Mathematics Interdisciplinary Research **7** (2022) 357 – 375

Original Scientific Paper

Some Results on Asymptotic Behavior of the Recalls of Random Median Quicksort

Mehri Javanian* *and Ali Mohammadian Mosammam*

Abstract

This paper investigates the asymptotic behavior of the number of recalls X_n of the Random Median Quicksort algorithm in order to sort a list of n distinct numbers. As $n \to \infty$, we provide the asymptotics of the expectation and variance of the recalls. Furthermore, by utilizing a refined version of the contraction method for degenerate limits, we show the limiting distribution of X_n correctly normalized is Gaussian. The theoretical results are demonstrated by a simulation study.

Keywords: median Quicksort, recalls of algorithm, contraction method, normal limiting distribution.

2020 Mathematics Subject Classification: 60F05, 60F15, 68P10, 68Q25.

How to cite this article
M. Javanian and A. Mohammadian Mosammam, Recalls of random median Quicksort, *Math. Interdisc. Res.* **7** (2022) 357-375.

1. Introduction

Sorting a list of numbers is obviously a fundamental problem. An efficient sorting algorithm is Quicksort that was invented and developed by Hoare [5] in 1962.

Let $S = [\pi_1, \pi_2, \ldots, \pi_n]$ be an array of n distinct numbers that we want to be sorted. Quicksort sorts S based on a divide and conquer strategy. Quicksort first picks a uniformly random element p from S and divide S (except p) into two subarrays: The array $S_<$ of elements smaller than p and the array $S_>$ of elements larger than p. The element p is called the pivot. Then Quicksort sorts $S_<$ and

*Corresponding author (E-mail: javanian@znu.ac.ir)
Academic Editor: Mohammad Amini Dehak
Received 14 April 2022, Accepted 23 November 2022
DOI:10.22052/MIR.2022.246282.1347

$S_>$ recursively. For many variants of Quicksort, the number of comparisons to sort the input numbers has been extensively studied in numerous papers; see for instance [7], [13], [14], [17], and thesis, such as [4], [8], [21] and [23].

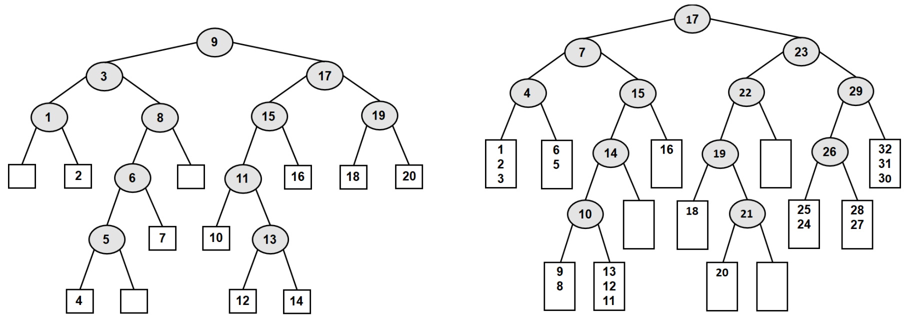
The efficiency of the Quicksort algorithm very much depends on the selection of the pivot element. Suppose we intend to sort the array $S = [\pi_1, \pi_2, \ldots, \pi_n]$, where $\pi_1 < \pi_2 < \cdots < \pi_n$, and our first selection of pivot is the maximum element π_n. In this case, we just divide and conquer in a trivial manner. One array will have one element and the other one will have $n-1$ elements. If this happens, then every element in the array of size $n-1$ is less than the pivot, and it has taken us $n-1$ comparisons with π_n. Now, again, assume the worst case occurs. Namely, this time, in the array of size $n-1$, we choose the largest element π_{n-1} as a pivot. Again the Quicksort makes $n-2$ comparisons and we are left with every element smaller than π_{n-1}. If we continue to select the largest element of what is left as a pivot, then, after $n-1$ recursive calls, the overall number of comparisons made is $1 + 2 + \cdots + (n-1) = \mathcal{O}(n^2)$. Therefore, Quicksort needs quadratic run-time to sort already sorted arrays if we select a pivot element in such a way that it gives the most unbalanced partitions, though this behavior is rare.

We can avoid the worst case in Quicksort by selecting each pivot element to be roughly in the middle of an input array of size n. Then, we make $n-1$ comparisons in the first round. Next, in the second round, we make about $n/2$ comparisons in each of two subarrays of size about $n/2$. Further, in the third round, we make $n/4$ comparisons in each of the four subarrays of size about $n/4$, and so forth. Thus, in each round, we make about n comparisons. At each round, we are splitting the arrays into roughly equally-sized subarrays, and it takes $\log_2(n)$ rounds to get down to trivially sorted arrays. Therefore, this case leads us to the $\mathcal{O}\big(n \log_2(n)\big)$ best-case asymptotic time complexity which is much smaller than $\mathcal{O}(n^2)$ in the worst case.

This informal argument makes it clear that the choice of pivot plays a fundamental role in the performance of Quicksort. Many authors have suggested various methods to rectify how to choose the pivot and to pass the worst-case manner, see [2], [5], [9], [20], [21] and [22]. These references include shuffling of the array randomly prior to initialisation of the Quicksort, selecting the median of the array as pivot, or selecting the median of a random sample of elements.

Scowen [20] proposed selecting as pivot the middle element among the array elements. Applying this strategy for the selection of pivot, the purpose is the splitting of the array into two subarrays of equal size. So the quadratic time is avoided in the case where the array is an almost-sorted array. Towards this pivot strategy, a naive idea would be computing the median as pivot among the array elements. Unfortunately, the computing the median imposes extra costs to the Quicksort algorithm. Hence, despite the always good selections, this strategy might not be better than selecting the pivot randomly.

Singleton [22] proposed a randomly select three elements from an array of n distinct numbers to be sorted and use their median as pivot, causing to a better estimate of the median of the array.



(a) A 1-median Quicksort tree of size 20. (b) A 3-median Quicksort tree of size 32.

Figure 1: Illustration of two $2k+1$-Median Quicksort trees for $k = 0$ and $k = 1$, in (a) and (b) respectively: The pivots are represented by gray ovals, i.e., the non-leaf nodes. The leaves are represented by white squares in (a) and white rectangles in (b). The recalls (recursive calls) of $2k + 1$-Median Quicksort are represented by the non-leaf nodes in the associated trees.

By Hoare [5], this modification generalized as to selecting a sample of $2k + 1$ elements at every recursive call, finding their median and applying that median as pivot, partitioning $n \geq 2k+2$ elements. The arrays of size at most $2k+1$ elements are sorted by a simpler algorithm, such as insertion sort. But he didn't analyse this pivot strategy.

Roesler [18] analyze the number of comparisons to sort an array of n distinct numbers by Hoare's $2k + 1$-Median version of Quicksort ($2k + 1$-MQ). Later on, Okasha and Roesler [14] studied the number of comparisons applied by a generalized version of $2k + 1$-Median Quicksort, where k is a random variable.

There is a connection between $2k + 1$-Median Quicksort ($2k + 1$-MQ) and random median-of-$2k + 1$ binary search trees, see Section 7.3 in [11]. Bell [1] and Walker and Wood [24] introduced the median-of-$2k+1$ binary search trees. Given an array S of $n \geq 2k + 2$ elements on which a total order is defined, a $2k+1$-MQ tree on S is constructed as follows. Take $2k + 1$ elements at random from S. The median of these elements serves as the root of the binary tree. For every pivot (median-of-$(2k + 1)$) encountered create a non-leaf node in the tree labeled with the pivot. The first pivot labels the root of the tree. As the first pivot splits the array S into two parts (subarrays), a pivot will be found in each subarray of size $\geq 2k + 2$ and will father a subtree. If the subarray of smaller elements is of size $\geq 2k + 2$, the pivot found in it is attached to the root as a left child. Similarly, if there are at least $2k + 2$ larger elements, the pivot found there will be attached to the root as a right child. As $2k + 1$-MQ continues recursively, nodes are added in this fashion to the tree until arrays of size at most $2k + 1$ elements are the subject of sorting; those are represented by leaves (bucket nodes). At the moment that a leaf node overflows (i.e., when it would grow to size $2k+1$), its bucket is split about the median, leaving two new leave nodes (buckets) of size k each and a non-leave node holding the median. For this binary tree construction, for example see the two $2k+1$-MQ trees for $k=0$ and $k=1$ in Figure 1, where the pivots are represented by gray ovals (i.e., the non-leaf nodes). In Figure 1, we illustrate the leaves by white squares and rectangles.

By "the number of recalls" of $2k+1$-MQ, we mean the number of times that the pivot (median-of-$2k+1$) splitting the arrays of size $\geq 2k+2$ is performed. As an example, the pivots (i.e., recalls) are shown by gray nodes (non-leaf nodes) in the $2k+1$-MQ trees illustrated by Figure 1.

A variant is the Random Median Quicksort (RMQ(μ)), where k is the outcome of a random variable A with distribution μ. For every recall take an independent random variable with the same distribution μ. The RMQ(μ) was introduced by Okasha [14]. Notice the set of all RMQ algorithms contains all MQ algorithms.

In this paper, we are interested in the number of recalls X_n performed by any $2k+1$-MQ or RMQ(μ) in order to sort a set of size n. μ is a probability measure on $\{0,1,2,\ldots,K\}, K \in \mathbb{N}_0$. For $n < 2K+1 := n_0$ we take any other sorting algorithm and do not recall RMQ(μ). We also call RMQ(μ) by **RMQ(K)**.

For the reader's convenience, some notation on RMQ(μ) is adopted from [14]. Let the random variable A has probability distribution $\mathbb{P}(A=k)=p_k$, $k=0,\ldots,K$. Let Z_n^{μ} be the rank ($=|S_<|+1$) of the pivot element and $Z_n^{(k)} = \big(Z_n^{\mu} | A=k\big)$ be the rank of the pivot element conditioned to the outcome $A=k$. The distribution of $Z_n^{(k)}$ is

$$p_{n,m}^{(k)} := \mathbb{P}\big(Z_n^{(k)}=m\big) = \frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}}, \quad m=k+1,\ldots,n-k. \tag{1}$$

The random variable $\frac{Z_n^{(k)}}{n}$ for fixed k converges in distribution to a random variable $U^{(k)}$ with Beta($k+1,k+1$) distribution, see Proposition 1 in [10]. The density of $U^{(k)}$ is

$$f_k(u) = \frac{\Gamma(2k+2)}{\big(\Gamma(k+1)\big)^2} u^k(1-u)^k, \qquad 0<u<1, \tag{2}$$

where $\Gamma(\cdot)$ denotes the gamma function that satisfies $\Gamma(k+1)=k\Gamma(k)=k!$.

The case μ the point measure on 0 is well known. Devroye [3], see also Example 8.1 in [6], gave an explicit formula for the expectation and variance and showed the limiting distribution of normalized X_n is the normal.

$$\mathbb{E}(X_n) = \frac{2n-1}{3}, \qquad \mathrm{Var}(X_n) = \frac{2}{45}(n+1), \qquad \frac{X_n-\mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. The argument runs by a connection between Quicksort and random binary search trees, see Section 7.3 in [11]. X_n gets the interpretation as the number of non-leaf nodes (nodes with outdegree $\neq 0$) in the associated random binary search tree.

The key equation is the recursion (3) in distribution of the $X_n, n \in I\!N_0$. In Section 2 we prove via an Euler differential equation the expectation and variance of X_n grows linearly with n. In Section 3 we show via the contraction method for degenerate limit [12], that the asymptotic distribution of standardized X_n is Gaussian. The last section gives some indication via simulation of the speed of convergence.

2. The Expectation and Variance

Let $a_n := \mathbb{E}(X_n)$ and $b_n := \mathrm{Var}(X_n)$ be the expectation and variance of the number X_n of recalls performed by RMQ(μ) to sort n distinct numbers. In this section, we show that $a_n = cn + o(n)$ and $b_n = \sigma^2 n + o(n)$ for some constants c, σ^2 depending only on μ.

Since the RMQ(μ) is a divide and conquer algorithm the $X_n, n \in I\!N$ satisfy the recursion

$$X_n \stackrel{d}{=} X_{Z_n^\mu - 1} + \overline{X}_{n - Z_n^\mu} + 1, \tag{3}$$

for $n \geq n_0 := 2K+1$. The symbol $\stackrel{d}{=}$ denotes equality in distribution. The random variables X_i, $\overline{X}_i$, Z_n^μ are independent for $n \geq n_0$. X_i has the same distribution as $\overline{X}_i$ for all $i \in I\!N_0$. Further $X_i \equiv 0$ for $i < n_0$.

The average behavior of RMQ(μ) can be obtained by taking expectation of (3), yielding the recurrence

$$a_n = \mathbb{E}\big(a_{Z_n^\mu - 1}\big) + \mathbb{E}\big(a_{n - Z_n^\mu}\big) + 1, \tag{4}$$

for $n \geq n_0$. Further $a_n = 0$ for $n < n_0$.

Hence conditioning on $A = k$ and using $P(Z_n^{(k)} = m) = P(Z_n^{(k)} = n - m + 1)$ we obtain from (4)

$$a_n = 2 \sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)} a_{m-1} + 1. \tag{5}$$

Theorem 2.1. Let X_n be the number of recalls for random median Quicksort, RMQ(μ), in order to sort n different numbers. Then for some constant c depending only on μ

$$\mathbb{E}(X_n) = cn + o(n). \tag{6}$$

Proof. We basically follow the paper [14]. Let D be the derivative operator of a functions f, i.e. $(D(f))(z) = \frac{d}{dz} f(z)$ for all z. We skip the brackets and use D also for formal power series. Define the generating function

$$a(z) := \sum_{n \geq 0} a_n z^n.$$

Then by a similar argument in the proof of Lemma 3.2 in [14], using (5) we obtain the Euler differential equation

$$D^{n_0}a(z)-z^{n_0}(1-z)^{-n_0-1}n_0! = 2\sum_{k=0}^{K} p_k D^{2K-2k} \sum_{m=k+1}^{n-k} \frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}} a_{m-1} D^{2k+1} z^n.$$

Multiplying both sides with $(1-z)^{n_0}$ provides the Euler differential equation

$$(1-z)^{n_0} D^{n_0} a(z) - \frac{z^{n_0}}{1-z} n_0! = 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \frac{(2K+1)!}{(2K-k-j)!} (1-z)^{2K-k-j} D^{2K-k-j} a(z) \quad (7)$$

Then by a similar argument in the proof of Lemma 3.2 in [14], using (5) we obtain the Euler differential equation

$$(1-z)^{2K+1} D^{2K+1} a(z) - \frac{(2K+1)!}{1-z} = 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \frac{(2K+1)!}{(2K-k-j)!} (1-z)^{2K-k-j} D^{2K-k-j} a(z). \quad (8)$$

Using a change of variables $z = 1-e^{-x}$, defining $y(x) := a(z) = a(1-e^{-x})$ and the polynomial P in the variable λ by

$$P(\lambda) = \binom{\lambda+2K}{2K+1} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \binom{\lambda+2K-k-j-1}{2K-k-j},$$

and dividing by $(2K+1)!$ in (8), yields

$$P(D)y(x) = e^x. \quad (9)$$

By Lemma 3.5 in [14], we obtain the special solution $y(x) = -e^x$ for (9) respectively the special solution $a_s(z)$ for (8)

$$a_s(z) = -\frac{1}{1-z} = -\sum_{n\geq 0} z^n, \quad |z| < 1. \quad (10)$$

The homogeneous linear differential equation $P(D)y(x) = 0$ has solutions

$$x^j e^{x\,\mathrm{Re}(\lambda_i)} \cos\big(x\,\mathrm{Im}(\lambda_i)\big),\; x^j e^{x\,\mathrm{Re}(\lambda_i)} \sin\big(x\,\mathrm{Im}(\lambda_i)\big),$$

$i = 1,\ldots,l$, $j = 0,\ldots,r_i-1$, where $\lambda_1,\ldots,\lambda_l \in \mathbb{C}$ be the eigenvalues of the characteristic polynomial P in λ and $1 \leq r_1,\ldots,r_l$ be the multiplicities, $\sum_i r_i = n$.

Therefore every solution of the associated homogeneous differential equation to (8) is a convex combination of the fundamental system

$$\begin{aligned} &\big(-\ln(1-z)\big)^j (1-z)^{-\operatorname{Re}(\lambda_i)} \cos\big(-\operatorname{Im}(\lambda_i)\ln(1-z)\big),\\ &\big(-\ln(1-z)\big)^j (1-z)^{-\operatorname{Re}(\lambda_i)} \sin\big(-\operatorname{Im}(\lambda_i)\ln(1-z)\big), \end{aligned} \tag{11}$$

$i = 1, \ldots, l$, $j = 0, \ldots, r_i - 1$. Proposition 3.4 and Lemma 3.8 in [14], respectively, show that $\lambda_1 = 2$ is an eigenvalue of P with multiplicity 1 and all other eigenvalues of P have a real part strictly less than 2.

Thus, from (10) and the above the fundamental system of solutions, the general solution $a(z)$ of (8) is

$$\begin{aligned} a(z) &= c\big(-\ln(1-z)\big)^0(1-z)^{-2}\\ &\quad+\sum_{i=2}^{l}\sum_{j=0}^{r_i-1}\Big[c_{i,j}\big(-\ln(1-z)\big)^j(1-z)^{-\operatorname{Re}(\lambda_i)}\cos\big(-\operatorname{Im}(\lambda_i)\ln(1-z)\big)\\ &\quad+\tilde{c}_{i,j}\big(-\ln(1-z)\big)^j(1-z)^{-\operatorname{Re}(\lambda_i)}\sin\big(-\operatorname{Im}(\lambda_i)\ln(1-z)\big)\Big]\\ &=: c\sum_{n\ge 0}(n+1)z^n + \omega(z), \qquad \big([z^n]\omega(z) = o(n)\big) \end{aligned}$$

for some constants c, $\{c_{i,j}\}$ and $\{\tilde{c}_{i,j}\}$, as $i = 1, \ldots, l$, $j = 0, \ldots, r_i - 1$. Finally, by Corollary 3.10 in [14], all eigenvalues $\lambda_i \neq 2$ contribute only an $o(n)$ term to a_n. Then $a_n = cn + o(n)$ and the result follows. □

By (6), we have $a_n = \mathbb{E}(X_n) = cn + g(n)$ where $g(n) = o(n)$. From (3) and defining $Q_n := X_n - cn - g(n)$ and $\overline{Q}_n := \overline{X}_n - cn - g(n)$, we obtain

$$Q_n \overset{d}{=} Q_{Z_n^\mu - 1} + \overline{Q}_{n-Z_n^\mu} + G_n\big(Z_n^\mu\big), \tag{12}$$

$$\begin{aligned} G_n(Z_n^\mu) &= (Z_n^\mu - 1)\Big(c + \frac{g(Z_n^\mu - 1)}{Z_n^\mu - 1}\Big)\\ &\quad+(n - Z_n^\mu)\Big(c + \frac{g(n - Z_n^\mu)}{n - Z_n^\mu}\Big) + 1 - n\Big(c + \frac{g(n)}{n}\Big)\\ &\sim (Z_n^\mu - 1)c + (n - Z_n^\mu)c + 1 - nc = 1 - c, \qquad (\because\ Z_n^\mu \overset{\text{a.s.}}{\longrightarrow} \infty,) \end{aligned} \tag{13}$$

as $n \to \infty$, where the notation $\overset{\text{a.s.}}{\longrightarrow}$ denotes the almost sure convergence.

Theorem 2.2. Let X_n be the number of recalls for random median Quicksort, RMQ(μ), in order to sort n different numbers. Then for some constant σ^2,

$$\operatorname{Var}(X_n) = \sigma^2 n + o(n). \tag{14}$$

Proof. Denote $b_n := \mathrm{Var}(X_n) = \mathbb{E}(Q_n^2)$. Hence squaring (12), conditioning on A and $Z_n^{(k)}$, using $X_i \stackrel{d}{=} \overline{X}_i$ for $i = 0, \ldots, n-1$, b_n satisfies the recurrence

$$b_n = 2\mathbb{E}\big(b_{Z_n^\mu - 1}\big) + \mathbb{E}\big[\big(G_n(Z_n^\mu)\big)^2\big] = 2\sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)} b_{m-1} + \mathbb{E}\big[\big(G_n(Z_n^\mu)\big)^2\big]. \tag{15}$$

Denote the generating functions of $g_n := \mathbb{E}\big[\big(G_n(Z_n^\mu)\big)^2\big]$ and b_n by

$$\mathcal{G}(z) := \sum_{n\geq 0} g_n z^n, \quad b(z) := \sum_{n\geq 0} b_n z^n.$$

Again, by a similar argument in the proof of Lemma 3.2 in [14], using (15) we obtain the Euler differential equation

$$(1-z)^{n_0} D^{n_0} b(z) - (1-z)^{n_0} D^{n_0} \mathcal{G}(z) = \sum_{k=0}^{K} \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \frac{2p_k(2K+1)!}{(2K-k-j)!} (1-z)^{2K-k-j} D^{2K-k-j} b(z). \tag{16}$$

By the same system in (11), every solution of the associated homogeneous differential equation to (16) is

$$b(z) = \sigma^2\big(-\ln(1-z)\big)^0 (1-z)^{-2} + \widetilde{\omega}(z) \sim \sigma^2(1-z)^{-2},$$

for some constant σ^2 and $[z^n]\widetilde{\omega}(z) = o(n)$. Besides, the special solution is $\mathcal{O}\big((1-z)^{-1}\big)$ (since $g_n \to (1-c)^2$ then $\mathcal{G}(z) = \mathcal{O}\big((1-z)^{-1}\big)$ and we have $(1-z)^{2K+1} D^{2K+1} \mathcal{G}(z) = \mathcal{O}\big((1-z)^{-1}\big)$). Namely, the general solution will be $b(z) = \sigma^2 \sum_{n\geq 0}(n+1)z^n + \widetilde{\omega}(z)$. Therefore, the assertion follows. □

3. Asymptotic Normality

In this section, we begin to prove the normality of limiting distribution of X_n. The proof was completed by applying the contraction method, which was first introduced by [17], in studying the Quicksort algorithm. The contraction technique is usually arranged as follows: first, a limit distributional equation of the random variable we considered is conjectured; then the conjecture is checked by showing that the distribution function of the random variable we studied converges to that of the conjectured limit under some metric space (see [15], [19]).

Here, we prefer the Zolotarev metric ζ_3 (see [16], [25]) as the metric space applied in the contraction method. Let the distribution of a random variable X denoted by $\mathcal{L}(X)$. Then, for any given random variables X and Y, the 3rd order Zolotarev metric between X and Y is defined as

$$\zeta_3(X,Y) := \zeta_3(\mathcal{L}(X), \mathcal{L}(Y)) := \sup\{|\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]| : f \in \mathcal{F}\},$$

where $\mathcal{F} = \{f : f \in C^{(2)}, |f^{(2)}(x) - f^{(2)}(y)| \leq |x - y|\}$ denotes the set of all twice differentiable functions, where the second derivative is Lipschitz continuous with Lipschitz constant equal to 1.

The following two lemmas give several properties of probability metric $\zeta_3(X, Y)$, which are quite useful in our proof of Theorem 3.5.

Lemma 3.1 (see [16], [25])**.** Let $\zeta_3(X, Y)$ be the 3rd order Zolotarev metric between the random variables X and Y. Then, we have

(i) For any real number $\theta > 0$,

$$\zeta_3(\theta X, \theta Y) = \theta^3 \zeta_3(X, Y); \tag{17}$$

(ii) If the random variables Y and (X_1, X_2) are independent mutually, then

$$\zeta_3(X_1 + Y, X_2 + Y) \leq \zeta_3(X_1, X_2); \tag{18}$$

(iii) For any pair (X, Y) of random variables,

$$\mathbb{E}(|X|^3) + \mathbb{E}(|Y|^3) < \infty,\ \mathbb{E}(X^k) = \mathbb{E}(Y^k),\ k = 1, 2 \implies \zeta_3(X, Y) < \infty; \tag{19}$$

(iv) For the random variables V and $\{V_n\}_{n \geq 1}$, as $n \to \infty$

$$\zeta_3(V_n, V) \to 0 \implies V_n \xrightarrow{\mathcal{D}} V. \tag{20}$$

Lemma 3.2 (see [13])**.** let X_1, X_2, T_1 and T_2 be random variables such that the pairs $(X_1 + T_1, X_2 + T_2)$ and (X_1, X_2) satisfies (19). Then

$$\zeta_3(X_1 + T_1, X_2 + T_2) \leq \zeta_3(X_1, X_2) + \sum_{i=1}^{2} \left\{ \frac{\|X_i\|_3^2 \|T_i\|_3}{2} + \frac{\|X_i\|_3 \|T_i\|_3^2}{2} + \frac{\|T_i\|_3^3}{6} \right\},$$

where we denote $\|X\|_3 := \mathbb{E}(|X|^3)^{1/3}$ for a random variable X.

Moreover, the proof of Theorem 3.5 requires the following upper bound for metric ζ_3:

$$\zeta_3(X, Y) \leq \frac{1}{2}\big(\|X\|_3^2 + \|X\|_3 \|Y\|_3 + \|Y\|_3^2\big)\ell_3(X, Y), \tag{21}$$

where the minimal L_3-metric ℓ_3 defined by

$$\ell_3(X, Y) := \ell_3(\mathcal{L}(X), \mathcal{L}(Y)) := \inf\{\|X' - Y'\|_3 : \mathcal{L}(X) = \mathcal{L}(X'), \mathcal{L}(Y) = \mathcal{L}(Y')\},$$

for random variables X and Y with $\|X\|_3 < \infty$, $\|Y\|_3 < \infty$.

We standardize X_n with its mean and variance, i.e.,

$$Y_n := \frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}}, \quad \mathrm{Var}(X_n) = n\big(\sigma^2 + \frac{h(n)}{n}\big) \sim \sigma^2 n, \quad \big(h(n) = o(n)\big). \tag{22}$$

Let $\tau(n) := \sqrt{\mathrm{Var}(X_n)}$. The Equations (12) and (13) give the distributional recurrence

$$Y_n \overset{d}{=} \frac{\tau(Z_n^\mu - 1)}{\tau(n)} Y_{Z_n^\mu - 1} + \frac{\tau(n - Z_n^\mu)}{\tau(n)} \overline{Y}_{n - Z_n^\mu} + \frac{1}{\sqrt{n}} G_n(Z_n^\mu),$$

$$=: u_n(Z_n^\mu) Y_{Z_n^\mu - 1} + \overline{u}_n(Z_n^\mu) \overline{Y}_{n - Z_n^\mu} + v_n(Z_n^\mu) \tag{23}$$

$$=: B_{Z_n^\mu - 1} + \overline{B}_{n - Z_n^\mu} + v_n(Z_n^\mu) \tag{24}$$

where $Y_i \overset{d}{=} \overline{Y}_i$, for $0 \le i \le n-1$. The random variables Y_i, $\overline{Y}_i$, $Z_n^{(k)}$, A, $0 \le k \le K$, $0 \le i \le n-1$, are independent for $n \ge n_0 = 2K + 2$.

To prove the Theorem 3.5, we still require some more arrangements. The following three lemmas are necessary.

Lemma 3.3. Let W, W_1 and W_2 be independent random variables with the standard normal distribution. If a random variable with the distribution function in (1), $Z_n^{(k)} = \big(Z_n^\mu \big| A = k\big)$ is independent of W, W_1 and W_2, then we have

$$W \overset{d}{=} \sqrt{\frac{Z_n^\mu}{n}}\, W_1 + \sqrt{\frac{n - Z_n^\mu}{n}}\, W_2. \tag{25}$$

Proof. It is sufficient to verify that the characteristic function of the right side of (25) is the same as that of a standard normal random variable. From the independence of the random variables W, W_1, W_2, $Z_n^{(A)}$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{it\Big(\sqrt{\frac{Z_n^\mu}{n}}\, W_1 + \sqrt{\frac{n - Z_n^\mu}{n}}\, W_2\Big)\Big\}\Big] \\
&= \sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)} \mathbb{E}\Big[\exp\Big\{it\Big(\sqrt{\frac{m}{n}}\, W_1 + \sqrt{\frac{n-m}{n}}\, W_2\Big)\Big\}\Big] \\
&= \sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)} \mathbb{E}\Big[\exp\Big\{i\Big(t\sqrt{\frac{m}{n}}\Big) W_1\Big\}\Big] \mathbb{E}\Big[\exp\Big\{i\Big(t\sqrt{\frac{n-m}{n}}\Big) W_2\Big\}\Big] \\
&= \sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)} \exp\Big(-\frac{m}{2n} t^2\Big) \exp\Big(-\frac{n-m}{2n} t^2\Big) \\
&= \exp\Big(-\frac{t^2}{2}\Big).
\end{aligned}$$

The function $e^{-\frac{t^2}{2}}$ is the characteristic function of a standard normal random variable. Hence we obtain the claim. □

Remark 1. *As $n \to \infty$, by* (2), (22) *and* (23), *we have*

$$\begin{aligned}\mathbb{E}\big[\big(u_n(Z_n^\mu)\big)^3\big] &\sim \mathbb{E}\Big[\Big(\frac{Z_n^\mu - 1}{n}\Big)^{\frac{3}{2}}\Big] = \sum_{k=0}^{K} p_k \mathbb{E}\Big[\Big(\frac{Z_n^{(k)} - 1}{n}\Big)^{\frac{3}{2}}\Big] \\ &\to \sum_{k=0}^{K} p_k \mathbb{E}\big[\big(U^{(k)}\big)^{\frac{3}{2}}\big] = \sum_{k=0}^{K} p_k \int_0^1 f_k(t) t^{\frac{3}{2}} \mathrm{d}t \\ &= \sum_{k=0}^{K} p_k \frac{\Gamma\big(k+1+\frac{3}{2}\big)\Gamma(2k+2)}{\Gamma\big(2k+2+\frac{3}{2}\big)\Gamma(k+1)} =: \sum_{k=0}^{K} p_k \varphi(k).\end{aligned}$$

The function $\varphi(\cdot)$ is strictly decreasing because,

$$\varphi(k+1) = \frac{4(2k+3)(2k+5)}{(4k+7)(4k+9)}\varphi(k) < \varphi(k) \le \varphi(0) = 0.4, \qquad k \ge 0.$$

Lemma 3.4. As $n \to \infty$, $\mathbb{E}[|Y_n^3|] = \mathcal{O}(1)$.

Proof. From (24) we obtain

$$\begin{aligned}\mathbb{E}[|Y_n^3|] \le\ & 2\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^3\big] + 6\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^2 \big|\overline{B}_{n - Z_n^\mu}\big|\big] \\ &+ 6\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^2 \big|v_n(Z_n^\mu)\big|\big] + 6\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|\big|v_n(Z_n^\mu)\big|^2\big] \qquad (26) \\ &+ 6\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\overline{B}_{n - Z_n^\mu} v_n(Z_n^\mu)\big|\big] + \mathbb{E}\big[\big|v_n(Z_n^\mu)\big|^3\big].\end{aligned}$$

Note that $\exists\, u^* > 0$ such that $u_n < u^*$ and $\overline{u}_n < u^*$ for $n \ge 1$, by (22) and (23). Let

$$\xi_n := 1 \vee \max_{0 \le j \le n} \mathbb{E}[|Y_j^3|].$$

Using (23), for $0 \le k \le K$ and $k+1 \le m \le n-k$, conditionally given $Z_n^{(k)} = m$,

$$\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^3\big] \le \mathbb{E}\big[\big(u_n(Z_n^\mu)\big)^3\big]\xi_{n-1} \qquad (27)$$

$$\begin{aligned}\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^2 \big|\overline{B}_{n - Z_n^\mu}\big|\big] &\le u^{*3}\big(\max_{0 \le j \le n-1} \mathbb{E}[|Y_j|^2]\big)\big(\max_{0 \le j \le n-1} \mathbb{E}[|Y_j|]\big) \\ &= \mathcal{O}(1). \qquad \text{(by (14))} \qquad (28)\end{aligned}$$

By Hölder inequality, the two summands in line (26) are bounded, e.g., for the first one, from (13) and (27), we have

$$\begin{aligned}\mathbb{E}\big[\big|B_{Z_n^\mu - 1}\big|^2 \big|v_n(Z_n^\mu)\big|\big] &\le u^{*2} \sum_{k=0}^{K} p_k \big\{\mathbb{E}\big[\big|Y_{Z_n^{(k)} - 1}\big|^3\big]\big\}^{\frac{2}{3}} \big\{\mathbb{E}\big[\big|v_n(Z_n^{(k)})\big|^3\big]\big\}^{\frac{1}{3}} \\ &\le u^{*2} \sum_{k=0}^{K} p_k \xi_{n-1}^{\frac{2}{3}} \big\{\mathbb{E}\big[\big|v_n(Z_n^{(k)})\big|^3\big]\big\}^{\frac{1}{3}} \\ &\le o(1)\xi_{n-1}.\end{aligned}$$

We similarly have

$$
\begin{aligned}
&\mathbb{E}\big[\big|B_{Z_n^\mu-1}\overline{B}_{n-Z_n^\mu}v_n(Z_n^\mu)\big|\big]\\
&\le {u^*}^2\sum_{k=0}^{K}p_k\big\{\mathbb{E}\big[\big|Y_{Z_n^{(k)}-1}\big|^3\big]\big\}^{\frac{1}{3}}\big\{\mathbb{E}\big[\big|Y_{Z_n^{(k)}-1}\big|^3\big]\big\}^{\frac{1}{3}}\big\{\mathbb{E}\big[\big|v_n(Z_n^{(k)})\big|^3\big]\big\}^{\frac{1}{3}}\\
&\le o(1)\xi_{n-1}.
\end{aligned}
$$

The summand $\mathbb{E}\big[\big|v_n(Z_n^\mu)\big|^3\big]$ tends to zero by (13). Collecting all terms, we obtain

$$
\begin{aligned}
\mathbb{E}[|Y_n^3|] &\le \big(2\mathbb{E}\big[\big(u_n(Z_n^\mu)\big)^3\big]+o(1)\big)\xi_{n-1}+\mathcal{O}(1)\\
&=\Big(2\sum_{k=0}^{K}p_k\mathbb{E}\big[\big(U^{(k)}\big)^{\frac{3}{2}}\big]+o(1)\Big)\xi_{n-1}+\mathcal{O}(1) \qquad (29)\\
&\le \big(0.8+o(1)\big)\xi_{n-1}+\mathcal{O}(1) \quad \text{(by Remark 1).}
\end{aligned}
$$

Hence, there exist an $n_0 \in \mathbb{N}$ and a constant $0 < \alpha < \infty$ such that for $n \ge n_0$

$$\mathbb{E}[|Y_n^3|] \le 0.9\xi_{n-1}+\alpha.$$

By induction, we have $\mathbb{E}[|Y_n^3|] \le \xi_{n_0} \vee (10\alpha)$ for all $n \ge 0$. This implies the claim. □

In the following, we begin to prove the asymptotic normality distribution for X_n.

Theorem 3.5. Let X_n be the number of recalls for RMQ(μ) on an array of size n. Then, as $n \to \infty$, for some constants c and σ^2 defined in (6) and (14), respectively,

$$\frac{X_n - cn}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2).$$

Proof. Let $\mathcal{N}$ be a standard normal random variable. Using Lemma 3.4 and (19), we have $\zeta_3(Y_n,\mathcal{N}) < \infty$. So, by (20), we just need to show that the Zolotarev metric between the random variables Y_n and $\mathcal{N}$, approaches 0, as $n \to \infty$.

For W_1 and W_2 independent standard normal random variables also independent of Z_n^μ, we set ($u_n(\cdot)$ and $\overline{u}_n(\cdot)$ defined in (23))

$$\Theta_n := u_n(Z_n^\mu)W_1 + \overline{u}_n(Z_n^\mu)W_2, \qquad n \ge n_0 := 2k+2.$$

Note that $\mathrm{Var}(\Theta_n) > 0$ for all $n \ge n_0$, and $\mathrm{Var}(\Theta_n) \to 1$ as $n \to \infty$. Hence there exists a deterministic sequence $(\delta_n)_{n\ge n_0}$ with $\delta_n \to 0$ as $n \to \infty$ such that $\mathrm{Var}\big((1+\delta_n)\Theta_n\big) = 1$ for all $n \ge n_0$. So, each pair from the random variables Y_n, $(1+\delta_n)\Theta_n$ and $\mathcal{N}$ satisfy(19). Thus we obtain

$$\zeta_3(Y_n,\mathcal{N}) \le \zeta_3\big(Y_n,(1+\delta_n)\Theta_n\big) + \zeta_3\big((1+\delta_n)\Theta_n,\mathcal{N}\big).$$

We introduce $\Phi_n := u_n(Z_n^\mu)Y_{Z_n^\mu-1} + \overline{u}_n(Z_n^\mu)\overline{Y}_{n-Z_n^\mu}$ and $\nu_n := v_n(Z_n^\mu)$. Then we have $Y_n \stackrel{d}{=} \Phi_n + \nu_n$ by (23). Now Lemma 3.2 yields

$$\zeta_3\big(Y_n,(1+\delta_n)\Theta_n\big) \le \zeta_3\big(\Phi_n,\Theta_n\big) + \frac{1}{2}\|\Phi_n\|_3^2\|\nu_n\|_3 + \frac{1}{2}\|\Phi_n\|_3\|\nu_n\|_3^2 + \frac{1}{6}\|\nu_n\|_3^3 + \Big(\frac{1}{2}|\nu_n| + \frac{1}{2}\nu_n^2 + \frac{1}{6}|\nu_n|^3\Big)\|\Phi_n\|_3^3.$$

Since $\sup_{n\ge n_0}\|\Phi_n\|_3 < \infty$ and $\|\nu_n\|_3 \to 0$ ($\nu_n \xrightarrow{\text{a.s.}} 0$) then

$$\zeta_3\big(Y_n,(1+\delta_n)\Theta_n\big) \le \zeta_3\big(\Phi_n,\Theta_n\big) + o(1).$$

The bound (21) implies $\zeta_3\big((1+\delta_n)\Theta_n,\mathcal{N}\big) \le M\ell_3\big((1+\delta_n)\Theta_n,\mathcal{N}\big)$ for all $n \ge n_0$ and a finite constant $M > 0$. Using Lemma 3.3 we obtain

$$\begin{aligned}\zeta_3\big((1+\delta_n)\Theta_n,\mathcal{N}\big) &\le M\ell_3\big((1+\delta_n)\Theta_n,\mathcal{N}\big)\\ &\le M\Bigg\|\Bigg((1+\delta_n)u_n(Z_n^\mu) - \sqrt{\frac{Z_n^\mu}{n}}\Bigg)W_1\\ &\qquad + \Bigg((1+\delta_n)\overline{u}_n(Z_n^\mu) - \sqrt{\frac{n-Z_n^\mu}{n}}\Bigg)W_2\Bigg\|_3\\ &\to 0.\end{aligned}$$

From (17), (18), (22) and Lemma 3.1, we can conclude that

$$\begin{aligned}\zeta_3(Y_n,\mathcal{N}) &\le \zeta_3\big(\Phi_n,\Theta_n\big) + o(1)\\ &\le \sum_{k=0}^{K} p_k\zeta_3\Big(u_n(Z_n^{(k)})Y_{Z_n^{(k)}-1} + \overline{u}_n(Z_n^{(k)})\overline{Y}_{n-Z_n^{(k)}},\\ &\qquad u_n(Z_n^{(k)})W_1 + \overline{u}_n(Z_n^{(k)})W_2\Big) + o(1)\\ &\le \sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)}\zeta_3\Big(u_n(m)Y_{m-1} + \overline{u}_n(m)\overline{Y}_{n-m},\\ &\qquad u_n(m)W_1 + \overline{u}_n(m)W_2\Big) + o(1)\\ &= 2\sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} p_{n,m}^{(k)}\big(u_n(m)\big)^3\zeta_3\big(Y_{m-1},\mathcal{N}\big) + o(1)\\ &= 2\sum_{k=0}^{K} p_k\mathbb{E}\big[\big(u_n(Z_n^{(k)})\big)^3\zeta_3(Y_{Z_n^{(k)}-1},\mathcal{N})\big] + o(1) \qquad (30)\\ &\le 2\sum_{k=0}^{K} p_k\Big(\mathbb{E}\big[\big(U^{(k)}\big)^{\frac{3}{2}}\big] + o(1)\Big)\sup_{0\le m\le n}\zeta_3(Y_{m-1},\mathcal{N}) + o(1).\end{aligned}$$

This implies, similarly to the inequality (29), $\big(\zeta_3(Y_n, \mathcal{N})\big)_{n\geq 0}$ that is bounded. We denote $\xi := \sup_{n\geq 0} \zeta_3(Y_n, \mathcal{N})$ and $s := \limsup_{n\to\infty} \zeta_3(Y_n, \mathcal{N}) \geq 0$. For any $\varepsilon > 0$ there exists an $n_1 \geq n_0$ such that $\zeta_3(Y_n, \mathcal{N}) \leq s + \varepsilon$ for all $n \geq n_1$. Hence, from (30) we obtain

$$\begin{aligned}\zeta_3(Y_n, \mathcal{N}) &\leq 2\sum_{k=0}^{K} p_k \mathbb{E}\big[\mathbb{1}_{\{Z_n^{(k)} \leq n_1\}} \big(u_n(Z_n^{(k)})\big)^3\big]\,\xi \\ &\quad + 2\sum_{k=0}^{K} p_k \mathbb{E}\big[\mathbb{1}_{\{Z_n^{(k)} > n_1\}} \big(u_n(Z_n^{(k)})\big)^3\big]\,(s+\varepsilon) + o(1) \\ &\quad \sim 2\sum_{k=0}^{K} p_k \mathbb{E}\big[\mathbb{1}_{\{Z_n^{(k)} > n_1\}} \big(u_n(Z_n^{(k)})\big)^3\big]\,(s+\varepsilon) + o(1).\end{aligned}$$

Therefore $0 \leq s = \limsup_{n\to\infty} \zeta_3(Y_n, \mathcal{N}) \leq 0.8(s+\varepsilon) < s + \varepsilon$. Since $\varepsilon > 0$ is arbitrary then we have $s = 0$. Namely, $\lim_{n\to\infty} \zeta_3(Y_n, \mathcal{N}) = 0$. By (20), the assertion holds. □

4. A Simulation Study

We now investigate the empirical performance of the various sorting algorithms proposed in this paper to sort n distinct numbers through an intensive simulation study. For each proposed sorting algorithm, the code to implement the method, run these simulations and produce the plots is provided with the R programming software (code available upon request). All simulations are carried out with `R` software and in Intel(R), Core(TM) i5-7200U CPU, 2.50GHz, 2701 Mhz, 32-bit processor PC laptop with 8 GB RAM memory, only.

The simulation consists of the generation of random permutations of $1, 2, \ldots, n$, and sort them in order to compare the number of recalls. The generated permutations for sorting are uniformly at random chosen from one to n integer numbers using `sample` function in `R` so that `sample(1:n, size=n, replace = FALSE)` generates a random permutation of the elements of $1 : n$. Indeed sample function takes a sample of the specified size from the elements of a vector using with or without replacement.

For each n, $n = 1, 2, \ldots, 4000$ number of elements, after generating the random permutations discussed above, the standard Quicksort, 3-MQ and RMQ(2) and RMQ(5) sorting algorithms are repeated 1000 times and the number of recalls are counted and saved in 4000×1000 matrices. The average number of recalls and variance of recalls versus the number of sorted elements for these 1000 replications were then displayed in Figure 2.

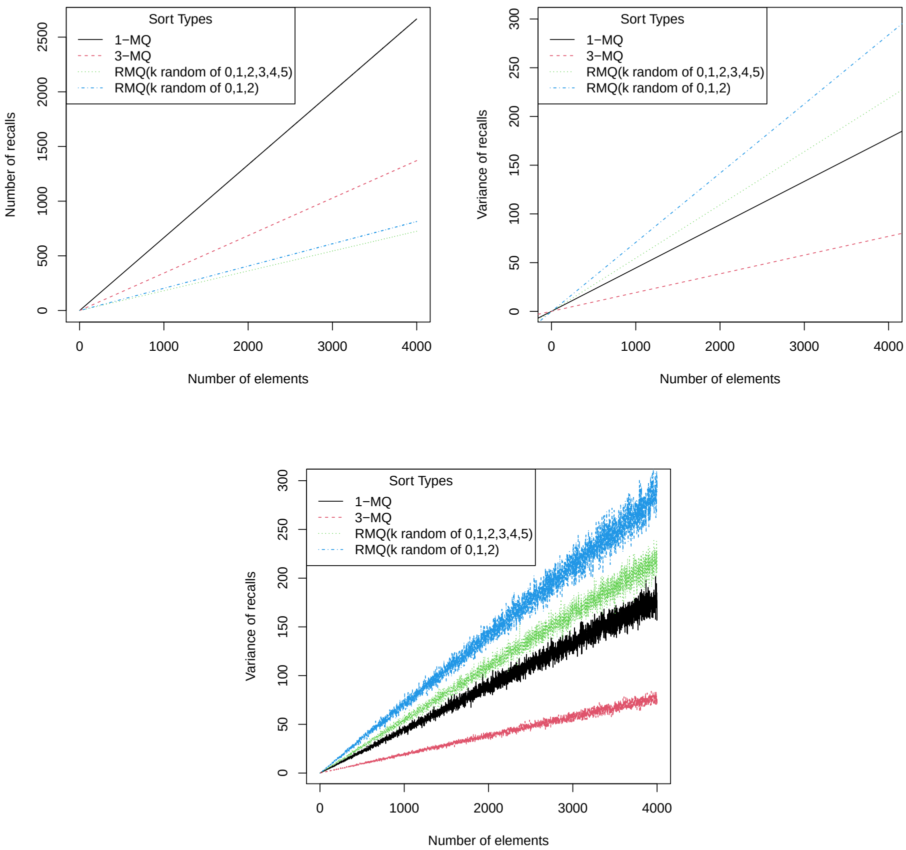


Figure 2: The average and variance of recalls for four algorithms with different number of elements.

Functions `rowMeans` and `apply(matrix, 1, var)` are applied to calculate the average number of recalls and variance of recalls for these 1000 replications for each size of n. It is easy to see that the algorithms run in linear time. As variance of recalls are noisy, linear regression with `lm` function fitted to the variance of recalls and shown in Figure 2. From Figure 2 it appears that the average number and variance of recalls increases linearly with estimated slopes resulting from `lm` function are shown in Table 1. Note also that the histogram for standardized number of recalls for $n = 4000$ and 1000 repeats are shown in Figure 3 which are consistent with the standard normal distribution. From these figures we conclude that, in term of number of recalls, the RMQ(5) algorithm performs better than RMQ(2), 3-MQ and 1-MQ, i.e., the standard Quicksort algorithm.

Table 1: Estimated slopes of the linear trends of the average and variance of recalls.

Algorithm	1-MQ	3-MQ	RMQ(2)	RMQ(5)
Slope (Average)	0.6666654	0.3428574	0.2035475	0.1813382
Slope (Variance)	0.04441642	0.01922950	0.0709953	0.054628859

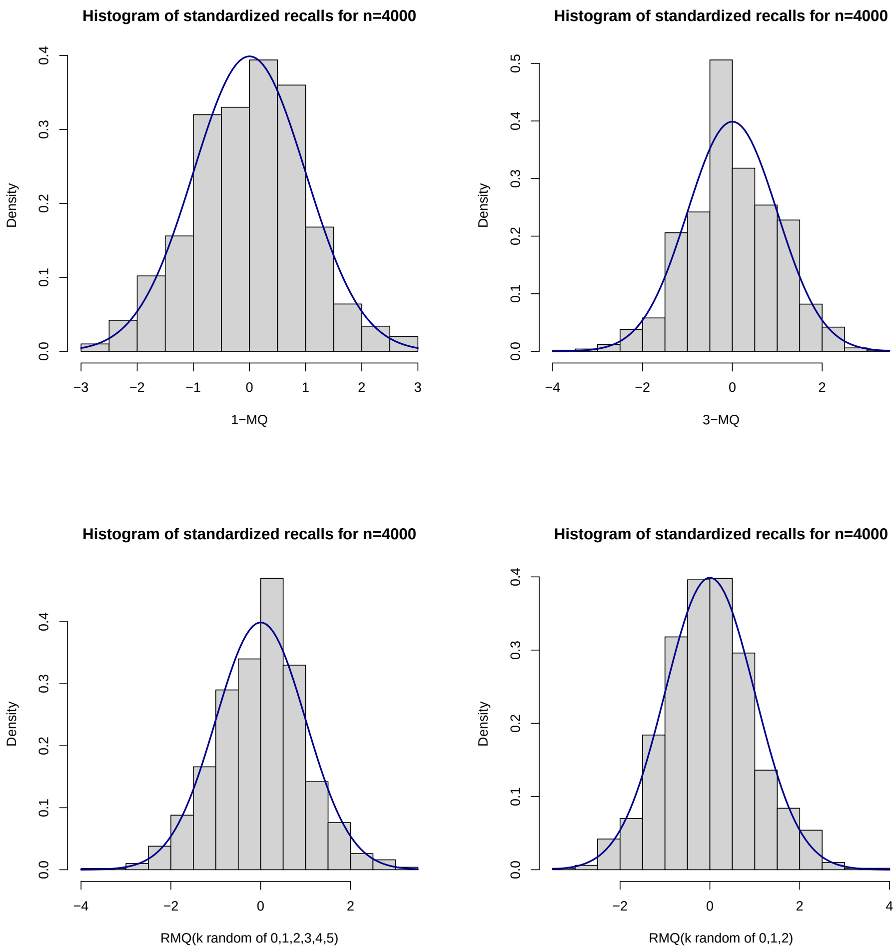


Figure 3: Histogram for the distribution of number of recalls for $n = 4000$ and 1000 repeats in different sorting algorithms.

5. Conclusion

Javanian and Roesler [10] have analyzed the correctly normalized number $X(n,l)$ of "comparisons" to sort the l smallest out of n elements in $2k+1$-Median version of Quicksort ($2k+1$-MQ). However, it has not been analyzed $Y(n,l)$, the number of "recalls" to sort the l smallest out of n elements in $2k+1$-MQ. The results of this paper is for the special case $l=n$, i.e., the number of "recalls" to sort the n elements in $2k+1$-MQ. In order to study the limit behavior of $Y(n,l)$ as $n\to\infty$, for $l=0,1,\ldots,n$, we can define

$$\Big(Y_n\Big(\frac{l}{n}\Big)\Big)_l := \Big(\frac{Y(n,l)-\mathbb{E}(Y(n,l))}{\sqrt{n}}\Big)_l.$$

Next, by extending the process Y_n to the process $Y_n(t) := Y_n\big(\frac{\lceil nt\rceil}{n}\big)$ on $[0,1]$, we can guess that the process Y_n converges weakly to a Brownian motion.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] C. J. Bell, *An Investigation into the Principles of the Classification and Analysis of Data of an Automatic Digital Computer*, Ph.D. Thesis, Leeds University, UK, 1965.

[2] T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein, *Introduction to Algorithms*, 2nd Ed., MIT Press, USA, 2001.

[3] L. Devroye, Limit laws for local counters in random binary search trees, *Random Struct. Algor.* **2** (1991) 303 − 316.

[4] P. Hennequin, *Analyse Enmoyenne D'algorithmes, Tri Rapide et Arbres de Recherche*, Ph.D. Thesis, École Polytechnique, Palaiseau, 1991.

[5] C. A. R. Hoare, Quicksort, *Comput. J.* **5** (1962) 10 − 15.

[6] C. Holmgren and S. Janson, Limit laws for functions of fringe trees for binary search trees and random recursive trees, *Electron. J. Probab.* **20** (4) (2015) 1 − 51.

[7] V. Iliopoulos, A note on multipivot Quicksort, *J. Info. Optim. Sci.* **39** (2018) 1139 − 1147.

[8] V. Iliopoulos, *The Quicksort Algorithm and Related Topics*, PhD Thesis, Department of Mathematical Sciences, University of Essex, 2013.

[9] D. E. Knuth, *The Art of Computer Programming, Vol. III: Sorting and Searching*, 2nd Ed., Addison-Wesley Publishing Company, Reading, MA, USA, 1998.

[10] M. Javanian and U. Röesler, Median Quicksort process, *Probab. Eng. Inf. Sci.* (2021) Submitted.

[11] H. M. Mahmoud, *Sorting: A Distribution Theory*, John Wiley & Sons, New York, 2000.

[12] R. Neininger and L. Rüschendorf, On the contraction method with degenerate limit equation, *Ann. Probab.* **32** (3B) (2004) 2838 – 2856.

[13] R. Neininger, Refined Quicksort asymptotics, *Random Struct. Algor.* **46** (2015) 346 – 361.

[14] H. M. Okasha and U. Röesler, Asymptotic distribution for random median Quicksort, *J. Discrete Algor.* **5** (2007) 592 – 608.

[15] S. Rachev and L. Rüschendorf, Probability metrics and recursive algorithms, *Adv. Appl. Probab.* **27** (1995) 770 – 799.

[16] S. Rachev, *Probability Metrics and the Stability of Stochastic Models*, John Wiley & Sons, New York, 1991.

[17] U. Röesler, A limit theorem for "Quicksort", *RAIRO Théor. Inform Appl.* **25** (1991) 85 – 100.

[18] U. Röesler, On the analysis of stochastic divide and conquer algorithms, *Algorithmica* **29** (12) (2001) 238 – 261.

[19] U. Röesler and L. Rüschendorf, The contraction method for recursive algorithms, *Algorithmica* **29** (2001) 3 – 33.

[20] R. S. Scowen, Algorithm 271: Quickersort, *Commun. ACM* **8** (1965) 669–670.

[21] R. Sedgewick, *Quicksort*, Ph.D. Thesis, Garland Pub. Co., New York, 1980.

[22] R. C. Singleton, Algorithm 347: An efficient algorithm for sorting with minimal storage, *Commun. ACM* **12** (3) (1969) 185 – 186.

[23] K. H. Tan, *An Asymptotic Analysis of the Number of Comparisons in Multipartition Quicksort*, Ph.D. Thesis, Carnegie Mellon University, 1993.

[24] A. Walker and D. Wood, Locally balanced binary trees, *Comput. J.* **19** (1976) 322 – 325.

[25] V. M. Zolotarev, Approximation of distributions of sums of independent random variables with values in infinite-dimensional spaces, *Theory Probab. Appl.* **21** (4) (1976) 721 – 737.

Mehri Javanian
Department of Statistics,
University of Zanjan,
Zanjan, I. R. Iran
e-mail: javanian@znu.ac.ir

Ali Mohammadian Mosammam
Department of Statistics,
University of Zanjan,
Zanjan, I. R. Iran
e-mail: a.m.mosammam@znu.ac.ir