0
χ_4	5	1	1	−3	5	−1	2	−1	−1	0	1	1	0	−1	0	0
χ_5	5	1	−3	1	5	2	−1	−1	−1	0	1	0	1	−1	0	0
χ_6	5	1	−1	3	5	−1	2	1	−1	0	1	−1	0	−1	0	0
χ_7	6	−2	0	0	−3	0	0	0	2	1	1	0	0	−1	d	$\bar{d}$
χ_8	6	−2	0	0	−3	0	0	0	2	1	1	0	0	−1	$\bar{d}$	d
χ_9	9	1	−3	−3	9	0	0	1	1	−1	1	0	0	1	−1	−1
χ_{10}	9	1	3	3	9	0	0	−1	1	−1	1	0	0	1	−1	−1
χ_{11}	10	−2	−2	2	10	1	1	0	0	0	−2	1	−1	0	0	0
χ_{12}	10	−2	2	−2	10	1	1	0	0	0	−2	−1	1	0	0	0
χ_{13}	12	4	0	0	−6	0	0	0	0	2	−2	0	0	0	−1	−1
χ_{14}	16	0	0	0	16	−2	−2	0	0	1	0	0	0	0	1	1
χ_{15}	18	2	0	0	−9	0	0	0	2	−2	−1	0	0	−1	1	1
χ_{16}	30	−2	0	0	−15	0	0	0	−2	0	1	0	0	1	0	0

Table 8: Character Table of $H_3 = 2 \times S_4$.

$[h]_{H_2}$	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$4a$	$4b$	$6a$
$C_{H_2}(h)$	48	8	8	48	16	16	6	8	8	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	−1	1	1	1	−1	−1
χ_3	1	−1	−1	1	1	1	1	−1	−1	1
χ_4	1	−1	1	−1	−1	1	1	−1	1	−1
χ_5	2	0	0	−2	−2	2	−1	0	0	1
χ_6	2	0	0	2	2	2	−1	0	0	−1
χ_7	3	1	−1	−3	1	−1	0	−1	1	0
χ_8	3	−1	−1	3	−1	−1	0	1	1	0
χ_9	3	−1	1	−3	1	−1	0	1	−1	0
χ_{10}	3	1	1	3	−1	−1	0	−1	−1	0

Table 9: Character Table of $H_2 = S_5$.

$[h]_{H_2}$	$1a$	$2a$	$2b$	$3a$	$4a$	$5a$	$6a$
$C_{H_2}(h)$	120	8	12	6	4	5	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	−1	1	−1	1	−1
χ_3	4	0	−2	1	0	−1	1
χ_4	4	0	2	1	0	−1	−1
χ_5	5	1	1	−1	−1	0	1
χ_6	5	1	−1	−1	1	0	−1
χ_7	6	−2	0	0	0	1	0

Table 10: Fischer Matrices of $2^6{:}(3^{\cdot}S_6)$.

Fischer Matrices of $2^6{:}(3^{\cdot}S_6)$	*Fischer matrices of* $2^6{:}(3^{\cdot}S_6)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 \\ 18 & -6 & 2 \\ 45 & 5 & -3 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 \\ 6 & -2 & 2 & -2 \\ 6 & -2 & -2 & 2 \end{pmatrix}$
$M(2B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$M(2C) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 4 & -4 & 0 \end{pmatrix}$
$M(3A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(3B) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(3C) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$

Continued on next page

Table 10 – continued from previous page.

Fischer Matrices of $2^6{:}A_8$	*Fischer Matrices of* $2^6{:}A_8$
$M(4B) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(5A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(6A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(6B) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(6C) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(12A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(15A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(15B) = \begin{pmatrix} 1 \end{pmatrix}$

Table 11: Character Table of $2^6 : (3^{\cdot}S_6)$.

$[g]_{\bar{G}}$	$1A$	$2A_1$	$2A_2$	$2A_3$	$2A_4$	$4A_1$	$4A_2$	$2A_5$	$4A_3$	$4A_4$	$2A_6$	$4A_5$	$4A_6$	$3A_1$	$3A_2$	$6A_1$	$3A_3$	$6A_2$
$C_{\bar{G}}(g)$	138240	7680	3072	768	256	128	128	384	128	96	384	384	64	1080	72	24	72	24
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1
χ_3	5	5	5	1	1	1	1	3	3	3	−1	−1	−1	5	2	2	−1	−1
χ_4	5	5	5	1	1	1	1	1	1	1	−3	−3	−3	5	−1	−1	2	2
χ_5	5	5	5	1	1	1	1	−3	−3	−3	1	1	1	5	2	2	−1	−1
χ_6	5	5	5	1	1	1	1	−1	−1	−1	3	3	3	5	−1	−1	2	2
χ_7	6	6	6	−2	−2	−2	−2	0	0	0	0	0	0	−3	0	0	0	0
χ_8	6	6	6	−2	−2	−2	−2	0	0	0	0	0	0	−3	0	0	0	0
χ_9	9	9	9	1	1	1	1	−3	−3	−3	−3	−3	−3	9	0	0	0	0
χ_{10}	9	9	9	1	1	1	1	3	3	3	3	3	3	9	0	0	0	0
χ_{11}	10	10	10	−2	−2	−2	−2	−2	−2	−2	2	2	2	10	1	1	1	1
χ_{12}	10	10	10	−2	−2	−2	−2	2	2	2	−2	−2	−2	10	1	1	1	1
χ_{13}	12	12	12	4	4	4	4	0	0	0	0	0	0	−6	0	0	0	0
χ_{14}	16	16	16	0	0	0	0	0	0	0	0	0	0	16	−2	−2	−2	−2
χ_{15}	18	18	18	2	2	2	2	0	0	0	0	0	0	−9	0	0	0	0
χ_{16}	30	30	30	−2	−2	−2	−2	0	0	0	0	0	0	−15	0	0	0	0
χ_{17}	18	−6	2	6	−2	−2	2	0	0	0	4	−4	0	0	0	0	3	−1
χ_{18}	18	−6	2	6	−2	−2	2	0	0	0	−4	4	0	0	0	0	3	−1
χ_{19}	72	−24	8	0	0	0	0	0	0	0	−8	8	0	0	0	0	3	−1
χ_{20}	72	−24	8	0	0	0	0	0	0	0	8	−8	0	0	0	0	3	−1
χ_{21}	90	−30	10	6	−2	−2	2	0	0	0	4	−4	0	0	0	0	−3	1
χ_{22}	90	−30	10	6	−2	−2	2	0	0	0	−4	4	0	0	0	0	−3	1
χ_{23}	108	−36	12	−12	4	4	−4	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	45	5	−3	9	1	1	−3	7	−1	−1	3	3	−1		3	−1	0	0
χ_{25}	45	5	−3	−3	5	−3	1	5	−3	1	−3	−3	1	0	3	−1	0	0
χ_{26}	45	5	−3	−3	5	−3	1	−5	3	−1	3	3	−1	0	3	−1	0	0
χ_{27}	45	5	−3	9	1	1	−3	−7	1	1	−3	−3	1	0	3	−1	0	0
χ_{28}	90	10	−6	6	6	−2	−2	−2	−2	2	−6	−6	2	0	−3	1	0	0
χ_{29}	90	10	−6	6	6	−2	−2	2	2	−2	6	6	−2	0	−3	1	0	0
χ_{30}	135	15	−9	−9	−1	−1	3	3	−5	3	3	3	−1	0	0	0	0	0
χ_{31}	135	15	−9	−9	−1	−1	3	−3	5	−3	−3	−3	1	0	0	0	0	0
χ_{32}	135	15	−9	3	−5	3	−1	−9	−1	3	3	3	−1	0	0	0	0	0
χ_{33}	135	15	−9	3	−5	3	−1	9	1	−3	−3	−3	1	0	0	0	0	0

Table 11: (*cont.*)

$[g]_{\bar{G}}$	$4A_7$	$4A_8$	$8A_1$	$4A_9$	$4A_{10}$	$5A_1$	$10A_1$	$6A_3$	$6A_4$	$12A_1$	$6A_5$	$12A_2$	$12A_3$	$15A_1$	$15A_2$
$C_{\bar{G}}(g)$	32	32	16	96	32	60	20	24	12	12	12	12	12	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	−1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	1
χ_3	1	1	1	−1	−1	0	0	1	0	0	−1	−1	−1	0	0
χ_4	−1	−1	−1	−1	−1	0	0	1	1	1	0	0	−1	0	0
χ_5	−1	−1	−1	−1	−1	0	0	1	0	0	1	1	−1	0	0
χ_6	1	1	1	−1	−1	0	0	1	−1	−1	0	0	−1	0	0
χ_7	0	0	0	2	2	1	1	1	0	0	0	0	−1	d	$\bar{d}$
χ_8	0	0	0	2	2	1	1	1	0	0	0	0	−1	$\bar{d}$	d
χ_9	1	1	1	1	1	−1	−1	1	0	0	0	0	1	−1	−1
χ_{10}	−1	−1	−1	1	1	−1	−1	1	0	0	0	0	1	−1	−1
χ_{11}	0	0	0	0	0	0	0	−2	1	1	−1	−1	0	0	0
χ_{12}	0	0	0	0	0	0	0	−2	−1	−1	1	1	0	0	0
χ_{13}	0	0	0	0	0	2	2	−2	0	0	0	0	0	−1	−1
χ_{14}	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
χ_{15}	0	0	0	2	2	−2	−2	−1	0	0	0	0	−1	1	1
χ_{16}	0	0	0	−2	−2	0	0	1	0	0	0	0	1	0	0
χ_{17}	2	−2	0	0	0	3	−1	0	0	0	1	−1	0	0	0
χ_{18}	−2	2	0	0	0	3	−1	0	0	0	−1	1	0	0	0
χ_{19}	0	0	0	0	0	−3	1	0	0	0	1	−1	0	0	0
χ_{20}	0	0	0	0	0	−3	1	0	0	0	−1	1	0	0	0
χ_{21}	−2	2	0	0	0	0	0	0	0	0	1	−1	0	0	0
χ_{22}	2	−2	0	0	0	0	0	0	0	0	−1	1	0	0	0
χ_{23}	0	0	0	0	0	3	−1	0	0	0	0	0	0	0	0
χ_{24}	1	1	−1	3	−1	0	0	0	1	−1	0	0	0	0	0
χ_{25}	1	1	−1	−3	1	0	0	0	−1	1	0	0	0	0	0
χ_{26}	−1	−1	1	−3	1	0	0	0	1	−1	0	0	0	0	0
χ_{27}	−1	−1	1	3	−1	0	0	0	−1	1	0	0	0	0	0
χ_{28}	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	−1	1	0	0	0	0	0
χ_{30}	−1	−1	1	3	−1	0	0	0	0	0	0	0	0	0	0
χ_{31}	1	1	−1	3	−1	0	0	0	0	0	0	0	0	0	0
χ_{32}	1	1	−1	−3	1	0	0	0	0	0	0	0	0	0	0
χ_{33}	−1	−1	1	−3	1	0	0	0	0	0	0	0	0	0	0

Table 12: Fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24}.

$[y]$		$1A$	$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$4C$	$5A$
$\lvert C_{M_{24}}(y)\rvert$		2448	21	76	10	5	3	1	9	6
		23040	504	80	80	04	84	28	6	0
$\chi(M_{24}\lvert G)(y)$		1771	91	51	16	7	11	7	7	1
$[x]$	$\lvert C_{\bar{G}}(x)\rvert$									
$1A$	138240	[1771]								
$2A_1$	7680			[1]						
$2A_2$	3072		[7]							
$2A_3$	768		[28]	10						
$2A_4$	256		84	[30]						
$2A_5$	384		56	[20]						
$2A_6$	384		[56]	20						
$3A_1$	1080				[1]					
$3A_2$	72				[15]	7				
$3A_3$	72				15	[7]				
$4A_1$	128						3	[1]		
$4A_2$	128						[3]	1		
$4A_3$	128						[3]	1		
$4A_4$	96						4		[1]	
$4A_5$	384						[1]			
$4A_6$	64						6	[2]		
$4A_7$	32						12	[4]	3	
$4A_8$	32						12	4	[3]	
$4A_9$	96						[4]		1	
$4A_{10}$	32						12	4	[3]	
$5A_1$	60									[1]

Table 12: (*cont.*)

$[y]$		$6A$	$6B$	$8A$	$10A$	$12A$	$12B$	$15A$	$15B$
$\lvert C_{M_{24}}(y)\rvert$		2	2	1	2	1	1	1	1
		4	4	6	0	2	2	5	5
$\chi(M_{24}\lvert G)(y)$		4	3	1	1	2	1	1	1
$[x]$	$\lvert C_{\bar{G}}(x)\rvert$								
$6A_1$	24	[1]	1						
$6A_2$	24	1	[1]						
$6A_3$	24	[1]	1						
$6A_4$	12	[2]	2						
$6A_5$	12	2	[2]						
$8A_1$	16			[1]					
$10A_1$	20				[1]				
$12A_1$	12					[1]	1		
$12A_2$	12					1	[1]		
$12A_3$	12					[1]	1		
$15A_1$	15							[1]	1
$15A_2$	15							1	[1]

Table 13: Summary for the Fusion of $2^6{:}(3^{\cdot}S_6)$ into M_{24}.

$[\bar{g}]_{\bar{G}}$	$\mapsto$	$[y]_{M_{24}}$	$[\bar{g}]_{\bar{G}}$	$\mapsto$	$[y]_{M_{24}}$	$[\bar{g}]_{\bar{G}}$	$\mapsto$	$[y]_{M_{24}}$
$1A$		$1A$	$4A_2$		$4A$	$6A_2$		$6B$
$2A_1$		$2B$	$4A_3$		$4A$	$6A_3$		$6A$
$2A_2$		$2A$	$4A_4$		$4C$	$6A_4$		$6A$
$2A_3$		$2A$	$4A_5$		$4A$	$6A_5$		$6B$
$2A_4$		$2B$	$4A_6$		$4B$	$8A_1$		$8A$
$2A_5$		$2B$	$4A_7$		$4B$	$10A_1$		$10A$
$2A_6$		$2A$	$4A_8$		$4C$	$12A_1$		$12A$
$3A_1$		$3A$	$4A_9$		$4A$	$12A_2$		$12B$
$3A_2$		$3A$	$4A_{10}$		$4C$	$12A_3$		$12A$
$3A_3$		$3B$	$5A_1$		$5A$	$15A_1$		$15A$
$4A_1$		$4B$	$6A_1$		$6A$	$15A_2$		$15B$

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